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A NOTE ON REIDEMEISTER TORSION AND PERIOD MATRIX OF RIEMANN SURFACES

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ABSTRACT. We consider compact Riemann surfaces Σ_g with genus at least 2. We explain the relation between the Reidemeister torsion of Σ_g and its period matrix.

1. Introduction

Reidemeister torsion is a topological invariant and was first introduced by Reidemeister [13] in 1935. He classified 3-dimensional lens spaces by using this combinatorial invariant of CW-complexes. Later, Franz [6] generalized Reidemeister torsion and classified the higher dimensional lens spaces; namely, S^{2n+1}/Γ where Γ is a cyclic group acting freely and isometrically on the sphere S^{2n+1}.

In 1964, de Rham [5] extended the results of Reidemeister and Franz to the spaces of constant curvature 1. To be more precise, *two isometries of S^n are diffeomorphic if and only if they are conjugate of each other by an isometry.*

The topological invariance of the torsion for manifold was proved in 1969 by Kirby and Siebenmann [8]. For arbitrary simplicial complex it was proved by Chapman [3, 4]. Thus, the classification of lens spaces of Reidemeister and Franz was actually topological (i.e. up to homeomorphism).

In 1961, by using the torsion, Milnor disproved *Hauptvermutung*. He constructed two homeomorphic but combinatorially distinct finite simplicial complexes. Later in 1962, Milnor [9] identified the Reidemeister torsion with Alexander polynomial. Since then, as a topological invariant, torsion has a very useful application in knot theory and links.

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We ([14]) presented an explanation of the claim mentioned ([15, pp. 187]) about the relation between a symplectic chain complex with ω-compatible bases and Reidemeister torsion of it (Theorem 2.2). We also ([14]) applied Theorem 2.2 to the chain-complex

$$0 \to \mathcal{C}_2(\Sigma_g; \mathrm{Ad}_\varrho) \overset{\partial_2 \otimes \mathrm{id}}{\longrightarrow} \mathcal{C}_1(\Sigma_g; \mathrm{Ad}_\varrho) \overset{\partial_1 \otimes \mathrm{id}}{\longrightarrow} \mathcal{C}_0(\Sigma_g; \mathrm{Ad}_\varrho) \to 0$$

where Σ_g is a compact Riemann surface of genus $g > 1$, $\varrho\colon \pi_1(\Sigma_g) \to \mathrm{PSL}_2(\mathbb{R})$ is discrete and faithful representation of the fundamental group $\pi_1(\Sigma_g)$ of Σ_g, where $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$, $\mathrm{SL}_2(\mathbb{R})$ is the 2×2 real matrices with determinant 1, where $\mathcal{C}_p(\Sigma_g; \mathrm{Ad}_\varrho)$ is (locally) $\mathcal{C}_p(\Sigma_g; \mathbb{R}) \otimes \mathfrak{sl}_2(\mathbb{R})$, and where $\mathfrak{sl}_2(\mathbb{R})$ denote the 2×2 trace-zero matrices with real entries.

In the present article, compact Riemann surfaces Σ_g of genus at least 1 are considered and the relation between the Reidemeister torsion and its period matrix is proved. The main result is:

Theorem 1.1. *Let $\mathfrak{h}^1 = \{\omega_i\}_{i=1}^{2g}$ be a basis for $\mathcal{H}^1(\Sigma_g)$. Let $\mathbf{K}$ be a cell decomposition of the compact Riemann surface Σ_g with genus $g \geq 1$, and let for $p = 0, 1, 2$, $\mathfrak{c}_p$ be the geometric bases of $\mathcal{C}_p(\mathbf{K}; \mathbb{Z})$. Then, $\mathcal{T}\big(\mathcal{C}_*, \{\mathfrak{c}_p\}_{p=0}^2, \{[\mathfrak{c}_0], \mathfrak{h}_1, [\mathfrak{c}_2]\}\big) = \big|\det \wp(\mathfrak{h}^1, \Gamma)\big|^{-1}$, where $\wp(\mathfrak{h}^1, \Gamma) = \Big[\int_{\Gamma_i} \omega_j\Big]$ is the period matrix of $\mathfrak{h}^1$ with respect to the canonical basis $\Gamma = \{\Gamma_i\}_{i=1}^{2g}$ of $\mathcal{H}_1(\Sigma_g)$, and $\mathfrak{h}_1 = \{\Omega_j\}_{j=1}^{2g}$ is the basis of $\mathcal{H}_1(\Sigma_g)$ corresponding to $\mathfrak{h}^1$.*

Our result can also be obtained as a special case of [2, Theorem 5.40] (taking $s = 0$ therein).

The organization of the paper is as follows. In §2, we explain Reidemeister torsion of a general chain complex and provide the basic facts about it. The symplectic chain complex associated to even dimensional manifolds M^{2m} with m odd is explained in §3. As an application, we also provide the proof of Theorem 1.1.

