VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-011-0001-x
Math. Slovaca **61** (2011), No. 2, 127–154

LOCAL PSEUDO-BCK ALGEBRAS WITH PSEUDO-PRODUCT

Lavinia Corina Ciungu

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. Pseudo-BCK algebras were introduced by G. Georgescu and A. Iorgulescu as a generalization of BCK algebras in order to give a corresponding structure to pseudo-MV algebras, since the bounded commutative BCK algebras correspond to MV algebras. Properties of pseudo-BCK algebras and their connections with other fuzzy structures were established by A. Iorgulescu and J. Kühr. The aim of this paper is to define and study the local pseudo-BCK algebras with pseudo-product. We will also introduce the notion of perfect pseudo-BCK algebras with pseudo-product and we will study their properties. We define the radical of a bounded pseudo-BCK algebra with pseudo-product and we prove that it is a normal deductive system. Another result consists of proving that every strongly simple pseudo-hoop is a local bounded pseudo-BCK algebra with pseudo-product.

1. Introduction

Pseudo-BCK algebras were introduced in [11] by G. Georgescu and A. Iorgulescu as a generalization of BCK algebras in order to give a corresponding structure to pseudo-MV algebras, since the bounded commutative BCK algebras correspond to MV algebras. Properties of pseudo-BCK algebras and their connections with others fuzzy structures were established by A. Iorgulescu in [15], [16], [17], [18]. The pseudo-product property (pP for short) proved to be very important to establish connections of pseudo-BCK algebras with other fuzzy structures. It was proved in [17] that the pseudo-BCK(pP) algebras are categorically equivalent with the partially ordered residuated integral monoids (porims) and it was proved in [15] that the pseudo-BCK(pP) lattices

2010 Mathematics Subject Classification: Primary 03G25, 06F05, 06F35.
Keywords: pseudo-BCK algebra, deductive system, local pseudo-BCK(pP) algebra, good pseudo-BCK(pP) algebra, maximal deductive system, normal deductive system, primary deductive system, perfect pseudo-BCK(pP) algebra, radical.

are termwise equivalent with the residuated lattices which generalize other structures such as pseudo-MTL algebras, bounded divisible non-commutative algebras (Rℓ-monoids), pseudo-BL algebras and pseudo-MV algebras. Pseudo-Iséki algebras were introduced in [18] and it was proved that they are categorically equivalent with the pseudo-BL algebras. J. Kühr proved in [20] that every pseudo-BCK algebra is a subreduct of a residuated lattice. Deductive systems of a pseudo-BCK algebra were introduced and studied in [14].

Local MV-algebras were studied in [1], local BL-algebras were studied in [25], while local bounded commutative Rℓ-monoids were investigated in [24]. For the case of non-commutative structures, local pseudo-MV algebras were presented in [22], local pseudo-BL algebras in [12], local pseudo-MTL algebras in [6] and local residuated lattices in [5]. Recently, properties of local bounded non-commutative Rℓ-monoids were investigated in [23]. In this paper we study new properties of the deductive systems of a pseudo-BCK(pP) algebra and we define and study the primary and the perfect deductive systems of a bounded pseudo-BCK(pP) algebra. We define and study the local pseudo-BCK algebras with pseudo-product. We will also introduce the notion of perfect pseudo-BCK(pP) algebra with pseudo-product and we will study their properties. The local bounded pseudo-BCK(pP) algebras are characterized in terms of primary deductive systems, while the perfect pseudo-BCK(pP) algebras are characterized in terms of perfect deductive systems. One of the main results consists of proving that the radical of a bounded pseudo-BCK(pP) algebra is normal. We also prove that every strongly simple pseudo-hoop is a local bounded pseudo-BCK(pP) algebra. Additionally, we prove some new properties of pseudo-BCK algebras.

2. Pseudo-BCK algebras and their basic properties

Definition 2.1. ([15]) A *pseudo-BCK algebra* (more precisely, *reversed left-pseudo-BCK algebra*) is a structure $\mathscr{A} = (A, \leq, \to, \rightsquigarrow, 1)$ where $\leq$ is a binary relation on A, $\to$ and $\rightsquigarrow$ are binary operations on A and 1 is an element of A satisfying, for all $x, y, z \in A$, the axioms:

(A_1) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$, $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$;

(A_2) $x \leq (x \to y) \rightsquigarrow y$, $x \leq (x \rightsquigarrow y) \to y$;

(A_3) $x \leq x$;

(A_4) $x \leq 1$;

(A_5) if $x \leq y$ and $y \leq x$, then $x = y$;

(A_6) $x \leq y$ iff $x \to y = 1$ iff $x \rightsquigarrow y = 1$.

Remark 2.2. ([15]) A pseudo-BCK algebra $\mathscr{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$ is *commutative* iff $\rightarrow = \rightsquigarrow$. Any commutative pseudo-BCK algebra is a BCK algebra.

Example 2.3. Consider $A = \{o_1, a_1, b_1, c_1, o_2, a_2, b_2, c_2, 1\}$ with $o_1 < a_1, b_1 < c_1 < 1$ and a_1, b_1 incomparable, $o_2 < a_2, b_2 < c_2 < 1$ and a_2, b_2 incomparable. Assume also that any element of the set $\{o_1, a_1, b_1, c_1\}$ is incomparable with any element of the set $\{o_2, a_2, b_2, c_2\}$. Consider the operations $\rightarrow$, $\rightsquigarrow$ given by the following tables:

$\rightarrow$	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1
o_1	1	1	1	1	o_2	a_2	b_2	c_2	1
a_1	o_1	1	b_1	1	o_2	a_2	b_2	c_2	1
b_1	a_1	a_1	1	1	o_2	a_2	b_2	c_2	1
c_1	o_1	a_1	b_1	1	o_2	a_2	b_2	c_2	1
o_2	o_1	a_1	b_1	c_1	1	1	1	1	1
a_2	o_1	a_1	b_1	c_1	o_2	1	b_2	1	1
b_2	o_1	a_1	b_1	c_1	c_2	c_2	1	1	1
c_2	o_1	a_1	b_1	c_1	o_2	c_2	b_2	1	1
1	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1

$\rightsquigarrow$	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1
o_1	1	1	1	1	o_2	a_2	b_2	c_2	1
a_1	b_1	1	b_1	1	o_2	a_2	b_2	c_2	1
b_1	o_1	a_1	1	1	o_2	a_2	b_2	c_2	1
c_1	o_1	a_1	b_1	1	o_2	a_2	b_2	c_2	1
o_2	o_1	a_1	b_1	c_1	1	1	1	1	1
a_2	o_1	a_1	b_1	c_1	b_2	1	b_2	1	1
b_2	o_1	a_1	b_1	c_1	b_2	c_2	1	1	1
c_2	o_1	a_1	b_1	c_1	b_2	c_2	b_2	1	1
1	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1

Then $\mathscr{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$ is a proper pseudo-BCK algebra.

Proposition 2.4. ([17], [18]) *In any pseudo-BCK algebra the following properties hold:*

(c$_1$) $x \leq y$ *implies* $y \rightarrow z \leq x \rightarrow z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$;

(c$_2$) $x \leq y, y \leq z$ *implies* $x \leq z$;

(c$_3$) $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$;

(c$_4$) $z \leq y \rightarrow x$ *iff* $y \leq z \rightsquigarrow x$;

(c_5) $z \to x \le (y \to z) \to (y \to x)$ *and* $z \rightsquigarrow x \le (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x)$;

(c_6) $x \le y \to x$, $x \le y \rightsquigarrow x$;

(c_7) $1 \to x = x = 1 \rightsquigarrow x$;

(c_8) $x \le y$ *implies* $z \to x \le z \to y$ *and* $z \rightsquigarrow x \le z \rightsquigarrow y$;

(c_9) $[(y \to x) \rightsquigarrow x] \to x = y \to x$, $[(y \rightsquigarrow x) \to x] \rightsquigarrow x = y \rightsquigarrow x$.

Definition 2.5. ([15]) If there is an element 0 of a pseudo-BCK algebra $\mathscr{A} = (A, \le, \to, \rightsquigarrow, 1)$ such that $0 \le x$ (i.e. $0 \to x = 0 \rightsquigarrow x = 1$), for all $x \in A$, then 0 is called the *zero* of $\mathscr{A}$. A pseudo-BCK algebra with zero is called *bounded pseudo-BCK algebra* and it is denoted by $\mathscr{A} = (A, \le, \to, \rightsquigarrow, 0, 1)$.

We note that $\le$ is a partial order on A, thus A is bounded if it has least element with respect to $\le$.

Example 2.6. Consider $A = \{0, a, b, c, 1\}$ with $0 < a$, $b < c < 1$ and a, b incomparable. Consider the operations $\to$, $\rightsquigarrow$ given by the following tables:

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	0	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $\mathscr{A} = (A, \le, \to, \rightsquigarrow, 0, 1)$ is a bounded pseudo-BCK algebra.

Definition 2.7. ([15]) A pseudo-BCK algebra with (pP) *condition* (i.e. with *pseudo-product* condition) or a *pseudo-BCK*(pP) *algebra* for short, is a pseudo-BCK algebra $\mathscr{A} = (A, \le, \to, \rightsquigarrow, 1)$ satisfying (pP) condition:

$$\text{(pP)}\quad (\forall x, y \in A)(\exists w \in A)\big(w = x \odot y := \min\{z :\ x \le y \to z\} = \min\{z :\ y \le x \rightsquigarrow z\}\big).$$

If A is a pseudo-BCK(pP) algebra, then for any $n \in \mathbb{N}$, $x \in A$ we put $x^0 = 1$ and $x^{n+1} = x^n \odot x = x \odot x^n$. If A is bounded, the *order* of $x \in A$, denoted $\mathrm{ord}(x)$ is the smallest $n \in \mathbb{N}$ such that $x^n = 0$. If there is no such n, then $\mathrm{ord}(x) = \infty$.

Definition 2.8. ([15])

(1) Let $\mathscr{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCK algebra. If the poset $(A, \leq)$ is a lattice, then we say that $\mathscr{A}$ is a *pseudo-BCK lattice*.

(2) Let $\mathscr{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCK(pP) algebra. If the poset $(A, \leq)$ is a lattice, then we say that $\mathscr{A}$ is a *pseudo-BCK*(pP) *lattice*.

A pseudo-BCK(pP) lattice $\mathscr{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$ will be denoted by

$$\mathscr{A} = (A, \vee, \wedge, \rightarrow, \rightsquigarrow, 1).$$

Remarks 2.9.

(1) ([17]) Pseudo-BCK(pP) algebras are categorically isomorphic with *left-porims* (partially ordered, residuated, integral left-monoids).

(2) ([15]) (Bounded) pseudo-BCK(pP) lattices are categorically isomorphic with (bounded) integral residuated lattices.

Example 2.10.

(1) If $\mathscr{A} = (A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$ is the bounded pseudo-BCK lattice from Example 2.6, then $\min\{z : \ b \leq a \rightarrow z\} = \min\{a, b, c, 1\}$ and $\min\{z : \ a \leq b \rightsquigarrow z\} = \min\{a, b, c, 1\}$ do not exist. Thus, $b \odot a$ does not exist, so $\mathscr{A}$ is not a pseudo-BCK(pP) algebra. Moreover, since $(A, \leq)$ is a lattice, it follows that $\mathscr{A}$ is a pseudo-BCK lattice.