2. Reidemeister torsion of a chain complex

We give the necessary definitions and explain the basic facts about the Reidemeister torsion in this section. For more information, we refer the reader [12, 14, 15] and the references therein.

We reserve $\Bbbk$ to denote the field $\mathbb{R}$ or $\mathbb{C}$.

Let $(\mathcal{C}_*, \partial_*) = (\mathcal{C}_n \overset{\partial_n}{\to} \mathcal{C}_{n-1} \to \cdots \to \mathcal{C}_1 \overset{\partial_1}{\to} \mathcal{C}_0 \to 0)$ be a chain complex of finite dimensional vector spaces over $\Bbbk$, where $\mathcal{B}_p = \mathrm{Im}\{\partial_{p+1}\colon \mathcal{C}_{p+1} \to \mathcal{C}_p\}$,

$\mathcal{Z}_p = \ker\{\partial_p : \mathcal{C}_p \to \mathcal{C}_{p-1}\}$, respectively. If $\mathcal{H}_p(\mathcal{C}_*) = \mathcal{Z}_p(\mathcal{C}_*)/\mathcal{B}_p(\mathcal{C}_*)$ denotes the pth homology of the chain complex, then clearly we have the following short-exact sequences: $0 \to \mathcal{Z}_p \hookrightarrow \mathcal{C}_p \twoheadrightarrow \mathcal{B}_{p-1} \to 0$ and $0 \to \mathcal{B}_p \hookrightarrow \mathcal{Z}_p \twoheadrightarrow \mathcal{H}_p \to 0$.

Let $\mathfrak{b}_p$ be a basis for $\mathcal{B}_p$, $\mathfrak{h}_p$ be a basis for $\mathcal{H}_p$, and $\ell_p \colon \mathcal{H}_p \to \mathcal{Z}_p$ and $s_p \colon \mathcal{B}_{p-1} \to \mathcal{C}_p$ be sections. Then, a new basis for $\mathcal{C}_p$, namely, $\mathfrak{b}_p \oplus \ell_p(\mathfrak{h}_p) \oplus s_p(\mathfrak{b}_{p-1})$ is obtained.

For $p = 0, \dots, n$, let $\mathfrak{c}_p$, $\mathfrak{b}_p$, and $\mathfrak{h}_p$ be bases for $\mathcal{C}_p$, $\mathcal{B}_p$ and $\mathcal{H}_p$, respectively. The *Reidemeister torsion* $\mathcal{T}(\mathcal{C}_*, \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n)$ of the chain complex $\mathcal{C}_*$ with respect to bases $\{\mathfrak{c}_p\}_{p=0}^n$, $\{\mathfrak{h}_p\}_{p=0}^n$ is $\prod_{p=0}^{n} [\mathfrak{b}_p \oplus \ell_p(\mathfrak{h}_p) \oplus s_p(\mathfrak{b}_{p-1}), \mathfrak{c}_p]^{(-1)^{(p+1)}}$, where $[\mathfrak{e}_p, \mathfrak{f}_p]$ denotes the determinant of the change-base-matrix from the basis $\mathfrak{f}_p$ to $\mathfrak{e}_p$ of $\mathcal{C}_p$.

In [10], Milnor proved that torsion is independent of the bases $\mathfrak{b}_p$, the sections s_p, ℓ_p. If $\mathfrak{c}'_p$, $\mathfrak{h}'_p$ are some other bases for $\mathcal{C}_p$ and $\mathcal{H}_p$ respectively, then an easy computation gives the following change-base-formula:

$$\mathcal{T}\big(\mathcal{C}_*, \{\mathfrak{c}'_p\}_{p=0}^n, \{\mathfrak{h}'_p\}_{p=0}^n\big) = \prod_{p=0}^{n} \left(\frac{[\mathfrak{c}'_p, \mathfrak{c}_p]}{[\mathfrak{h}'_p, \mathfrak{h}_p]}\right)^{(-1)^p} \mathcal{T}\big(\mathcal{C}_*, \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n\big). \quad (2.1)$$

Formula (2.1) easily follows from the independence of torsion from $\mathfrak{b}_p$ and sections s_p, ℓ_p. Note that if, for instance, $[\mathfrak{c}'_p, \mathfrak{c}_p] = 1$ and $[\mathfrak{h}'_p, \mathfrak{h}_p] = 1$, then we have the same torsion.

By the Zig-Zag Lemma, for the short-exact sequence of chain complexes

$$0 \to \mathcal{A}_* \overset{\imath}{\hookrightarrow} \mathcal{B}_* \overset{\pi}{\twoheadrightarrow} \mathcal{D}_* \to 0, \quad (2.2)$$

there is also the long-exact sequence of vector space $\mathcal{C}_*$ of length $3n+2$. More precisely,

$$\mathcal{C}_* \colon \cdots \to \mathcal{H}_p(\mathcal{A}_*) \overset{\imath_*}{\to} \mathcal{H}_p(\mathcal{B}_*) \overset{\pi_*}{\to} \mathcal{H}_p(\mathcal{D}_*) \overset{\Delta}{\to} \mathcal{H}_{p-1}(\mathcal{A}_*) \to \cdots, \quad (2.3)$$

where $\mathcal{C}_{3p} = \mathcal{H}_p(\mathcal{D}_*)$, $\mathcal{C}_{3p+1} = \mathcal{H}_p(\mathcal{A}_*)$ and $\mathcal{C}_{3p+2} = \mathcal{H}_p(\mathcal{B}_*)$. Clearly, the bases $\mathfrak{h}_p(\mathcal{D}_*)$, $\mathfrak{h}_p(\mathcal{A}_*)$, and $\mathfrak{h}_p(\mathcal{B}_*)$ serve as bases for $\mathcal{C}_{3p}$, $\mathcal{C}_{3p+1}$, and $\mathcal{C}_{3p+2}$, respectively.

The following theorem is due to Milnor and states that the alternating product of the torsions of the chain complexes in (2.2) is equal to the torsion of (2.3). Namely:

Theorem 2.1. ([10]) *Let $\mathfrak{c}_p^{\mathcal{A}}$, $\mathfrak{c}_p^{\mathcal{B}}$, $\mathfrak{c}_p^{\mathcal{D}}$ be bases respectively for $\mathcal{A}_p$, $\mathcal{B}_p$, $\mathcal{D}_p$, and let $\mathfrak{h}_p^{\mathcal{A}}$, $\mathfrak{h}_p^{\mathcal{B}}$, $\mathfrak{h}_p^{\mathcal{D}}$ be bases for $\mathcal{H}_p(\mathcal{A}_*)$, $\mathcal{H}_p(\mathcal{B}_*)$, $\mathcal{H}_p(\mathcal{D}_*)$. Furthermore, suppose $\mathfrak{c}_p^{\mathcal{A}}$, $\mathfrak{c}_p^{\mathcal{B}}$, $\mathfrak{c}_p^{\mathcal{D}}$ are compatible in the sense that $[\mathfrak{c}_p^{\mathcal{B}}, \mathfrak{c}_p^{\mathcal{A}} \oplus \widetilde{\mathfrak{c}_p^{\mathcal{D}}}] = \pm 1$, where $\pi\left(\widetilde{\mathfrak{c}_p^{\mathcal{D}}}\right) = \mathfrak{c}_p^{\mathcal{D}}$. Then, $\mathcal{T}\big(\mathcal{B}_*, \{\mathfrak{c}_p^{\mathcal{B}}\}_{p=0}^n, \{\mathfrak{h}_p^{\mathcal{B}}\}_{p=0}^n\big) = \mathcal{T}\big(\mathcal{A}_*, \{\mathfrak{c}_p^{\mathcal{A}}\}_{p=0}^n, \{\mathfrak{h}_p^{\mathcal{A}}\}_{p=0}^n\big) \times \mathcal{T}\big(\mathcal{D}_*, \{\mathfrak{c}_p^{\mathcal{D}}\}_{p=0}^n, \{\mathfrak{h}_p^{\mathcal{D}}\}_{p=0}^n\big) \times \mathcal{T}\big(\mathcal{C}_*, \{\mathfrak{c}_{3p}\}_{p=0}^{3n+2}, \{0\}_{p=0}^{3n+2}\big)$.*

In [1], [14], it is independently explained that a general chain complex can (unnaturally) be splitted as a direct sum of two chain complexes, one of which is exact and the other is ∂-zero. Moreover, it is proved independently in [1, Proposition 1.5] and [14, Theorem 2.0.4] that torsion $\mathcal{T}(\mathcal{C}_*)$ of a general complex can be interpreted as element of $\bigotimes_{p=0}^{n} \big(\det(\mathcal{H}_p(\mathcal{C}_*))\big)^{(-1)^{p+1}}$. For the details, we refer the reader to [1], [14].

The chain-complex $\mathcal{C}_*\colon 0 \to \mathcal{C}_q \xrightarrow{\partial_q} \mathcal{C}_{q-1} \to \cdots \to \mathcal{C}_{q/2} \to \cdots \to \mathcal{C}_1 \xrightarrow{\partial_1} \mathcal{C}_0 \to 0$ is called a *symplectic chain complex*, if the following two conditions are satisfied:

(1) $q \equiv 2 \pmod 4$,

(2) There are non-degenerate anti-symmetric ∂-compatible bilinear maps $\omega_{p,q-p}\colon \mathcal{C}_p \times \mathcal{C}_{q-p} \to \mathbb{R}$. To be more precisely,

$$\omega_{p,q-p}(a,b) = (-1)^{p(q-p)}\omega_{q-p,p}(b,a) \qquad \text{and}$$
$$\omega_{p,q-p}(\partial_{p+1}a,b) = (-1)^{p+1}\omega_{p+1,q-(p+1)}(a,\partial_{q-p}b).$$

Clearly, it follows from $q \equiv 2 \pmod 4$ that $\omega_{p,q-p}(a,b) = (-1)^p\omega_{q-p,p}(b,a)$.