(2) If $\mathscr{A} = (A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$ is a reduct of a residuated lattice, then it is obvious that $\mathscr{A}$ is a bounded pseudo-BCK(pP) algebra.

Example 2.11. ([16]) Take $A = \{0, a_1, a_2, s, a, b, n, c, d, m, 1\}$ with $0 < a_1 < a_2 < s < a,\ b < n < c,\ d < m < 1$ (a is incomparable with b and c is incomparable with d). Consider the operations $\rightarrow$, $\rightsquigarrow$ given by the following tables:

$\rightarrow$	0	a_1	a_2	s	a	b	n	c	d	m	1
0	1	1	1	1	1	1	1	1	1	1	1
a_1	a_1	1	1	1	1	1	1	1	1	1	1
a_2	a_1	a_1	1	1	1	1	1	1	1	1	1
s	0	a_1	a_2	1	1	1	1	1	1	1	1
a	0	a_1	a_2	m	1	m	1	1	1	1	1
b	0	a_1	a_2	m	m	1	1	1	1	1	1
n	0	a_1	a_2	m	m	m	1	1	1	1	1
c	0	a_1	a_2	m	m	m	m	1	m	1	1
d	0	a_1	a_2	m	m	m	m	m	1	1	1
m	0	a_1	a_2	m	m	m	m	m	m	1	1
1	0	a_1	a_2	s	a	b	n	c	d	m	1

$\rightsquigarrow$	0	a_1	a_2	s	a	b	n	c	d	m	1
0	1	1	1	1	1	1	1	1	1	1	1
a_1	a_2	1	1	1	1	1	1	1	1	1	1
a_2	0	a_1	1	1	1	1	1	1	1	1	1
s	0	a_1	a_2	1	1	1	1	1	1	1	1
a	0	a_1	a_2	m	1	m	1	1	1	1	1
b	0	a_1	a_2	m	m	1	1	1	1	1	1
n	0	a_1	a_2	m	m	m	1	1	1	1	1
c	0	a_1	a_2	m	m	m	m	1	m	1	1
d	0	a_1	a_2	m	m	m	m	m	1	1	1
m	0	a_1	a_2	m	m	m	m	m	m	1	1
1	0	a_1	a_2	s	a	b	n	c	d	m	1

Then $\mathscr{A} = (A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$ is a bounded pseudo-BCK(pP) algebra. The operation $\odot$ is given by the following table:

$\odot$	0	a_1	a_2	s	a	b	n	c	d	m	1
0	0	0	0	0	0	0	0	0	0	0	0
a_1	0	0	0	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_2	0	a_1	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2
s	0	a_1	a_2	s	s	s	s	s	s	s	s
a	0	a_1	a_2	s	s	s	s	s	s	s	a
b	0	a_1	a_2	s	s	s	s	s	s	s	b
n	0	a_1	a_2	s	s	s	s	s	s	s	n
c	0	a_1	a_2	s	s	s	s	s	s	s	c
d	0	a_1	a_2	s	s	s	s	s	s	s	d
m	0	a_1	a_2	s	s	s	s	s	s	s	m
1	0	a_1	a_2	s	a	b	n	c	d	m	1

Proposition 2.12. ([18]) *In any pseudo-BCK*(pP) *algebra the following properties hold:*

(c_{10}) $x \odot y \leq x, y$;

(c_{11}) $(x \rightarrow y) \odot x \leq x, y$, $x \odot (x \rightsquigarrow y) \leq x, y$;

(c_{12}) $y \leq x \rightarrow (y \odot x)$, $y \leq x \rightsquigarrow (x \odot y)$;

(c_{13}) $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$, $x \rightsquigarrow y \leq (z \odot x) \rightsquigarrow (z \odot y)$;

(c_{14}) $x \odot (y \rightarrow z) \leq y \rightarrow (x \odot z)$, $(y \rightsquigarrow z) \odot x \leq y \rightsquigarrow (z \odot x)$;

(c_{15}) $(y \rightarrow z) \odot (x \rightarrow y) \leq x \rightarrow z$, $(x \rightsquigarrow y) \odot (y \rightsquigarrow z) \leq x \rightsquigarrow z$;

(c_{16}) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$, $x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z$;

(c_{17}) $(x \odot z) \rightarrow (y \odot z) \leq x \rightarrow (z \rightarrow y)$, $(z \odot x) \rightsquigarrow (z \odot y) \leq x \rightsquigarrow (z \rightsquigarrow y)$;

(c_{18}) $x \to y \leq (x \odot z) \to (y \odot z) \leq x \to (z \to y)$,
$x \rightsquigarrow y \leq (z \odot x) \rightsquigarrow (z \odot y) \leq x \rightsquigarrow (z \rightsquigarrow y)$;

(c_{19}) $x \leq y$ *implies* $x \odot z \leq y \odot z$ *and* $z \odot x \leq z \odot y$.

Let $\mathscr{A} = (A, \leq, \to, \rightsquigarrow, 0, 1)$ be a bounded pseudo-BCK algebra. We define two negations $^-$ and $^\sim$ ([18]): for all $x \in A$,

$$x^- = x \to 0, \qquad x^\sim = x \rightsquigarrow 0.$$

Proposition 2.13. ([18]) *In a bounded pseudo-BCK algebra the following hold:*

(c_{20}) $1^- = 0 = 1^\sim$, $0^- = 1 = 0^\sim$;

(c_{21}) $x \leq (x^-)^\sim$, $x \leq (x^\sim)^-$;

(c_{22}) $x \to y \leq y^- \rightsquigarrow x^-$, $x \rightsquigarrow y \leq y^\sim \to x^\sim$;

(c_{23}) $x \leq y$ *implies* $y^- \leq x^-$ *and* $y^\sim \leq x^\sim$;

(c_{24}) $x \to y^\sim = y \rightsquigarrow x^-$;

(c_{25}) $((x^-)^\sim)^- = x^-$, $((x^\sim)^-)^\sim = x^\sim$.

Proposition 2.14. *In a bounded pseudo-BCK algebra the following hold:*

(c_{26}) $x \to y^{-\sim} = y^- \rightsquigarrow x^- = x^{-\sim} \to y^{-\sim}$ *and*
$x \rightsquigarrow y^{\sim -} = y^\sim \to x^\sim = x^{\sim -} \rightsquigarrow y^{\sim -}$;

(c_{27}) $x \to y^\sim = y^{\sim -} \rightsquigarrow x^- = x^{-\sim} \to y^\sim$ *and*
$x \rightsquigarrow y^- = y^{-\sim} \to x^\sim = x^{\sim -} \rightsquigarrow y^-$;

(c_{28}) $(x \to y^{\sim -})^{\sim -} = x \to y^{\sim -}$ *and* $(x \rightsquigarrow y^{-\sim})^{-\sim} = x \rightsquigarrow y^{-\sim}$.

Proof.

(c_{26}): By (c_{24}) we have: $y \rightsquigarrow x^- = x \to y^\sim$.
Replacing y with y^- we get: $y^- \rightsquigarrow x^- = x \to y^{-\sim}$.
Replacing x with $x^{-\sim}$ in the last equality we get: $y^- \rightsquigarrow x^{-\sim -} = x^{-\sim} \to y^{-\sim}$.
Hence, applying (c_{25}) it follows that: $y^- \rightsquigarrow x^- = x^{-\sim} \to y^{-\sim}$.
Thus, $x \to y^{-\sim} = y^- \rightsquigarrow x^- = x^{-\sim} \to y^{-\sim}$.
Similarly, $x \rightsquigarrow y^{\sim -} = y^\sim \to x^\sim = x^{\sim -} \rightsquigarrow y^{\sim -}$.

(c_{27}): The assertions follow replacing in (c_{26}), y with $y^\sim$ and respectively y with y^- and applying (c_{25}).

(c_{28}): Applying (c_3) and (c_{27}) we have:

$$1 = (x \to y^{\sim -}) \rightsquigarrow (x \to y^{\sim -}) = x \to ((x \to y^{\sim -}) \rightsquigarrow y^{\sim -})$$
$$= x \to ((x \to y^{\sim -})^{\sim -} \rightsquigarrow y^{\sim -}) = (x \to y^{\sim -})^{\sim -} \rightsquigarrow (x \to y^{\sim -}).$$

Hence, $(x \to y^{\sim -})^{\sim -} \leq x \to y^{\sim -}$.
On the other hand, by (c_{21}) we have $x \to y^{\sim -} \leq (x \to y^{\sim -})^{\sim -}$, so $(x \to y^{\sim -})^{\sim -} = x \to y^{\sim -}$. Similarly, $(x \rightsquigarrow y^{-\sim})^{-\sim} = x \rightsquigarrow y^{-\sim}$. $\square$

Proposition 2.15. ([15]) *In a bounded pseudo-BCK*(pP) *algebra the following hold:*

(c_{29}) $(x_{n-1} \to x_n) \odot (x_{n-2} \to x_{n-1}) \odot \cdots \odot (x_1 \to x_2) \leq x_1 \to x_n$ *and* $(x_1 \rightsquigarrow x_2) \odot (x_2 \rightsquigarrow x_3) \odot \cdots \odot (x_{n-1} \rightsquigarrow x_n) \leq x_1 \rightsquigarrow x_n$*;*

(c_{30}) $x \odot 0 = 0 \odot x = 0$*;*

(c_{31}) $x \odot 1 = 1 \odot x = x$*;*

(c_{32}) $x^- \odot x = 0$ *and* $x \odot x^\sim = 0$*;*

(c_{33}) $x \leq y^-$ *iff* $x \odot y = 0$ *and* $x \leq y^\sim$ *iff* $y \odot x = 0$*;*

(c_{34}) $x \to y^- = (x \odot y)^-$ *and* $x \rightsquigarrow y^\sim = (y \odot x)^\sim$*;*

(c_{35}) $x \leq y^-$ *iff* $y \leq x^\sim$*;*

(c_{36}) $x \leq x^\sim \to y$ *and* $x \leq x^- \rightsquigarrow y$*.*

Definition 2.16. ([15]) A bounded pseudo-BCK algebra $\mathscr{A} = (A, \leq, \to, \rightsquigarrow, 0, 1)$ is with (pDN) (*pseudo-Double Negation*) condition if it satisfies the following condition:

(pDN) $(\forall x \in A)\big((x^-)^\sim = (x^\sim)^- = x\big)$.

Proposition 2.17. ([15]) *Let $\mathscr{A}$ be a pseudo-BCK algebra with* (pDN) *condition. Then for all $x, y \in A$ the following hold:*

(c_{37}) $x \leq y$ *iff* $y^- \leq x^-$ *iff* $y^\sim \leq x^\sim$*;*

(c_{38}) $x \to y = y^- \rightsquigarrow x^-$, $x \rightsquigarrow y = y^\sim \to x^\sim$*;*

(c_{39}) $x^\sim \to y = y^- \rightsquigarrow x$*;*

(c_{40}) $(x \to y^-)^\sim = (y \rightsquigarrow x^\sim)^-$*.*

Theorem 2.18. ([15]) *A bounded pseudo-BCK algebra $\mathscr{A} = (A, \leq, \to, \rightsquigarrow, 0, 1)$ with* (pDN) *condition is with* (pP) *condition, where*

$$x \odot y = (x \to y^-)^\sim = (y \rightsquigarrow x^\sim)^-$$

(by (c_{40})).

Definition 2.19. A bounded pseudo-BCK algebra $\mathscr{A}$ is called *good* if

$$(x^-)^\sim = (x^\sim)^- \qquad \text{for all} \quad x \in A.$$

Remark 2.20. It is easy to show that any bounded pseudo-BCK algebra can be extended to a good one. Indeed, consider the bounded pseudo-BCK algebra $\mathscr{A} = (A, \leq, \to, \rightsquigarrow, 0, 1)$ and an element $0_1 \notin A$. Consider a new pseudo-BCK algebra $\mathscr{A}_1 = (A_1, \leq, \to_1, \rightsquigarrow_1, 0_1, 1)$, where $A_1 = A \cup \{0_1\}$ and the operations $\to_1$ and $\rightsquigarrow_1$ are defined as follows:

$$x \to_1 y = \begin{cases} x \to y, & \text{if } x, y \in A, \\ 1, & \text{if } x = 0_1,\ \ y \in A_1, \\ 0_1, & \text{if } x \in A,\ \ y = 0_1, \end{cases}$$

$$x \rightsquigarrow_1 y = \begin{cases} x \rightsquigarrow y, & \text{if } x, y \in A, \\ 1, & \text{if } x = 0_1,\ \ y \in A_1, \\ 0_1, & \text{if } x \in A,\ \ y = 0_1. \end{cases}$$

One can easily check that $\mathscr{A}_1$ is a good pseudo-BCK algebra.

Example 2.21. Consider the pseudo-BCK lattice $\mathscr{A}$ from Example 2.11. Since $(a_1^-)^\sim = a_2$ and $(a_1^\sim)^- = a_1$, it follows that $\mathscr{A}$ is not good. $\mathscr{A}$ is extended to the good pseudo-BCK algebra (see [16]) $\mathscr{A}_1 = (A_1, \leq, \rightarrow, \rightsquigarrow, 0, 1)$, where $A = \{0, a_1, a_2, b_2, s, a, b, n, c, d, m, 1\}$ with $0 < a_1 < a_2 < b_2 < s < a$, $b < n < c$, $d < m < 1$ (a is incomparable with b and c is incomparable with d). The operations $\rightarrow$ and $\rightsquigarrow$ are constructed in the way described in Remark 2.20.

Proposition 2.22. *In any good pseudo-BCK*(pP) *algebra the following properties hold:*

(1) $(x^\sim \odot y^\sim)^- = (x^- \odot y^-)^\sim$;

(2) $x^{-\sim} \odot y^{-\sim} \leq (x \odot y)^{-\sim}$.

Proof. Applying (c_{34}), (c_{24}), (c_{25}) we have:

(1): $(x^\sim \odot y^\sim)^- = x^\sim \rightarrow y^{\sim-} = x^\sim \rightarrow y^{-\sim} = y^{-\sim-} \rightsquigarrow x^{\sim-} = y^- \rightsquigarrow x^{\sim-}$ $= y^- \rightsquigarrow x^{-\sim} = (x^- \odot y^-)^\sim$.

(2): Because the pseudo-BCK(pP) algebra is good and by (c_{11}), we have: $(x \odot y)^{-\sim} = (x \odot y)^{\sim-} \geq x^{\sim-} \odot (x^{\sim-} \rightsquigarrow (x \odot y)^{\sim-}) = x^{\sim-} \odot (x^{\sim-} \rightsquigarrow (x \odot y)^{-\sim}) = x^{\sim-} \odot (x^{\sim-} \rightsquigarrow (x \rightarrow y^-)^\sim)$.

Applying (c_{16}) we get: $x^{\sim-} \rightsquigarrow (x \rightarrow y^-)^\sim = x^{\sim-} \rightsquigarrow ((x \rightarrow y^-) \rightsquigarrow 0) = [(x \rightarrow y^-) \odot x^{\sim-}] \rightsquigarrow 0 = [(x \rightarrow y^-) \odot x^{\sim-}]^\sim = [(x^{\sim-} \rightarrow y^-) \odot x^{\sim-}]^\sim$.
(By (c_{26}) replacing y with y^- we have $x \rightarrow y^- = x^{-\sim} \rightarrow y^-$).

Applying (c_{11}) we have $(x^{\sim-} \rightarrow y^-) \odot x^{\sim-} \leq y^-$, hence

$$[(x^{\sim-} \rightarrow y^-) \odot x^{\sim-}]^\sim \geq y^{-\sim}.$$

Thus, $(x \odot y)^{-\sim} \geq x^{\sim-} \odot (x^{\sim-} \rightsquigarrow (x \rightarrow y^-)^\sim) = x^{\sim-} \odot [(x^{\sim-} \rightarrow y^-) \odot x^{\sim-}]^\sim$ $\geq x^{\sim-} \odot y^{-\sim}$. □

Similarly as in [23] for the case of bounded non-commutative Rℓ-monoids, a good pseudo-BCK(pP) algebra A which satisfies the identity $(x \odot y)^{-\sim} = x^{-\sim} \odot y^{-\sim}$ for all $x, y \in A$ will be called *normal* pseudo-BCK(pP) algebra.

Proposition 2.23. *Let $\mathscr{A} = (A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$ be a good pseudo-BCK algebra. We define a binary operation $\oplus$ on A by $x \oplus y := y^\sim \rightarrow x^{\sim-}$. Then, for all $x, y \in A$ the following hold:*

(1) $x \oplus y = x^- \rightsquigarrow y^{\sim-}$,

(2) $x, y \leq x \oplus y$,

(3) $x \oplus 0 = 0 \oplus x = x^{\sim-}$,

(4) $x \oplus 1 = 1 \oplus x = 1$,
(5) $(x \oplus y)^{-\sim} = x \oplus y = x^{-\sim} \oplus y^{-\sim}$,
(6) $\oplus$ *is associative.*

Proof.
(1) It follows by (c_{26}), second identity, replacing x with x^-.
(2) Since $x \le x^{\sim -} \le y^\sim \to x^{\sim -}$, it follows that $x \le x \oplus y$.
Similarly, $y \le y^{\sim -} \le x^- \rightsquigarrow y^{\sim -}$, so $y \le x \oplus y$.
(3) $x \oplus 0 = 0^\sim \to x^{\sim -} = 1 \to x^{\sim -} = x^{\sim -}$.
Similarly, $0 \oplus x = x^\sim \to 0^{\sim -} = x^\sim \to 0 = x^{\sim -}$.
(4) $1 \oplus x = x^\sim \to 1^{\sim -} = x^\sim \to 1 = 1$. Similarly, $x \oplus 1 = 1$.
(5) $(x \oplus y)^{-\sim} = (y^\sim \to x^{\sim -})^{-\sim} = y^\sim \to x^{\sim -} = x \oplus y$ (we applied (c_{28})).
We also have: $x^{-\sim} \oplus y^{-\sim} = (y^{-\sim})^\sim \to (x^{-\sim})^{-\sim} = y^\sim \to x^{-\sim} = x \oplus y$.
(6) Applying (c_{28}) and (c_3) we get:
$(x \oplus y) \oplus z = (x^- \rightsquigarrow y^{\sim -}) \oplus z = z^\sim \to (x^- \rightsquigarrow y^{\sim -})^{\sim -} = z^\sim \to (x^- \rightsquigarrow y^{\sim -}) = x^- \rightsquigarrow (z^\sim \to y^{\sim -}) = x^- \rightsquigarrow (y \oplus z) = x^- \rightsquigarrow (y \oplus z)^{\sim -} = x \oplus (y \oplus z)$. □

Proposition 2.24. *If $\mathscr{A} = (A, \le, \to, \rightsquigarrow, 0, 1)$ is a good pseudo-BCK*(pP) *algebra, then*

$$x \oplus y = (y^- \odot x^-)^\sim = (y^\sim \odot x^\sim)^-.$$

Proof. It follows applying (c_{34}). □

For any $n \in \mathbb{N}$, $x \in A$ we put $0x = 0$, $1x = x$ and $(n+1)x = nx \oplus x = x \oplus nx$ for $n \ge 1$.

Proposition 2.25. *If $\mathscr{A} = (A, \le, \to, \rightsquigarrow, 0, 1)$ is a normal pseudo-BCK*(pP) *algebra, then the following hold for all $x, y \in A$ and $n \in \mathbb{N}$:*

(1) $(x \odot y)^- = y^- \oplus x^-$ *and* $(x \odot y)^\sim = y^\sim \oplus x^\sim$;
(2) $((x \odot y)^n)^- = n(y^- \oplus x^-)$ *and* $((x \odot y)^n)^\sim = n(y^\sim \oplus x^\sim)$;
(3) $(x^n)^- = nx^-$ *and* $(x^n)^\sim = nx^\sim$.

Proof.
(1) $(x \odot y)^- = (x \odot y)^{-\sim -} = (x^{-\sim} \odot y^{-\sim})^- = y^{-\sim -} \oplus x^{-\sim -} = y^- \oplus x^-$;
$(x \odot y)^\sim = (x \odot y)^{\sim -\sim} = (x^{\sim -} \odot y^{\sim -})^\sim = y^{-\sim -} \oplus x^{-\sim -} = y^\sim \oplus x^\sim$;
(2) For $n = 2$ we have:
$((x \odot y)^2)^- = [(x \odot y) \odot (x \odot y)]^- = [(x \odot y) \odot (x \odot y)]^{-\sim -} = [(x \odot y)^{-\sim} \odot (x \odot y)^{-\sim}]^- = (x \odot y)^{-\sim -} \oplus (x \odot y)^{-\sim -} = (x \odot y)^- \oplus (x \odot y)^- = (y^- \oplus x^-) \oplus (y^- \oplus x^-) = 2(y^- \oplus x^-)$.

By induction we get $((x \odot y)^n)^- = n(y^- \oplus x^-)$ and similarly $((x \odot y)^n)^\sim = n(y^\sim \oplus x^\sim)$;

(3) It follows from (2) for $y = 1$. □

3. Deductive systems of pseudo-BCK algebras with pseudo-product

In this section we will define the notion of deductive system for a pseudo-BCK(pP) algebra and we will extend some results proved in [8], [9], [12], [5], [6] for the case of pseudo-BL algebras, pseudo-MTL algebras and residuated lattices.

Definition 3.1. Let $\mathscr{A}$ be pseudo-BCK algebra. The subset $D \subseteq A$ is called *deductive system* of A if it satisfies the following conditions:

(DS_1) $1 \in D$;

(DS_2) for all $x, y \in A$, if $x, x \to y \in D$, then $y \in D$.

The condition (DS_2) is equivalent with the following condition:

(DS_2') for all $x, y \in A$, if $x, x \rightsquigarrow y \in D$, then $y \in D$.

We will denote by $\mathscr{D}S(A)$ the set of all deductive systems of A.