Using the ∂-compatibility of the non-degenerate anti-symmetric bilinear maps $\omega_{p,q-p}\colon \mathcal{C}_p \times \mathcal{C}_{q-p} \to \mathbb{R}$, one can easily extend these to homologies ([14]).

Let $\mathcal{C}_*$ be a symplectic chain complex. The bases $\mathfrak{c}_p$, $\mathfrak{c}_{q-p}$ of $\mathcal{C}_p$, $\mathcal{C}_{q-p}$ are said to be *ω-compatible* if the matrix of $\omega_{p,q-p}$ in bases $\mathfrak{c}_p$, $\mathfrak{c}_{q-p}$ is $\mathrm{I}_{k\times k}$ when $p \neq q/2$ and $\begin{bmatrix} 0_{l\times l} & \mathrm{I}_{l\times l} \\ -\mathrm{I}_{l\times l} & 0_{l\times l} \end{bmatrix}$ when $p = q/2$, where k is $\dim \mathcal{C}_p = \dim \mathcal{C}_{q-p}$ and $2l = \dim \mathcal{C}_{q/2}$.

In a similar manner, considering $[\omega_{p,q-p}]\colon \mathcal{H}_p(\mathcal{C}_*) \times \mathcal{H}_{q-p}(\mathcal{C}_*) \to \mathbb{R}$, the $[\omega_{p,q-p}]$-compatibility of bases $\mathfrak{h}_p$, $\mathfrak{h}_{q-p}$ of $\mathcal{H}_p(\mathcal{C}_*)$, $\mathcal{H}_{q-p}(\mathcal{C}_*)$ is defined.

By using ω-compatible bases $\mathfrak{c}_p$, we showed in [14] that a general symplectic chain complex $\mathcal{C}_*$ can be splitted ω-orthogonally as a direct sum of an exact and ∂-zero symplectic complexes. We also proved Theorem 2.2, one of the main results of [14]. More precisely:

Theorem 2.2. ([14]) *Let $\mathcal{C}_*$ be a general symplectic chain complex. Let $\mathfrak{c}_p$, $\mathfrak{h}_p$ be bases for $\mathcal{C}_p$, $\mathcal{H}_p(\mathcal{C}_*)$. Then,*

$$\mathcal{T}\big(\mathcal{C}_*, \{\mathfrak{c}_p\}_{p=0}^{q}, \{\mathfrak{h}_p\}_{p=0}^{q}\big) = \prod_{p=0}^{(q/2)-1} (\det[\omega_{p,q-p}])^{(-1)^p} \sqrt{\det[\omega_{q/2,q/2}]}^{\,(-1)^{q/2}},$$

where $\det[\omega_{p,q-p}]$ *denotes the determinant of the matrix of the non-degenerate pairing* $[\omega_{p,q-p}]\colon \mathcal{H}_p(\mathcal{C}_*) \times \mathcal{H}_{q-p}(\mathcal{C}_*) \to \mathbb{R}$ *in bases* $\mathfrak{h}_p, \mathfrak{h}_{q-p}$.

For detailed proof, we may refer the reader to [14].

3. Application

In this section, we apply the results of §2 to compact Riemann surfaces Σ_g of genus $g \geq 1$ and explain the relation between the Reidemeister torsion and the intersection number pairing of Σ_g. We refer the reader to [7, 14, 15] for unexplained subjects.

3.1. The torsion of a manifold

If M^n is an n-manifold, $\mathbf{K}$ is a cell-decomposition of M^n with the *geometric basis* $\mathfrak{c}_p = \{c_1^p, \ldots, c_{m_p}^p\}$ for the p-cells $\mathcal{C}_p(\mathbf{K}; \mathbb{Z})$, $p = 0, \ldots, n$, then we have the following chain-complex associated to M^n

$$0 \to \mathcal{C}_n(\mathbf{K}) \overset{\partial_n}{\to} \mathcal{C}_{n-1}(\mathbf{K}) \to \cdots \to \mathcal{C}_1(\mathbf{K}) \overset{\partial_1}{\to} \mathcal{C}_0(\mathbf{K}) \to 0,$$

where ∂_p denotes the boundary operator.

$\mathcal{T}\big(\mathcal{C}_*(\mathbf{K}), \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n\big)$ is called the *Reidemeister torsion* of M^n, where $\mathfrak{h}_p$ is a $\mathbb{k}$-basis for $\mathcal{H}_p(\mathbf{K}; \mathbb{k})$.

Following similar arguments as in [14, Lemma 2.0.5], one can prove:

Lemma 3.1. $\mathcal{T}\big(\mathcal{C}_*(\mathbf{K})\big)$ *does not depend on the cell-decomposition* $\mathbf{K}$ *of the manifold* M^n.

Therefore, for an n-manifold M^n, if $\mathbf{K}$ is a cell-decomposition of M^n and for each $p = 0, \ldots, n$, $\mathfrak{c}_p = \{c_1^p, \ldots, c_{m_p}^p\}$ is a basis for the p-cells $\mathcal{C}_p(\mathbf{K}; \mathbb{Z})$, then Reidemeister torsion $\mathcal{T}\big(\mathcal{C}_*(\mathbf{K}), \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n\big)$ of M^n is well-defined, where $\mathfrak{h}_p$ is a $\mathbb{k}$-basis for $\mathcal{H}_p(\mathbf{K}; \mathbb{k})$.

Note also that from [1, Proposition 1.5] and [14, Theorem 2.0.4] it follows that $\mathcal{T}(\mathcal{C}_*(\mathbf{K}))$ is an element of the dual of the one dimensional vector space $\bigotimes_{p=0}^{n} (\det(\mathcal{H}_p(M))^{(-1)^p}$.

3.2. Symplectic chain complex for a manifold

In this subsection, symplectic chain complex associated to a compact even dimensional manifold M is explained.

Let M^{2m} with m odd be a compact oriented $2m$-manifold. If $\mathbf{K}$ is a cell decomposition of M^{2m}, then let $\mathbf{K}'$ be the corresponding dual cell-decomposition of M^{2m} associated to $\mathbf{K}$. The dual cell-decomposition $\mathbf{K}'$ can be obtained as follows. Let $\mathbf{K} = \{\sigma_\alpha^k\}_{\alpha,k}$ and let $\{\tau_\alpha^k\}_{\alpha,k}$ be the first barycentric subdivision of $\mathbf{K}$. Then, for each vertex $\sigma_\alpha^0 \in \mathbf{K}$, let $(\sigma_\alpha^0)' = \bigcup_{\sigma_\alpha^0 \in \tau_\beta^{2m}} \tau_\beta^{2m}$ be the $2m$-cell given as the union of all $2m$-simplices τ_β^{2m} in the subdivision with σ_α^0 as a vertex. For each k-simplex in the cell-decomposition $\mathbf{K}$, we let $(\sigma_\alpha^k)' = \bigcap_{\sigma_\beta^0 \in \sigma_\alpha^k} (\sigma_\beta^0)'$ be the intersection of the $2m$-cells $(\sigma_\beta^0)'$ associated to the $k+1$ vertices of σ_α^k.

In this way, one can associate the dual cell-decomposition $\mathbf{K}' = \{\Delta_\alpha^{2m-k} = (\sigma_\alpha^k)'\}$ of M^{2m} corresponding to $\mathbf{K}$. Note that $\Delta_\alpha^{2m-k} = (\sigma_\alpha^k)'$ and σ_α^k meet transversely. Given an orientation on σ_α^k, we can take the dual orientation on Δ_α^{2m-k} to be the one such that at $P \in \sigma_\alpha^k \cap (\sigma_\alpha^k)'$, $\imath_P(\sigma_\alpha^k, (\sigma_\alpha^k)') = 1$, where $\imath_P$ denotes the intersection number index at P.

Clearly, the intersection pairings $(\cdot,\cdot)_{k,2m-k} \colon \mathcal{C}_k(\mathbf{K};\mathbb{Z}) \times \mathcal{C}_{2m-k}(\mathbf{K}';\mathbb{Z}) \to \mathbb{R}$ satisfies the following: for all $\alpha \in \mathcal{C}_k(\mathbf{K};\mathbb{Z})$, $\beta \in \mathcal{C}_{2m-k}(\mathbf{K}';\mathbb{Z})$

(1) $(\alpha,\beta)_{k,2m-k} = (-1)^{k(2m-k)}(\beta,\alpha)_{2m-k,k}$,
(2) $(\alpha, \partial_{2m-k}\beta)_{(k+1),2m-(k+1)} = (-1)^{2m-k+1}(\partial_{k+1}\alpha, \beta)_{k,2m-k}$,

where ∂ denotes the boundary operator.

In the above, (1) follows from the similar property of the intersection index, and (2) follows from $\partial_{2m-k}(\Delta_\alpha^{2m-k}) = (-1)^{2m-k+1}(\partial_k(\alpha_\alpha^k))'$ (see, for example, [7, p. 55]) for details.

This, in particular, means that intersection number pairings $(\cdot,\cdot)_{k,2m-k}$ are ∂-compatible, anti-symmetric bilinear maps.

If we set $\mathcal{D}_k = \mathcal{C}_k(\mathbf{K};\mathbb{Z}) \oplus \mathcal{C}_k(\mathbf{K}';\mathbb{Z})$ and define $(\cdot,\cdot)_{k,2m-k}$ as 0 on $\mathcal{C}_k(\mathbf{K};\mathbb{Z}) \times \mathcal{C}_{2m-k}(\mathbf{K};\mathbb{Z})$ and $\mathcal{C}_k(\mathbf{K}';\mathbb{Z}) \times \mathcal{C}_{2m-k}(\mathbf{K}';\mathbb{Z})$, then the chain-complex $0 \to \mathcal{D}_{2m} \to \mathcal{D}_{2m-1} \to \cdots \to \mathcal{D}_m \to \cdots \to \mathcal{D}_1 \to \mathcal{D}_0 \to 0$ becomes a symplectic chain-complex.