Obviously, $\{1\}, A \in \mathscr{D}S(A)$.

A deductive system D of a pseudo-BCK algebra $\mathscr{A}$ is called *proper* if $D \neq A$.

Definition 3.2. A deductive system D of a pseudo-BCK algebra $\mathscr{A}$ is called *normal* if it satisfies the condition:

(DS_3) for all $x, y \in A$, $x \to y \in D$ iff $x \rightsquigarrow y \in D$.

The normal deductive system is called *compatible deductive* system in [19], but for an easier connection with the previous results, in this paper we will use the notion of normal deductive system.

We will denote by $\mathscr{D}S_n(A)$ the set of all normal deductive systems of A.

It is obvious that $\{1\}, A \in \mathscr{D}S_n(A)$ and $\mathscr{D}S_n(A) \subseteq \mathscr{D}S(A)$.

Definition 3.3. Let $\mathscr{A}$ be pseudo-BCK(pP) algebra. The subset $\emptyset \neq F \subseteq A$ is called *filter* of A if it satisfies the following conditions:

(F_1) $x, y \in F$ implies $x \odot y \in F$;

(F_2) $x \in F$, $y \in A$, $x \leq y$ implies $y \in F$.

One can easily check that in the case of a pseudo-BCK(pP) algebra the definition of the filter is equivalent with the definition of the deductive system.

Proposition 3.4. ([7]) *If A is a bounded pseudo-BCK*(pP) *algebra, then the sets*

$$A_0^- = \{x \in A : \; x^- = 0\} \qquad \textit{and} \qquad A_0^\sim = \{x \in A : \; x^\sim = 0\}$$

are proper deductive systems of A.

Proposition 3.5. ([7]) *Let A be a bounded pseudo-BCK algebra and $H \in \mathscr{DS}_n(A)$. Then:*

(1) *$x^- \in H$ iff $x^\sim \in H$;*

(2) *$x \in H$ implies $(x^-)^- \in H$ and $(x^\sim)^\sim \in H$.*

Definition 3.6. A deductive system is called *maximal* if it is proper and not strictly contained in any other deductive system. Denote:

$$\mathrm{Max}(A) := \{F : \; F \text{ is maximal deductive system of } A\},$$

$$\mathrm{Max}_n(A) := \{F : \; F \text{ is maximal normal deductive system of } A\}.$$

Clearly, $\mathrm{Max}_n(A) \subseteq \mathrm{Max}(A)$.

Proposition 3.7. ([7]) *Any proper deductive system of a bounded pseudo-BCK algebra A can be extended to a maximal deductive system of A.*

Examples 3.8.

(1) Let A be the pseudo-BCK(pP) algebra A from Example 2.11 and $D_1 = \{s, a, b, n, c, d, m, 1\}$, $D_2 = \{a_2, s, a, b, n, c, d, m, 1\}$. Then:

$$\mathscr{DS}(A) = \{\{1\}, D_1, D_2, A\}, \qquad \mathrm{Max}(A) = \{D_2\},$$

$$\mathscr{DS}_n(A) = \{\{1\}, D_1, A\}, \qquad \mathrm{Max}_n(A) = \emptyset.$$

(2) In the case of the pseudo-BCK(pP) algebra A_1 from Example 2.21, denoting by $D_1 = \{a_1, a_2, b_2, s, a, b, n, c, d, m, 1\}$, $D_2 = \{b_2, s, a, b, n, c, d, m, 1\}$ and $D_3 = \{s, a, b, n, c, d, m, 1\}$, we have:

$$\mathscr{DS}(A) = \big\{\{1\}, D_1, D_2, D_3, A\big\}, \qquad \mathrm{Max}(A) = \{D_1\},$$

$$\mathscr{DS}_n(A) = \{\{1\}, D_1, D_3, A\}, \qquad \mathrm{Max}_n(A) = \{D_1\}.$$

Definition 3.9. For every subset $X \subseteq A$, the smallest deductive system of A containing X (i.e. the intersection of all deductive systems $D \in \mathscr{DS}(A)$ such that $X \subseteq D$) is called the deductive system *generated by* X and will be denoted by $\langle X \rangle$. If $X = \{x\}$ we write $\langle x \rangle$ instead of $\langle \{x\} \rangle$.

Lemma 3.10. ([12]) *Let A be a bounded pseudo-BCK*(pP) *algebra and $x, y \in A$. Then:*

(1) *$\langle x \rangle$ is proper iff* $\mathrm{ord}(x) = \infty$;

(2) *if $x \le y$ and* $\mathrm{ord}(y) < \infty$, *then* $\mathrm{ord}(x) < \infty$;

(3) *if $x \le y$ and* $\mathrm{ord}(x) = \infty$, *then* $\mathrm{ord}(y) = \infty$.

Proposition 3.11. ([8]) *If A is a pseudo-BCK(pP) algebra and $X \subseteq A$, then*

$$\begin{aligned}\langle X\rangle &= \big\{y \in A:\ y \geq x_1 \odot x_2 \odot \cdots \odot x_n \ \textit{for some}\ n \geq 1 \\ &\qquad \textit{and}\ x_1, x_2, \ldots, x_n \in X\big\} \\ &= \big\{y \in A:\ x_1 \to (x_2 \to (\ldots(x_n \to y)\ldots)) = 1\ \textit{for some}\ n \geq 1 \\ &\qquad \textit{and}\ x_1, x_2, \ldots, x_n \in X\big\} \\ &= \big\{y \in A:\ x_1 \rightsquigarrow (x_2 \rightsquigarrow (\ldots(x_n \rightsquigarrow y)\ldots)) = 1\ \textit{for some}\ n \geq 1 \\ &\qquad \textit{and}\ x_1, x_2, \ldots, x_n \in X\big\}.\end{aligned}$$

Remarks 3.12. ([8]) Let A be a pseudo-BCK(pP) algebra. Then:

(1) If X is a deductive system of A, then $\langle X\rangle = X$;

(2) $\langle x\rangle = \{y \in A:\ y \geq x^n \text{ for some } n \geq 1\}$.
$\langle x\rangle$ is called *principal* deductive system;

(3) If D is a deductive system of A and $x \in A$, then

$$D(x) = \langle D \cup \{x\}\rangle = \big\{y \in A:\ y \geq (d_1 \odot x^{n_1}) \odot (d_2 \odot x^{n_2}) \odot \cdots \odot (d_m \odot x^{n_m}) \\ \text{for some } m \geq 1,\ n_1, n_2, \ldots, n_m \geq 0,\ d_1, d_2, \ldots, d_m \in D\big\}.$$

The next result is obvious.

Lemma 3.13. *Let A be a pseudo-BCK(pP) algebra and D a proper deductive system of A. Then the following are equivalent:*

(a) *D is maximal;*

(b) *for all $x \in A$, if $x \notin D$ then $\langle D \cup \{x\}\rangle = A$.*

Proposition 3.14. ([4]) *If D_1, D_2 are nonempty subsets of a pseudo-BCK(pP) algebra A such that $1 \in D_1 \cap D_2$, then*

$$\langle D_1 \cup D_2\rangle = \big\{x \in A:\ x \geq (d_1 \odot d_1') \odot (d_2 \odot d_2') \odot \cdots \odot (d_n \odot d_n') \\ \textit{for some}\ n \geq 1,\ d_1, d_2, \ldots, d_n \in D_1,\ d_1', d_2', \ldots, d_n' \in D_2\big\}.$$

The next result can be proved similarly as in [5] for the case of the residuated lattices.

Lemma 3.15. *Let A be a pseudo-BCK(pP) algebra and $H \in \mathscr{DS}_n(A)$. Then:*

(1) *For any $x \in A$ and $h \in H$ there is $h' \in H$ such that $x \odot h \geq h' \odot x$;*

(2) *For any $x \in A$ and $h \in H$ there is $h'' \in H$ such that $h \odot x \geq x \odot h''$.*

Proposition 3.16. *Let A be a pseudo-BCK*(pP) *algebra, $H \in \mathscr{D}S_n(A)$ and $x \in A$. Then*

$$\begin{aligned} H(x) = \langle H \cup \{x\}\rangle &= \{y \in A : \ y \geq h \odot x^n \text{ for some } n \in \mathbb{N},\ h \in H\} \\ &= \{y \in A : \ y \geq x^n \odot h \ \text{ for some } \ n \in \mathbb{N},\ \ h \in H\} \\ &= \{y \in A : \ x^n \to y \in H \ \text{ for some } \ n \geq 1\} \\ &= \{y \in A : \ x^n \rightsquigarrow y \in H \ \text{ for some } \ n \geq 1\}. \end{aligned}$$

Corollary 3.17. *Let A be a pseudo-BCK*(pP) *algebra and H a proper normal deductive system of A. Then the following are equivalent:*

(a) $H \in \mathrm{Max}_n(A)$;
(b) *for all $x \in A$, if $x \notin H$, then for any $y \in A$, $x^n \to y \in H$ for some $n \in \mathbb{N}$, $n \geq 1$;*
(c) *for all $x \in A$, if $x \notin H$, then for any $y \in A$, $x^n \rightsquigarrow y \in H$ for some $n \in \mathbb{N}$, $n \geq 1$.*

Proof.
(a) $\Longrightarrow$ (b): Since H is maximal, then by Lemma 3.13, $\langle H \cup \{x\}\rangle = A$ and applying Proposition 3.16 we get the assertion (b);
(b) $\Longrightarrow$ (a): Let $x \in A \setminus H$. By (b), for all $y \in A$ we have $x^n \to y \in H$ for some $n \in \mathbb{N}$, $n \geq 1$. Since $(x^n \to y) \odot x^n \leq y$, then by Proposition 3.16 it follows that $y \in \langle H \cup \{x\}\rangle$. Hence, $\langle H \cup \{x\}\rangle = A$. Applying Lemma 3.13 we get that $H \in \mathrm{Max}_n(A)$;
(a) $\Longleftrightarrow$ (c): Similarly as (a) $\Longleftrightarrow$ (b). □

Based on Proposition 3.14 and Lemma 3.15 we can prove the following result.

Proposition 3.18. *If A is a pseudo-BCK*(pP) *algebra and $D_1, D_2 \in \mathscr{D}S_n(A)$, then*

$$\langle D_1 \cup D_2\rangle = \{x \in A : \ x \geq u \odot v \ \text{ for some } \ u \in D_1,\ \ v \in D_2\}.$$

Definition 3.19. A bounded pseudo-BCK(pP) algebra A is *locally finite* if for any $x \in A$, $x \neq 1$ implies $\mathrm{ord}(x) < \infty$.

Proposition 3.20. ([7]) *Let A be a bounded pseudo-BCK*(pP) *algebra. The following are equivalent:*

(a) *A is locally finite;*
(b) *$\{1\}$ is the unique proper deductive system of A.*

Theorem 3.21. *If D is a proper deductive system of A, then the following are equivalent:*

(a) $D \in \mathrm{Max}(A)$;

(b) *For any $x \notin D$ there is $d \in D$, $n, m \in \mathbb{N}$, $n, m \geq 1$ such that* $(d \odot x^n)^m = 0$.

Proof.