One can easily extend the intersection number pairings to homologies

$$(\cdot,\cdot)_{k,2m-k} \colon \mathcal{H}_k(M) \times \mathcal{H}_{2m-k}(M) \to \mathbb{R}.$$

Poincaré duality gives the following commutative diagram

$$\begin{array}{ccccc} \mathcal{H}^k(M) & \times & \mathcal{H}^{2m-k}(M) & \xrightarrow{\wedge_{k,2m-k}} & \mathcal{H}^{2m}(M) \\ \uparrow \mathrm{PD} & & \uparrow \mathrm{PD} & \circlearrowleft & \uparrow \\ \mathcal{H}_k(M) & \times & \mathcal{H}_{2m-k}(M) & \xrightarrow{(\cdot,\cdot)_{k,2m-k}} & \mathbb{R}, \end{array}$$

where $\wedge_{k,2m-k}$ denotes the wedge product.

3.3. Proof of main theorem

In this subsection, we apply the above to compact Riemann surfaces Σ_g of genus $g \geq 1$ and prove our main result.

Theorem 3.2. *Let* $\mathfrak{h}^1 = \{\omega_i\}_{i=1}^{2g}$ *be a basis for* $\mathcal{H}^1(\Sigma_g)$. *Let* $\mathbf{K}$ *be a cell decomposition of the compact Riemann surface* Σ_g *with genus* $g \geq 1$, *and let for* $p = 0, 1, 2$, $\mathfrak{c}_p$ *be the geometric bases of* $\mathcal{C}_p(\mathbf{K};\mathbb{Z})$. *Then,* $\mathcal{T}\big(\mathcal{C}_*, \{\mathfrak{c}_p\}_{p=0}^{2}, \{[\mathfrak{c}_0], \mathfrak{h}_1, [\mathfrak{c}_2]\}\big) = \big|\det \wp(\mathfrak{h}^1, \Gamma)\big|^{-1}$, *where* $\wp(\mathfrak{h}^1,\Gamma) = \Big[\int_{\Gamma_i} \omega_j\Big]$ *is the period matrix of* $\mathfrak{h}^1$ *with respect to the canonical basis* $\Gamma = \{\Gamma_i\}_{i=1}^{2g}$ *of* $\mathcal{H}_1(\Sigma_g)$, *and* $\mathfrak{h}_1 = \{\Omega_j\}_{j=1}^{2g}$ *is the basis of* $\mathcal{H}_1(\Sigma_g)$ *corresponding to* $\mathfrak{h}^1$.

Proof. Considering the Poincaré dual isomorphism $\mathcal{H}_1(\Sigma_g) \overset{\mathrm{PD}}{\to} \mathcal{H}^1(\Sigma_g)$, let $\mathfrak{h}_1 = \{\Omega_i\}_{i=1}^{2g}$ be the basis for $\mathcal{H}_1(\Sigma_g)$ corresponding to the basis $\mathfrak{h}^1 = \{\omega_i\}_{i=1}^{2g}$ of $\mathcal{H}^1(\Sigma_g)$.

Let $\mathbf{K}'$ be the dual cell-decomposition of Σ_g corresponding to the cell-decomposition $\mathbf{K}$ of Σ_g, let $\mathfrak{c}'_p$ be the basis for $\mathcal{C}_p(\mathbf{K}';\mathbb{Z})$ corresponding to the basis $\mathfrak{c}_p$ of $\mathcal{C}_p(\mathbf{K};\mathbb{Z})$.

As explained in Section §3.2, the intersection number pairings $(\cdot,\cdot)_{p,2n-p}$ enable us to consider $\mathcal{D}_* = \mathcal{C}_*(\mathbf{K};\mathbb{Z}) \oplus \mathcal{C}_*(\mathbf{K}';\mathbb{Z})$ as a symplectic chain complex. Moreover, the geometric basis $\mathfrak{c}_p \oplus \mathfrak{c}'_p$ is ω-compatible basis for the symplectic chain-complex $\mathcal{D}_*$.