(a) $\Longrightarrow$ (b): Since $0 \in A = \langle D \cup \{x\} \rangle$, by Remark 3.12 it follows that there exist $m \geq 1$, $n_1, n_2, \ldots, n_m \geq 0$, $d_1, d_2, \ldots, d_m \in D$ such that

$$(d_1 \odot x^{n_1}) \odot (d_2 \odot x^{n_2}) \odot \cdots \odot (d_m \odot x^{n_m}) = 0.$$

Taking $n = \max\{n_1, n_2, \ldots, n_m\}$ and $d = d_1 \odot d_2 \odot \cdots \odot d_m \in D$ we get

$$(d \odot x^n)^m \leq (d_1 \odot x^{n_1}) \odot (d_2 \odot x^{n_2}) \odot \cdots \odot (d_m \odot x^{n_m}) = 0.$$

It follows that $(d \odot x^n)^m = 0$.

(b) $\Longrightarrow$ (a): Assume that there is a proper deductive system E of A such that $D \subset E$, $D \neq E$. Then, there exists $x \in E$ such that $x \notin D$. By the hypothesis, there exist $d \in D$, $n, m \in \mathbb{N}$ such that $(d \odot x^n)^m = 0$. Since $x, d \in E$, it follows that $0 \in E$, hence $E = A$ which is a contradiction. Thus, $D \in \mathrm{Max}(A)$. □

The next result follows from Corollary 3.17.

Theorem 3.22. *If H is a proper normal deductive system of a bounded pseudo-BCK*(pP) *algebra A, then the following are equivalent:*

(a) $H \in \mathrm{Max}_n(A)$;

(b) *For any $x \in A$, $x \notin H$* iff $(x^n)^- \in H$ *for some $n \in \mathbb{N}$;*

(c) *For any $x \in A$, $x \notin H$* iff $(x^n)^\sim \in H$ *for some $n \in \mathbb{N}$.*

According to [21], the class of pseudo-BCK algebras is not closed under homomorphic images. In other words, there exist congruences $\theta \in \mathrm{Con}(A)$ such that the quotient algebra $(A/\theta, \to, \rightsquigarrow, 1/\theta)$ is not a pseudo-BCK algebra (see [21], Example 2.2.3).

A congruence $\theta \in \mathrm{Con}(A)$ such that the quotient algebra $(A/\theta, \to, \rightsquigarrow, 1/\theta)$ is a pseudo-BCK algebra is called in [21] *relative congruence*. With any $H \in \mathscr{DS}_n(A)$ we associate a binary relation $\equiv_H$ on A by defining $x \equiv_H y$ iff $x \to y, y \to x \in H$ iff $x \rightsquigarrow y, y \rightsquigarrow x \in H$.

For a given $H \in \mathscr{DS}_n(A)$ the relation $\equiv_H$ is an equivalence relation on A.

It was proved in [21] that $\theta_H = \equiv_H$ is a relative congruence of $(A, \to, \rightsquigarrow, 1)$, that is A/θ_H becomes a pseudo-BCK algebra with the natural operations induced from those of A. Moreover, the congruence θ_H is also compatible with the operation $\odot$. Indeed, if $x \equiv_H y$ and $a \equiv_H b$, we prove that $x \odot a \equiv_H y \odot b$. From $x \geq (x \to y) \odot x$ and $a \geq (b \to a) \odot b$, it follows that $x \odot a \geq (x \to y) \odot x \odot (b \to a) \odot b$. Since $b \to a \in H$, by Lemma 3.15 there exists $h' \in H$ such that $x \odot (b \to a) \odot b \geq h' \odot x \odot b$. It follows that $x \odot a \geq (x \to y) \odot h' \odot x \odot b$, hence $(x \to y) \odot h' \leq x \odot b \to x \odot a$. Since $(x \to y) \odot h' \in H$, we get that $x \odot b \to x \odot a \in H$. Similarly, $x \odot a \to x \odot b \in H$, so $x \odot a \equiv_H x \odot b$. One can analogously show that $x \odot b \equiv_H y \odot b$ whence $x \odot a \equiv_H y \odot b$.

Thus, A/θ_H is a pseudo-BCK(pP) algebra. This algebra is called the *quotient* of A by θ_H and it will be denoted shortly A/H. For any $x \in A$, let x/H be the congruence class $x/_{\equiv_H}$ of x, hence $A/H = \{x/H : x \in A\}$.

The next result is obvious.

Lemma 3.23. *If H be a normal deductive system of a bounded pseudo-BCK*(pP) *algebra A, then:*

(1) $x/H = 1/H$ *iff* $x \in H$;

(2) $x/H = 0/H$ *iff* $x^- \in H$ *iff* $x^\sim \in H$;

(3) $x/H \leq y/H$ iff $x \to y \in H$ iff $x \rightsquigarrow y \in H$.

Proposition 3.24. *If H is a proper normal deductive system of a bounded pseudo-BCK*(pP) *algebra A, then the following are equivalent:*

(a) $H \in \mathrm{Max}_n(A)$;

(b) A/H *is locally finite.*

Proof. H is maximal iff the condition (b) from Theorem 3.22 is satisfied. This condition is equivalent with: for any $x \in A, x/H \neq 1/H$ iff $(x^n)^-/H = 1/H$ for some $n \in \mathbb{N}$ iff $(x/H)^n = 0/H$ for some $n \in \mathbb{N}$ iff A/H is locally finite. □

Proposition 3.25. *If A is a bounded pseudo-BCK*(pP) *algebra and $D = A \setminus \{0\} \in \mathrm{Max}(A)$, then A is good.*

Proof. Obviously $(0^-)^\sim = (0^\sim)^- = 0$. Assume $x > 0$, that is, $x \in D$. If $x^-, x^\sim \in D$ it follows that $x^- \odot x, x \odot x^\sim \in D$, that is $0 \in D$, a contradiction.

Thus, $x^- = x^\sim = 0$, hence $(x^-)^\sim = (x^\sim)^- = 1$. Therefore, $(x^-)^\sim = (x^\sim)^-$ for all $x \in A$, so A is a good pseudo-BCK(pP) algebra. □

Proposition 3.26. *Let A be a linearly ordered pseudo-BCK*(pP) *algebra, $D \in \mathrm{Max}(A)$ and $x, y \in A$. Then:*

(1) $y \notin D$ *and* $y \odot x = x$ *implies* $x = 0$;

(2) $y \notin D$ *and* $x \odot y = x$ *implies* $x = 0$.

Proof.

(1) Consider $y \in A \setminus D$ such that $y \odot x = x$. Assume $x \in A$, $x > 0$ and consider $E = \{z \in A : z \odot x = x\}$. First we prove that E is a proper deductive system. Obviously, $1, y \in E$ and $0 \notin E$. Consider $z \in A$ such that $y \to z \in E$, so $(y \to z) \odot x = x$. Since $(y \to z) \odot y \odot x = (y \to z) \odot x = x$, it follows that $x = [(y \to z) \odot y] \odot x \leq z \odot x \leq x$. Thus, $z \odot x = x$, hence $z \in E$. Therefore,

E is a proper deductive system. Since $y \in E$ and D is maximal, it follows that $y \in D$, a contradiction. Thus, $x = 0$.

(2) Similarly as in (1). □

Definition 3.27. Let A and B be two bounded pseudo-BCK(pP) algebras. A function $f\colon A \longrightarrow B$ is a *homomorphism* if it satisfies the following conditions, for all $x, y \in A$:

(H_1) $f(x \odot y) = f(x) \odot f(y)$;

(H_2) $f(x \to y) = f(x) \to f(y)$;

(H_3) $f(x \rightsquigarrow y) = f(x) \rightsquigarrow f(y)$;

(H_4) $f(0) = 0$.

Remark 3.28. If $f\colon A \longrightarrow B$ is a bounded pseudo-BCK(pP) algebras homomorphism, then one can easily prove that the following hold for all $x \in A$:

(H_5) $f(1) = 1$;

(H_6) $f(x^-) = (f(x))^-$;

(H_7) $f(x^\sim) = (f(x))^\sim$;

(H_8) if $x, y \in A$, $x \le y$, then $f(x) \le f(y)$.

The *kernel* of f is the set $\ker(f) = f^{-1}(1) = \{x \in A: \ f(x) = 1\}$.

The function $\pi_H\colon A \longrightarrow A/H$ defined by $\pi_H(x) = x/H$ for any $x \in A$ is a surjective homomorphism which is called the *canonical projection* from A to A/H. One can easily prove that $\ker(\pi_H) = H$.

The proofs of the results in the next proposition are obvious.

Proposition 3.29. *Let A and B be non-trivial pseudo-BCK*(pP) *algebras. If $f\colon A \longrightarrow B$ is a homomorphism, then the following hold:*

(1) $\ker(f)$ *is a proper deductive system of A.*

(2) *f is injective iff* $\ker(f) = \{1\}$.

(3) *If $G \in \mathscr{DS}(B)$, then $f^{-1}(G) \in \mathscr{DS}(A)$ and* $\ker(f) \subseteq f^{-1}(G)$.
If $G \in \mathscr{DS}_n(B)$, then $f^{-1}(G) \in \mathscr{DS}_n(A)$. In particular $\ker(f) \in \mathscr{DS}_n(A)$.

(4) *If f is surjective and $D \in \mathscr{DS}(A)$ such that* $\ker(f) \subseteq D$, *then $f(D) \in \mathscr{DS}(B)$.*

Proposition 3.30. *If $f\colon A \longrightarrow B$ is a surjective bounded pseudo-BCK*(pP) *algebras homomorphism, then there is a bijective correspondence between $\{D: \ D \in \mathscr{DS}(A), \ \ker(f) \subseteq D\}$ and $\mathscr{DS}(B)$.*

Proof. By Proposition 3.29, for any $D \in \mathscr{DS}(A)$ such that $\ker(f) \subseteq D$ and $G \in \mathscr{DS}(B)$ there is the correspondence $D \mapsto f(D)$ and $G \mapsto f^{-1}(G)$ between the two sets.

We have to prove that $f^{-1}(f(D)) = D$ and $f(f^{-1}(G)) = G$. Since f is surjective, it follows that $f(f^{-1}(G)) = G$. Obviously, $D \subseteq f^{-1}(f(D))$ always holds.

Suppose that $x \in f^{-1}(f(D))$, then $f(x) \in f(D)$, so there is $x' \in D$ such that $f(x) = f(x')$. We have $f(x') \to f(x) = 1$, so $f(x' \to x) = 1$, that is $x' \to x \in \ker(f) \subseteq D$.

From $x', x' \to x \in D$ we get $x \in D$. Thus, $f^{-1}(f(D)) = D$. □

Corollary 3.31. *If $D \in \mathscr{D}S_n(A)$, then:*

(1) *$\pi_D(E) \in \mathscr{D}S(A/D)$, where $E \in \mathscr{D}S(A)$ such that $D \subseteq E$;*

(2) *the correspondence $E \mapsto \pi_D(E)$ is a bijection between $\{F: \ F \in \mathscr{D}S(A), \ \ D \subseteq F\}$ and $\mathscr{D}S(A/D)$.*

Proof.

(1) It follows from Proposition 3.29(4);

(2) It follows from Proposition 3.30. □

Proposition 3.32. *If $D, H \in \mathscr{D}S_n(A)$ such that $H \subseteq D$, then $D \in \mathrm{Max}(A)$ iff $\pi_H(D) \in \mathrm{Max}(A/H)$.*

Proof. We will apply Theorem 3.22. Suppose that $D \in \mathrm{Max}(A)$ and let $y \in A/H$, $y \notin \pi_H(D)$. It follows that there is $x \in A$ such that $y = \pi_H(x) = x/H$. Obviously, $x \notin D$. Since $D \in \mathrm{Max}(A)$, it follows that:
$(x^n)^- \in D$ for some $n \in \mathbb{N}$ iff $\pi_H((x^n)^-) \in \pi_H(D)$ for some $n \in \mathbb{N}$ iff $\pi_H(((x/H)^n)^-) \in \pi_H(D)$ for some $n \in \mathbb{N}$ iff $(y^n)^- \in \pi_H(D)$ for some $n \in \mathbb{N}$.

Thus, $\pi_H(D) \in \mathrm{Max}(A/H)$. The converse can be proved in a similar way. □

Corollary 3.33. *If H is a proper normal deductive system of a bounded pseudo-BCK*(pP) *algebra A, then there is a bijection between $\{D: \ D \in \mathrm{Max}(A),$ $H \subseteq D\}$ and* $\mathrm{Max}(A/H)$.

Proposition 3.34. *If P is a proper normal deductive system of a bounded pseudo-BCK*(pP) *algebra A, then the following are equivalent:*

(a) *for all $x, y \in A$, $((x \odot y)^n)^- \in P$ for some $n \in \mathbb{N}$ implies $(x^m)^- \in P$ or $(y^m)^- \in P$ for some $m \in \mathbb{N}$;*

(b) *for all $x, y \in A$, $((x \odot y)^n)^\sim \in P$ for some $n \in \mathbb{N}$ implies $(x^m)^\sim \in P$ or $(y^m)^\sim \in P$ for some $m \in \mathbb{N}$.*

Proof. It is obvious taking into consideration that, since P is a normal deductive system, then $x^- \in P$ iff $x^\sim \in P$ for all $x \in A$. □

Definition 3.35. A proper normal deductive system of a bounded pseudo-BCK(pP) algebra A is called *primary* if it satisfies one of the above equivalent conditions.

Remark 3.36. If the bounded pseudo-BCK(pP) algebra A is normal, then its primary deductive systems can be dually characterized by means of the operation $\oplus$. Indeed, if P is a proper normal deductive system of A, applying Proposition 2.25 we have:

$$((x \odot y)^n)^- = n(y^- \oplus x^-), \quad (x^m)^- = mx^- \quad \text{and} \quad (y^m)^- = my^-$$

for all $n, m \in \mathbb{N}$.
Therefore, a proper normal deductive system P of the normal pseudo-BCK(pP) algebra A is primary if it satisfies the following condition for all $x, y \in A$:
if $n(y^- \oplus x^-) \in P$ for some $n \in \mathbb{N}$, then $mx^- \in P$ or $my^- \in P$ for some $m \in \mathbb{N}$.

Obviously, the above conditon is equivalent with the following:
if $n(y^\sim \oplus x^\sim) \in P$ for some $n \in \mathbb{N}$, then $mx^\sim \in P$ or $my^\sim \in P$ for some $m \in \mathbb{N}$.

4. Local pseudo-BCK algebras with pseudo-product

Definition 4.1. A pseudo-BCK(pP) algebra is called *local* if it has a unique maximal deductive system.

In this section by a pseudo-BCK(pP) algebra we mean a bounded pseudo-BCK(pP) algebra, even though some notions and properties are valid for an arbitrary pseudo-BCK(pP) algebra.

We will denote:

$$D(A) = \{x \in A : \operatorname{ord}(x) = \infty\} \quad \text{and} \quad D(A)^* = \{x \in A : \operatorname{ord}(x) < \infty\}.$$

Obviously, $D(A) \cap D(A)^* = \emptyset$ and $D(A) \cup D(A)^* = A$.

We also can remark that $1 \in D(A)$ and $0 \in D(A)^*$.

Let A be a pseudo-BCK(pP) algebra and $D \in \mathscr{DS}(A)$. We will use the following notations:

$$D^*_- = \{x \in A : x \leq y^- \text{ for some } y \in D\},$$
$$D^*_\sim = \{x \in A : x \leq y^\sim \text{ for some } y \in D\}.$$

The next results can be proved similarly as in [5] for the case of the residuated lattices.

Proposition 4.2. ([7]) *Let A be a local pseudo-BCK*(pP) *algebra. Then:*

1. *any proper deductive system of A is included in the unique maximal deductive system of A;*
2. *A_0^- and $A_0^\sim$ are included in the unique maximal deductive system of A.*

Theorem 4.3. *Let A be a pseudo-BCK(pP) algebra. Then the following are equivalent:*

(a) *$D(A)$ is a deductive system of A;*

(b) *$D(A)$ is a proper deductive system of A;*

(c) *A is local;*

(d) *$D(A)$ is the unique maximal deductive system of A;*

(e) *for all $x, y \in A$, $\mathrm{ord}(x \odot y) < \infty$ implies $\mathrm{ord}(x) < \infty$ or $\mathrm{ord}(y) < \infty$.*

Corollary 4.4. *If A is a local pseudo-BCK(pP) algebra, then:*

(1) *for any $x \in A$, $\mathrm{ord}(x) < \infty$ or $[\mathrm{ord}(x^-) < \infty$ and $\mathrm{ord}(x^\sim) < \infty]$;*

(2) *$D(A)^*_- \subseteq D(A)^*$ and $D(A)^*_\sim \subseteq D(A)^*$;*

(3) *$D(A) \cap D(A)^*_- = D(A) \cap D(A)^*_\sim = \emptyset$.*

Example 4.5. Consider the pseudo-BCK(pP) algebra A from Example 2.11. One can easily check that $D(A) = \{a_2, s, a, b, n, c, d, m, 1\}$ and it is a deductive system of A, so A is a local pseudo-BCK(pP) algebra.

Proposition 4.6. ([7]) *Any linearly ordered pseudo-BCK(pP) algebra is local.*

Proposition 4.7. ([7]) *Any locally finite pseudo-BCK(pP) algebra is local.*

Proposition 4.8. *If P is a proper normal deductive system of a bounded pseudo-BCK(pP) A, then the following are equivalent:*

(a) *P is primary;*

(b) *A/P is a local pseudo-BCK(pP) algebra;*

(c) *P is contained in a unique maximal deductive system of A.*

Proof.

(a) $\iff$ (b): Applying Theorem 4.3 (e) and Lemma 3.23 (2), we have: A/P is local iff for all $x, y \in A$, $\mathrm{ord}(x/P \odot y/P) < \infty$ implies $\mathrm{ord}(x/P) < \infty$ or $\mathrm{ord}(y/P) < \infty$ iff for all $x, y \in A$, $(x/P \odot y/P)^n = 0/P$ for some $n \in \mathbb{N}$ implies $(x/P)^m = 0/P$ or $(y/P)^m = 0/P$ for some $m \in \mathbb{N}$ iff for all $x, y \in A$, $(x/P \odot y/H)^n = 0/P$ for some $n \in \mathbb{N}$ implies $x^m/P = 0/P$ or $y^m/P = 0/P$ for some $m \in \mathbb{N}$ iff for all $x, y \in A$, $((x \odot y)^n)^- \in P$ for some $n \in \mathbb{N}$ implies $(x^m)^- \in P$ or $(y^m)^- \in P$ for some $m \in \mathbb{N}$ iff P is primary.

(a) $\iff$ (c): By (a) $\iff$ (b), P is primary iff A/P is local iff A/P has a unique maximal deductive system. By Corollary 3.33 there is a bijection between $\mathrm{Max}(A/P)$ and $\{D : \ D \in \mathrm{Max}(A), \ P \subseteq D\}$. It follows that P is primary if and only if there is a unique maximal deductive system of A containing P. $\square$

Theorem 4.9. *If A is a pseudo-BCK*(pP) *algebra, then the following are equivalent:*

(a) *A is local;*

(b) *any proper normal deductive system of A is primary;*

(c) *$\{1\}$ is a primary deductive system of A.*

Proof.

(a) $\Longrightarrow$ (b): Let H be a proper normal deductive system of A. By Theorem 4.3 (d), $D(A)$ is the unique maximal deductive system of A. Hence, $H \subseteq D(A)$ and according to Proposition 4.8 it follows that H is primary;

(b) $\Longrightarrow$ (c): Since $\{1\}$ is a proper normal deductive system of A, then by (b) we get that $\{1\}$ is primary;

(c) $\Longrightarrow$ (a): Since $\{1\}$ is primary, applying Proposition 4.8 it follows that $A/\{1\}$ is local. Taking into consideration that $A \cong A/\{1\}$, it follows that A is local. □

Definition 4.10. A primary deductive system P of a bounded pseudo-BCK(pP) algebra A is called *perfect* if for all $x \in A$, $(x^n)^- \in P$ for some $n \in \mathbb{N}$ implies $((x^-)^m)^- \notin P$ for all $m \in \mathbb{N}$.

An element x of a pseudo-BCK(pP) algebra A is said to be *zero divisor* if there exists an element $0 \neq y \in A$ such that $x \odot y = 0$ or $y \odot x = 0$. The set of all zero divisors of A is denoted by $\mathrm{Div}(A)$. Obviously, $0 \in \mathrm{Div}(A)$ and $1 \notin \mathrm{Div}(A)$.

Proposition 4.11. *Let A be a bounded pseudo-BCK*(pP) *algebra satisfying the conditions:* $\mathrm{Div}(A) = \{0\}$, $\mathrm{ord}(x) = \infty$ *and* $x^- = x^\sim = 0$ *for all* $x \in A \setminus \{0\}$. *Then, any proper normal deductive system of A is perfect.*

Proof. We first prove that any proper normal deductive system P of A is primary.

Let $x, y \in A$ and consider the following cases:

(1) If $x, y > 0$, then $x \odot y > 0$, so $\mathrm{ord}(x \odot y) = \infty$. It follows that $(x \odot y)^n \neq 0$ for all $n \in \mathbb{N}$. Hence, $((x \odot y)^n)^- = 0 \notin P$;

(2) If $x = 0$, then $((x \odot y)^n)^- = 0^- = 1 \in P$ for all $n \in \mathbb{N}$. Moreover, $(x^m)^- = 0^- = 1 \in P$ for all $m \in \mathbb{N}$;

(3) If $y = 0$, then similarly as in (2) we get that $(y^m)^- = 0^- = 1 \in P$ for all $m \in \mathbb{N}$.

Thus, P is a primary deductive system of A.

Since $x^n \neq 0$ for all $x \in A \setminus \{0\}$, it follows that $(x^n)^- = 0 \notin P$ for all $n \in \mathbb{N}$. For $x = 0$ we have $(0^n)^- = 1 \in P$ for all $n \in \mathbb{N}$ and $((0^-)^m)^- = 0 \notin P$ for all $m \in \mathbb{N}$. Thus, P is a perfect deductive system of A. □

Examples 4.12.