Thus, by Theorem 2.2, we obtain

$$\mathcal{T}\big(D_*, \{\mathfrak{c}_p \oplus \mathfrak{c}'_p\}_{p=0}^{2}, \{[\mathfrak{c}_0] \oplus [\mathfrak{c}_0], \mathfrak{h}_1 \oplus \mathfrak{h}_1, [\mathfrak{c}_2] \oplus [\mathfrak{c}_2]\}\big) = \big(\mathrm{Pfaf}([\omega_{1,1}])\big)^{-1} \tag{3.1}$$

where $[\omega_{1,1}]\colon \mathcal{H}_1(\mathcal{D}_*) \times \mathcal{H}_1(\mathcal{D}_*) \to \mathbb{R}$ is $\begin{bmatrix} 0 & (\cdot,\cdot)_{1,1} \\ -(\cdot,\cdot)_{1,1} & 0 \end{bmatrix}$,
and $(\cdot,\cdot)_{1,1}\colon \mathcal{H}_1(\Sigma_g) \times \mathcal{H}_1(\Sigma_g) \to \mathbb{R}$ is the extension of the intersection form $(\cdot,\cdot)_{1,1}\colon \mathcal{C}_1(\mathbf{K};\mathbb{Z}) \times \mathcal{C}_1(\mathbf{K}';\mathbb{Z}) \to \mathbb{R}$,
and where $\mathrm{Pfaf}\big([\omega_{1,1}]\big) = \sqrt{\det \begin{bmatrix} [\omega_{1,1}] \\ \text{in basis } \ \mathfrak{h}_1 \oplus \mathfrak{h}_1 \end{bmatrix}}$.

For $(\cdot,\cdot)_{1,1}\colon \mathcal{H}_1(\Sigma_g) \times \mathcal{H}_1(\Sigma_g) \to \mathbb{R}$ being non-degenerate anti-symmetric, then determinant of $(\cdot,\cdot)_{1,1}$ in basis $\mathfrak{h}_1$ is positive. More precisely, it equals to $\mathrm{Pfaf}((\cdot,\cdot)_{1,1})^2$ and hence positive. Therefore, $\mathrm{Pfaf}([\omega_{1,1}]) = \det \begin{bmatrix} (\cdot,\cdot)_{1,1} \\ \text{in basis } \ \mathfrak{h}_1 \end{bmatrix}$.

Let us now consider the chain complex

$$0 \to \mathcal{D}_2 = \mathcal{C}_2(\mathbf{K};\mathbb{Z}) \oplus \mathcal{C}_2(\mathbf{K}';\mathbb{Z}) \to \mathcal{D}_1 \to \mathcal{D}_0 \to 0.$$

Clearly, using the inclusion $\mathcal{C}_p(\mathbf{K};\mathbb{Z}) \hookrightarrow \mathcal{D}_p$ and the projection $\mathcal{D}_p \to \mathcal{C}_p(\mathbf{K}';\mathbb{Z})$, we have the following short-exact sequence of chain complexes

$$0 \to \mathcal{C}_*(\mathbf{K};\mathbb{Z}) \hookrightarrow \mathcal{D}_* \twoheadrightarrow \mathcal{C}_*(\mathbf{K}';\mathbb{Z}) \to 0.$$

If we consider the inclusion $s_p\colon \mathcal{C}_p(\mathbf{K}';\mathbb{Z}) \to \mathcal{D}_p$ as a section, then the bases $\mathfrak{c}_p$ of $\mathcal{C}_p(\mathbf{K};\mathbb{Z})$, $\mathfrak{c}_p \oplus s_p(\mathfrak{c}'_p)$ of $\mathcal{D}_p$, and $\mathfrak{c}'_p$ of $\mathcal{C}_p(\mathbf{K}';\mathbb{Z})$ are compatible bases. Namely, the determinant of the change-base-matrix from $\mathfrak{c}_p \oplus s_p(\mathfrak{c}'_p)$ to $\mathfrak{c}_p \oplus \mathfrak{c}'_p$ is 1.

Hence, by Theorem 2.1, we have

$$\begin{aligned}&\mathcal{T}\big(\mathcal{D}_*, \{\mathfrak{c}_p \oplus \mathfrak{c}'_p\}_{p=0}^2, \{\mathfrak{h}_p \oplus \mathfrak{h}_p\}_{p=0}^2\big)\\ &= \mathcal{T}\big(\mathcal{C}_*(\mathbf{K};\mathbb{Z}), \{\mathfrak{c}_p\}_{p=0}^2, \{\mathfrak{h}_p\}_{p=0}^2\big) \times \mathcal{T}\big(\mathcal{C}_*(\mathbf{K}';\mathbb{Z}), \{\mathfrak{c}'_p\}_{p=0}^2, \{\mathfrak{h}_p\}_{p=0}^2\big) \times \mathcal{T}(\mathcal{H}_*),\end{aligned}$$

where $\mathcal{H}_*$ denotes the long-exact sequence $0 \to \mathcal{H}_2(\mathcal{C}_*(\mathbf{K})) \to \mathcal{H}_2(D_*) \to \mathcal{H}_2(\mathcal{C}_*(\mathbf{K}')) \to \mathcal{H}_1(\mathcal{C}_*(\mathbf{K})) \to \mathcal{H}_1(\mathcal{D}_*) \to \mathcal{H}_1(\mathcal{C}_*(\mathbf{K}')) \to \mathcal{H}_0(\mathcal{C}_*(\mathbf{K})) \to \mathcal{H}_0(\mathcal{D}_*) \to \mathcal{H}_0(\mathcal{C}_*(\mathbf{K}')) \to 0$, where $\mathcal{H}_2(D_*) = \mathcal{H}_p(\mathcal{C}_*(\mathbf{K})) \oplus \mathcal{H}_p(\mathcal{C}_*(\mathbf{K}'))$.