(1) It is a simple routine to check that the normal deductive system $D = \{s, a, b, n, c, d, m, 1\}$ of the pseudo-BCK(pP) algebra A from Example 2.11 is primary, but D it is not perfect $((a_1^2)^- = 0^- = 1 \in D$ and $((a_1^-)^2)^- = (a_1^2)^- = 0^- = 1 \in D)$;

(2) According to Proposition 4.11, the normal deductive systems

$$D_1 = \{a_1, a_2, b_2, s, a, b, n, c, d, m, 1\} \quad \text{and} \quad D_3 = \{s, a, b, n, c, d, m, 1\}$$

of the pseudo-BCK(pP) algebra A_1 from Example 2.21 are perfect deductive systems.

Definition 4.13. A pseudo-BCK(pP) algebra A is called *perfect* if it satisfies the following conditions:

(1) A is a local good pseudo-BCK(pP) algebra;

(2) for any $x \in A$, $\mathrm{ord}(x) < \infty$ iff $\mathrm{ord}(x^-) = \infty$ and $\mathrm{ord}(x^\sim) = \infty$.

Proposition 4.14. *Let A be a local good pseudo-BCK*(pP) *algebra. Then the following are equivalent:*

(a) *A is perfect;*

(b) $D(A)^*_- = D(A)^*_\sim = D(A)^*$.

Proof.

(a) $\Longrightarrow$ (b): Since A is a local pseudo-BCK(pP) algebra, applying Corollary 4.4(2) we get $D(A)^*_- \subseteq D(A)^*$ and $D(A)^*_\sim \subseteq D(A)^*$.

Conversely, consider $x \in D(A)^*$, that is $\mathrm{ord}(x) < \infty$. By the definition of a perfect pseudo-BCK(pP) algebra we get $\mathrm{ord}(x^-) = \infty$ and $\mathrm{ord}(x^\sim) = \infty$, that is $x^-, x^\sim \in D(A)$. Applying the properties $x \le x^{\sim -}$ and $x \le x^{-\sim}$ we get $x \in D(A)^*_-$ and $x \in D(a)^*_\sim$. It follows that $D(A)^* \subseteq D(A)^*_-$ and respectively $D(A)^* \subseteq D(A)^*_\sim$. Thus, $D(A)^*_- = D(A)^*$ and $D(A)^*_\sim = D(A)^*$.

(b) $\Longrightarrow$ (a): Consider $x \in A$ such that $\mathrm{ord}(x) < \infty$, that is $x \in D(A)^*$.

Since $D(A)^*_- = D(A)^*$, there exists $y \in D(A)$ such that $x \le y^-$, so $y^{-\sim} \le x^\sim$. By $y \le y^{-\sim}$ and $\mathrm{ord}(y) = \infty$, we get $\mathrm{ord}(y^{-\sim}) = \infty$. From $y^{-\sim} \le x^\sim$ we get $\mathrm{ord}(x^\sim) = \infty$. Since $D(A)^*_\sim = D(A)^*$, there exists $y \in D(A)$ such that $x \le y^\sim$, so $y^{\sim -} \le x^-$. By $y \le y^{\sim -}$ and $\mathrm{ord}(y) = \infty$, we get $\mathrm{ord}(y^{\sim -}) = \infty$. From $y^{\sim -} \le x^-$ we get $\mathrm{ord}(x^-) = \infty$.

Conversely, consider $x \in A$ such that $\mathrm{ord}(x^-) = \infty$ and $\mathrm{ord}(x^\sim) = \infty$.

Since A is local, by Corollary 4.4(1) it follows that $\mathrm{ord}(x) < \infty$. Thus, A is a perfect pseudo-BCK(pP) algebra. □

Examples 4.15.

(1) Consider the pseudo-BCK(pP) algebra A from Example 2.11. Since A is not good, it follows that it is not a perfect pseudo-BCK(pP) algebra.

(2) If A_1 is the good pseudo-BCK(pP) algebra A from Example 2.21, we have $D(A) = \{a_1, a_2, b_2, s, a, b, n, c, d, m, 1\}$ and $D(A)^* = \{0\}$. Since $\mathrm{ord}(0^-) = \mathrm{ord}(0^\sim) = \infty$, it follows that A is a perfect pseudo-BCK(pP) algebra.

Proposition 4.16. *Let A be a good pseudo-BCK(pP) algebra and P a proper normal deductive system of A. Then the following are equivalent:*

(a) *P is a perfect deductive system of A;*

(b) *A/P is a perfect pseudo-BCK(pP) algebra;*

(c) *for all $x \in A$, $(x^n)^\sim \in P$ for some $n \in \mathbb{N}$ implies $((x^\sim)^m)^\sim \notin P$ for all $m \in \mathbb{N}$.*

Proof. By Proposition 4.8, A/P is local iff P is primary. Also, A/P is perfect iff the following condition is satisfied:

$$\mathrm{ord}(x/P) < \infty \quad \text{iff} \quad \mathrm{ord}((x/P)^-) = \infty \quad \text{and} \quad \mathrm{ord}((x/P)^\sim) = \infty.$$

But, applying Lemma 3.23, we have:

$\mathrm{ord}(x/P) < \infty$ iff $(x/P)^n = 0/P$ for some $n \in \mathbb{N}$
iff $(x^n)^- \in P$ for some $n \in \mathbb{N}$ and $(x^n)^\sim \in P$ for some $n \in \mathbb{N}$.

We also have:

$\mathrm{ord}((x/P)^-) = \infty$ iff $((x/P)^-)^m \neq 0/P$ for all $m \in \mathbb{N}$
iff $((x^-)^m)^- \notin P$ for all $m \in \mathbb{N}$.

Taking into consideration the definition of a perfect deductive system it follows that (a) $\iff$ (b).
Similarly,

$\mathrm{ord}((x/P)^\sim) = \infty$ iff $((x/P)^\sim)^m \neq 0/P$ for all $m \in \mathbb{N}$
iff $((x^\sim)^m)^\sim \notin P$ for all $m \in \mathbb{N}$.

Thus, (a) $\iff$ (c). □

Proposition 4.17. *If P is a perfect deductive system of A, then:*

(1) *for all $x \in A$, $(x^n)^- \in P$ for some $n \in \mathbb{N}$ iff $((x^-)^m)^- \notin P$ for all $m \in \mathbb{N}$;*

(2) *for all $x \in A$, $(x^n)^\sim \in P$ for some $n \in \mathbb{N}$ iff $((x^\sim)^m)^\sim \notin P$ for all $m \in \mathbb{N}$.*

Proof.

(1) The first implication follows immediately, since P is perfect.

Consider $x \in A$ such that $((x^-)^m)^- \notin P$ for all $m \in \mathbb{N}$. By (c_{32}), $x^- \odot x = 0$, so $((x^- \odot x)^m)^- = 0^- = 1 \in P$ for all $m \in \mathbb{N}$. Since P is primary, it follows that $((x^-)^n)^- \in P$ or $(x^n)^- \in P$ for some $n \in \mathbb{N}$. Taking into consideration that $((x^-)^n)^- \notin P$ for all $n \in \mathbb{N}$, we conclude that $(x^n)^- \in P$ for some $n \in \mathbb{N}$;

(2) Similarly as (1). □

Theorem 4.18. *If A is a local good pseudo-BCK*(pP) *algebra, then the following are equivalent:*

(a) *A is perfect;*

(b) *any proper normal deductive system of A is perfect;*

(c) *$\{1\}$ is a perfect deductive system of A.*

Proof.

(a) $\Longrightarrow$ (b): Let D be a proper normal deductive system of A. By Theorem 4.9 it follows that D is primary. Let $x \in A$ such that $(x^n)^- \in D$ for some $n \in \mathbb{N}$ and suppose that $((x^-)^m)^- \in D$ for some $m \in \mathbb{N}$. Since D is proper, then $\langle (x^n)^- \rangle, \langle ((x^-)^m)^- \rangle \subseteq D$ are also proper deductive systems of A. By Lemma 3.10(1) it follows that $\mathrm{ord}((x^n)^-) = \mathrm{ord}(((x^-)^m)^-) = \infty$. Since A is perfect, $\mathrm{ord}(x^n) < \infty$ and $\mathrm{ord}((x^-)^m) < \infty$, hence $\mathrm{ord}(x) < \infty$ and $\mathrm{ord}(x^-) < \infty$, a contradiction with the fact that A is perfect.

Thus, $(x^n)^- \in D$ for $n \in \mathbb{N}$ implies $((x^-)^m)^- \notin D$ for all $m \in \mathbb{N}$, that is D is perfect.

(b) $\Longrightarrow$ (c): It is obvious, since $\{1\}$ is a proper normal deductive system of A.

(c) $\Longrightarrow$ (a): Since $\{1\}$ is a perfect deductive system of A, applying Proposition 4.16 it follows that $A/\{1\}$ is perfect. Taking into consideration that $A \cong A/\{1\}$ we get that A is perfect. $\square$

Definition 4.19. Let A be a pseudo-BCK(pP) algebra. The intersection of all maximal deductive systems of A is called the *radical* of A and it is denoted by $\mathrm{Rad}(A)$.

Proposition 4.20. ([7]) *If A is a perfect pseudo-BCK*(pP) *algebra, then* $\mathrm{Rad}(A) = D(A)$.

Example 4.21. Consider the perfect pseudo-BCK(pP) A_1 from Example 2.21. One can easily check that $\mathrm{Rad}(A_1) = D(A_1) = \{a_1, a_2, b_2, s, a, b, n, c, d, m, 1\}$.

Remark 4.22. If A is a perfect pseudo-BCK(pP) algebra and $x \in \mathrm{Rad}(A)^*$, $y \in A$ such that $y \leq x$, then $y \in \mathrm{Rad}(A)^*$.

Theorem 4.23. *If A is a perfect pseudo-BCK*(pP) *algebra, then* $\mathrm{Rad}(A)$ *is a normal deductive system of A.*

Proof. We have to prove that $x \to y \in \mathrm{Rad}(A)$ iff $x \rightsquigarrow y \in \mathrm{Rad}(A)$ for all $x, y \in A$. Consider $x, y \in A$ such that $x \to y \in \mathrm{Rad}(A)$ and suppose $x \rightsquigarrow y \notin \mathrm{Rad}(A)$.

From $y \leq y^{-\sim}$ we get $x \to y \leq x \to y^{-\sim}$ (by (c_{21}) and (c_8)). Since $\mathrm{Rad}(A)$ is a deductive system of A, it follows that $x \to y^{-\sim} \in \mathrm{Rad}(A)$, that is $(x \odot y^{\sim})^- \in \mathrm{Rad}(A)$ (by (c_{34}) and from the fact that A is good). Hence, $x \odot y^{\sim} \in \mathrm{Rad}(A)^*$.

On the other hand, from $x \rightsquigarrow y \notin \mathrm{Rad}(A)$, it follows that $x \rightsquigarrow y \in \mathrm{Rad}(A)^*$. Since $x \leq x^{-\sim}$, by (c_1) we get $x^{-\sim} \rightsquigarrow y \leq x \rightsquigarrow y$, so $x^{-\sim} \rightsquigarrow y \in \mathrm{Rad}(A)^*$ (by Remark 4.22). By (c_{36}) we have $x^{\sim} \leq x^{\sim -} \rightsquigarrow y$, so $x^{\sim} \in \mathrm{Rad}(A)^*$, that is $x \in \mathrm{Rad}(A)$. But $y \leq x \rightsquigarrow y$, so $y \in \mathrm{Rad}(A)^*$, that is $y^{\sim} \in \mathrm{Rad}(A)$. Since $\mathrm{Rad}(A)$ is a deductive system of A and $x, y^{\sim} \in \mathrm{Rad}(A)$, we get $x \odot y^{\sim} \in \mathrm{Rad}(A)$ which is a contradiction. Thus, $x \to y \in \mathrm{Rad}(A)$ implies $x \rightsquigarrow y \in \mathrm{Rad}(A)$. Similarly, $x \rightsquigarrow y \in \mathrm{Rad}(A)$ implies $x \to y \in \mathrm{Rad}(A)$ and we conclude that $\mathrm{Rad}(A)$ is a normal deductive system of A. □

Remark 4.24. If the pseudo-BCK(pP) algebra A is not perfect, then the above result is not always valid. Indeed, consider the pseudo-BCK(pP) algebra A from Example 2.11. Since A is not good, it is not a perfect pseudo-BCK(pP) algebra. Moreover, $D = \{a_2, s, a, b, n, c, d, 1\}$ is the unique maximal deductive system of A, so $\mathrm{Rad}(A) = D$. But D is not a normal deductive system.

Corollary 4.25. *If A is a perfect pseudo-BCK(pP) algebra, then $A/\mathrm{Rad}(A)$ is perfect too.*

Proof. By Theorem 4.23, $\mathrm{Rad}(A)$ is a proper normal deductive system of A and by Theorem 4.18 it follows that $\mathrm{Rad}(A)$ is perfect. Applying Proposition 4.16 we get that $A/\mathrm{Rad}(A)$ is a perfect pseudo-BCK(pP) algebra. □

5. Connection with pseudo-hoops

Pseudo-hoops were originally introduced by Bosbach in [2] and [3] under the name of *complementary semigroups* and their properties were recently studied in [13] and [10].

Definition 5.1. ([13]) A *pseudo-hoop* is an algebra $(A, \odot, \to, \rightsquigarrow, 1)$ of type $(2, 2, 2, 0)$ such that, for all $x, y, z \in A$:

(psH_1) $x \odot 1 = 1 \odot x = x$;

(psH_2) $x \to x = x \rightsquigarrow x = 1$;

(psH_3) $(x \odot y) \to z = x \to (y \to z)$;

(psH_4) $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$;

(psH_5) $(x \to y) \odot x = (y \to x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$.

If the operation $\odot$ is commutative, or equivalently $\to = \rightsquigarrow$, then the pseudo-hoop is said to be *hoop*. On the pseudo-hoop A we define $x \leq y$ iff $x \to y = 1$ (equivalent to $x \rightsquigarrow y = 1$) and $\leq$ is a partial order on A. A pseudo-hoop A is bounded if there is an element $0 \in A$ such that $0 \leq x$ for all $x \in A$.

PROPOSITION 5.2. ([13]) *In every pseudo-hoop* $(A, \odot, \to, \rightsquigarrow, 1)$ *the following hold:*

(h_1) $(A, \leq)$ *is a meet-semillatice with* $x \wedge y = (x \to y) \odot x = x \odot (x \rightsquigarrow y)$*;*

(h_2) $x \odot y \leq z$ *iff* $x \leq y \to z$ *iff* $y \leq x \rightsquigarrow z$*;*

(h_3) $x \to x = x \rightsquigarrow x = 1$*;*

(h_4) $1 \to x = 1 \rightsquigarrow x = x$*;*

(h_5) $x \to 1 = x \rightsquigarrow 1 = 1$*;*

(h_6) $x \leq (x \to y) \rightsquigarrow y$*;*

(h_7) $x \leq (x \rightsquigarrow y) \to y$*;*

(h_8) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$*;*

(h_9) $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$.

The proofs of he next two results are obvious from Proposition 5.2.

PROPOSITION 5.3. *Every pseudo-BCK*(pP) *algebra satisfying* (psH_5) *is a pseudo-hoop.*

PROPOSITION 5.4. *Every pseudo-hoop is a pseudo-BCK*(pP) *algebra.*

COROLLARY 5.5. *A pseudo-BCK*(pP) *algebra with* (psH_5) *is termwise equivalent with a pseudo-hoop.*

THEOREM 5.6. *Every locally finite pseudo-hoop is with* (pDN).

P r o o f. Let A be a locally finite pseudo-hoop and $x \in A$. If $x = 0$, then $0^{-\sim} = 0^{\sim -} = 0$.

Suppose $x \neq 0$ and we prove that $x^{-\sim} = x$. By (c_{21}) we have $x \leq x^{-\sim}$. Suppose that $x^{-\sim} \not\leq x$, hence $x^{-\sim} \to x \neq 1$. Since A is locally finite, there is $m \in \mathbb{N}$, $n \geq 1$ such that $(x^{-\sim} \to x)^n = 0$. We have:

$$\begin{aligned}(x^{-\sim} \to x) \to x^{-} &= (x^{-\sim} \to x) \to x^{-\sim -} = (x^{-\sim} \to x) \to (x^{-\sim} \to 0)\\ &= (x^{-\sim} \to x) \odot x^{-\sim} \to 0 = (x \wedge x^{-\sim}) \to 0\\ &= x \to 0 = x^{-}.\end{aligned}$$

$$\begin{aligned}(x^{-\sim} \to x)^2 \to x^{-} &= (x^{-\sim} \to x) \to ((x^{-\sim} \to x) \to x^{-})\\ &= (x^{-\sim} \to x) \to x^{-} = x^{-}.\end{aligned}$$

By induction we get $(x^{-\sim} \to x)^n \to x^{-} = x^{-}$. Thus, $0 \to x^{-} = x^{-}$, so $x^{-} = 1$. Hence $x = 0$, a contradiction. Therefore, $x^{-\sim} = x$ and similarly $x^{\sim -} = x$. $\square$

DEFINITION 5.7. ([13]) A pseudo-hoop A is called *simple* if $\{1\}$ is the unique proper normal deductive system of A. The pseudo-hoop A is called *strongly simple* if $\{1\}$ is the unique proper deductive system of A.

Obviously, any strongly simple pseudo-hoop is simple.

Theorem 5.8. *Every strongly simple bounded pseudo-hoop is local.*

P r o o f. By Proposition 3.20 it follows that a strongly simple bounded pseudo-hoop A is locally finite and by Proposition 4.7 we get that A is local. □

REFERENCES

[1] BELLUCE, L. P.—DI NOLA, A.—LETTIERI, A.: *Local MV algebras*, Rend. Circ. Mat. Palermo (2) **42** (1993), 347–361.

[2] BOSBACH, B.: *Komplementäre Halbgruppen. Axiomatik und Aritmetik*, Fund. Math. **64** (1969), 257–287.

[3] BOSBACH, B.: *Komplementäre Halbgruppen. Kongruenzen and Quotienten*, Fund. Math. **69** (1970), 1–14.

[4] BUŞNEAG, D.—PICIU, D.: *On the lattice of filters of a pseudo BL-algebra*, J. Mult.-Valued Logic Soft Comput. **12** (2006), 217–248.

[5] CIUNGU, L. C.: *Classes of residuated lattices*, An. Univ. Craiova Ser. Mat. Inform. **33** (2006), 189–207.

[6] CIUNGU, L. C.: *Some classes of pseudo-MTL algebras*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **50(98)** (2007), 223–247.

[7] CIUNGU, L. C.: *On perfect pseudo-BCK algebras with pseudo-product*, An. Univ. Craiova Ser. Mat. Inform. **34** (2007), 29–42.

[8] DI NOLA, A.—GEORGESCU, G.—IORGULESCU, A.: *Pseudo-BL algebras: Part I*, Mult.-Valued Log. **8** (2002), 673–714.

[9] DI NOLA, A.—GEORGESCU, G.—IORGULESCU, A.: *Pseudo-BL algebras: Part II*, Mult.-Valued Log. **8** (2002), 717–750.

[10] DVUREČENSKIJ, A.: *Aglianò-Montagna type decomposition of linear pseudo-hoops and its applications*, J. Pure Appl. Algebra **211** (2007), 851–861.

[11] GEORGESCU, G.—IORGULESCU, A.: *Pseudo-BCK algebras: An extension of BCK algebras*. In: Proceedings of DMTCS'01: Combinatorics, Computability and Logic, Springer, London, 2001, pp. 97–114.

[12] GEORGESCU, G.—LEUŞTEAN, L.: *Some classes of pseudo-BL algebras*, J. Aust. Math. Soc. **73** (2002), 127–153.

[13] GEORGESCU, G.—LEUŞTEAN, L.—PREOTEASA, V.: *Pseudo-hoops*, J. Mult.-Valued Logic Soft Comput. **11** (2005), 153–184.

[14] HALAŠ, R.—KÜHR, J.: *Deductive systems and annihilators of pseudo-BCK algebras*, Ital. J. Pure Appl. Math. **25** (2009), 83–94.

[15] IORGULESCU, A.: *Classes of pseudo-BCK algebras – Part I*, J. Mult.-Valued Logic Soft Comput. **12** (2006), 71–130.

[16] IORGULESCU, A.: *Classes of pseudo-BCK algebras – Part II*, J. Mult.-Valued Logic Soft Comput. **12** (2006), 575–629.

[17] IORGULESCU, A.: *On pseudo-BCK algebras and porims*, Sci. Math. Jpn. **16** (2004), 293–305.

[18] IORGULESCU, A.: *Pseudo-Iséki algebras. Connection with pseudo-BL algebras*, J. Mult.-Valued Logic Soft Comput. **3-4** (2005), 263–308.

[19] KÜHR, J.: *Commutative pseudo-BCK algebras*, Southeast Asian Bull. Math. **33** (2009), 451–475.

[20] KÜHR, J.: *Pseudo-BCK algebras and residuated lattices*, Contrib. Gen. Algebra **16** (2005), 139–144.
[21] KÜHR, J.: *Pseudo-BCK Algebras and Related Structures.* Habilitation Thesis, Univerzita Palackého v Olomouci, Olomouc, 2007.
[22] LEUŞTEAN, I.: *Local pseudo-MV algebras*, Soft Comput. **5** (2001), 386–395.
[23] RACHŮNEK, J.—ŠALOUNOVÁ, D.: *A generalization of local fuzzy structures*, Soft Comput. **11** (2007), 565–571.
[24] RACHŮNEK, J.—ŠALOUNOVÁ, D.: *Local bounded commutative residuated ℓ-monoids*, Czechoslovak Math. J. **57** (2007), 395–406.
[25] TURUNEN, E.—SESSA, S.: *Local BL algebras*, Mult.-Valued Log. **6** (2001), 229–249.

Received 31. 10. 2008
Accepted 30. 9. 2009

Department of Mathematics
Polytechnical University of Bucharest
Splaiul Independenţei 313
Bucharest
ROMANIA
E-mail: lavinia_ciungu@math.pub.ro