Considering the inclusion $\mathcal{H}_p(\mathcal{C}_*(\mathbf{K}');\mathbb{Z}) \to \mathcal{H}_p(\mathcal{D}_*)$ and the projection $\mathcal{H}_p(\mathcal{D}_*) \to \mathcal{H}_p(\mathcal{C}_*(\mathbf{K});\mathbb{Z})$ as sections, we get $\mathcal{T}(\mathcal{H}_*) = 1$ and hence we have

$$\begin{aligned}&\mathcal{T}\big(\mathcal{D}_*, \{\mathfrak{c}_p \oplus \mathfrak{c}'_p\}_{p=0}^2, \{[c_0] \oplus [c_0], \mathfrak{h}_1 \oplus \mathfrak{h}_1, [c_2] \oplus [c_2]\}\big)\\ =\ &\big[\mathcal{T}\big(\mathcal{C}_*, \{\mathfrak{c}_p\}_{p=0}^2, \{[c_0], \mathfrak{h}_1, [c_2]\}\big)\big]^2 . \qquad (3.2)\end{aligned}$$

Combining (3.1) and (3.2), we have

$$\mathcal{T}\big(\mathcal{C}_*, \{\mathfrak{c}_p\}_{p=0}^2, \{[c_0], \mathfrak{h}_1, [c_2]\}\big) = \pm\sqrt{\det\begin{bmatrix} (\cdot,\cdot)_{1,1} \\ \text{in basis } \ \mathfrak{h}_1 \end{bmatrix}}^{(-1)} .$$

By Theorem 2.2 and the existence of ω-compatible bases obtained from the natural bases ([14]), we actually obtain

$$\mathcal{T}\big(\mathcal{C}_*, \{\mathfrak{c}_p\}_{p=0}^2, \{[c_0], \mathfrak{h}_1, [c_2]\}\big) = \sqrt{\det\begin{bmatrix} (\cdot,\cdot)_{1,1} \\ \text{in basis } \ \mathfrak{h}_1 \end{bmatrix}}^{(-1)} .$$

Let $H = [h_{ij}]$ be the $2g \times 2g$-square matrix with $h_{ij} = (\Omega_i, \Omega_j)_{1,1}$. Note also that from the commutative diagram

$$\begin{array}{ccccc} \mathcal{H}^1(\Sigma_g) & \times & \mathcal{H}^1(\Sigma_g) & \overset{\wedge_{1,1}}{\longrightarrow} & \mathcal{H}^2(\Sigma_g) \\ \uparrow \text{PD} & & \uparrow \text{PD} & \circlearrowleft & \uparrow \\ \mathcal{H}_1(\Sigma_g) & \times & \mathcal{H}_1(\Sigma_g) & \overset{(\cdot,\cdot)_{1,1}}{\longrightarrow} & \mathbb{R}, \end{array}$$

the non-degenerate skew-symmetric matrix H satisfies $h_{ij} = \int\limits_{\Sigma_g} \omega_i \wedge \omega_j$.

Let $\{\Gamma_1, \dots, \Gamma_g, \Gamma_{1+g}, \dots, \Gamma_{2g}\}$ be a canonical basis for $\mathcal{H}_1(\Sigma_g)$, namely, Γ_i intersects Γ_{i+g} once positively and does not intersect others. Then, we clearly obtain $\mathcal{T}\big(\mathcal{C}_*, \{\mathfrak{c}_p\}_{p=0}^2, \{[\mathfrak{c}_0], \mathfrak{h}_1, [\mathfrak{c}_2]\}\big) = \sqrt{\det(H)}^{(-1)} = |\det[(\Omega_j, \Gamma_i)_{1,1}]|^{-1}$.

From the Poincaré duality it follows that $(\Omega_j, \Gamma_j)_{1,1} = \int\limits_{\Sigma_g} \omega_j \wedge \gamma_i = \int\limits_{\Gamma_i} \omega_j$, where $\gamma_i \in \mathcal{H}^1(\Sigma_g)$ is corresponding to $\Gamma_i \in \mathcal{H}_1(\Sigma_g)$.

Therefore, we prove:

$$\mathcal{T}\big(\mathcal{C}_*, \{\mathfrak{c}_p\}_{p=0}^2, \{[\mathfrak{c}_0], \mathfrak{h}_1, [\mathfrak{c}_2]\}\big) = \big|\det \wp(\mathfrak{h}^1, \Gamma)\big|^{-1},$$

where $\wp(\mathfrak{h}^1, \Gamma) = \Big[\int\limits_{\Gamma_i} \omega_j\Big]$ is the period matrix of $\mathfrak{h}^1$ with respect to the canonical basis $\Gamma = \{\Gamma_i\}_{i=1}^{2g}$ of $\mathcal{H}_1(\Sigma_g)$. □

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