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KOROVKIN TYPE APPROXIMATION THEOREM FOR BBH TYPE OPERATORS VIA $\mathscr{I}$-CONVERGENCE

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ABSTRACT. This paper provides a Korovkin type approximation theorem for a class of positive linear operators including Bleimann-Butzer and Hahn operators via $\mathscr{I}$-convergence.

1. Introduction

The idea of statistical convergence was introduced by Fast (see [12]) and has been studied by several authors (see [15], [21]). The concept of statistical convergence is based on the natural density of subsets of $\mathbb{N}$, the set of natural numbers. If $F \subset \mathbb{N}$ then $\delta(F)$, the natural density of F is defined as $\delta(F) = \lim\limits_{j\to\infty} n^{-1} |\{n \le j:\ n \in F\}|$ provided that the limit exist. The symbol $|E|$ denotes the cardinality of the set E. The number sequence $x := (x_n)$ is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{n \in \mathbb{N}:\ |x_n - L| \ge \varepsilon\}) = 0$. Statistical convergence of x to L is denoted by st-$\lim\limits_j x_j = L$. It is well known that convergence implies statistical convergence but statistical convergence does not imply convergence.

Now we recall that, for an infinite nonnegative regular summability matrix $A = (a_{jn})$, the A-density of a subset F of $\mathbb{N}$ is given by,

$$\delta_A(F) = \lim_j \sum_{n\in F} a_{jn} = 0$$

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whenever limit exists. A sequence x is called A-statistically convergent to a number L if, for every $\varepsilon > 0$, $\delta_A\big(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}\big) = 0$. This limit is denoted by $\mathrm{st}_A\text{-}\lim x = L$ ([10], see also [19]). In the case $A = C_1$, the Cesáro matrix, C_1-statistical convergence coincides with statistical convergence (see [12], [13]). Moreover if $A = I$, the identity matrix, then it reduces to the ordinary convergence. Similar to statistical convergence, every convergent sequence is A-statistical convergent, but its converse is not always true.

The notions of $\mathscr{I}$-convergence and $\mathscr{I}^*$-convergence of sequences of real numbers has been initiated by the school of late professor Šalát [17] [18]. The concept of $\mathscr{I}$-convergence is based on the ideal of subsets of $\mathbb{N}$. A collection $\mathscr{I}$ of subsets of $\mathbb{N}$ satisfying the conditions

i) $\emptyset \in \mathscr{I}$,
ii) $A, B \in \mathscr{I} \implies A \cup B \in \mathscr{I}$,
iii) $A \in \mathscr{I}$ & $B \subset A \implies B \in \mathscr{I}$,

is called an ideal in $\mathbb{N}$. An ideal $\mathscr{I}$ with $\mathbb{N} \notin \mathscr{I}$ is called a nontrivial ideal. $\mathscr{I}$ is called admissible if $\{n\} \in \mathscr{I}$ for all $n \in \mathbb{N}$. Let $\mathscr{I}$ be a nontrivial ideal in $\mathbb{N}$, then a sequence $x := (x_n)$ is said to be $\mathscr{I}$-convergent to L if for every $\varepsilon > 0$, $\{n : |x_n - L| \geq \varepsilon\} \in \mathscr{I}$.

Note that, if $\mathscr{I}$ is the family of all finite subsets of $\mathbb{N}$, then $\mathscr{I}$-convergence reduces to ordinary convergence. Also choosing $\mathscr{I} = \{F \subset \mathbb{N} : \delta_A(F) = 0\}$ then $\mathscr{I}$-convergence coincides with A-statistical convergence where $A = (a_{jn})$ is a nonnegative regular summability matrix.

A sequence $x := (x_n)$ is said to be $\mathscr{I}^*$-convergent to L if and only if there exists a set $M = \{m_1 < m_2 < \dots < m_k < \dots\}$ with $\mathbb{N} \backslash M \in \mathscr{I}$ such that $\lim\limits_k x_{m_k} = L$. It is known that for an admissible ideal $\mathscr{I}$, $\mathscr{I}^*$-convergence implies $\mathscr{I}$-convergence. However inverse implication is not always true. An admissible ideal $\mathscr{I}$ is said to satisfy additive property (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \dots\} \subset \mathscr{I}$ there exists a countable family of sets $\{B_1, B_2, \dots\} \subset \mathscr{I}$ such that $A_j \Delta B_j = A_j \backslash B_j \cup B_j \backslash A_j$ is a finite set for all $j = 1, 2, \dots$ and $B = \bigcup\limits_{j=1}^{\infty} B_j \in \mathscr{I}$. For an admissible ideal $\mathscr{I}$ which has AP property, $\mathscr{I}$-convergence implies $\mathscr{I}^*$-convergence.

Recently, A-statistical extensions of Korovkin theorems and their applications has been studied in [5], [6], [7], [8], [9], [14], [20]. Also Duman gave a Korovkin type approximation theorem via $\mathscr{I}$-convergence by using the usual test functions $f_i(y) = y^i$, $i = 0, 1, 2$ (see [4]).

The goal of this paper is to prove a Korovkin type theorem via $\mathscr{I}$-convergence for a class of linear positive operators in several variables including BBH operators by using different test functions and, defining a generalization of BBH operators, to investigate the approximation properties of these operators in $\mathscr{I}$-convergence sense.

2. Construction of the operators

The aim of this section is to introduce a generalization of BBH operators in several variables. Let $C_B(K^r)$ be the space of all real valued continuous and bounded functions defined on the set $K^r = [0,\infty)^r$ endowed with the usual supremum norm

$$\|f\| := \sup_{x_1,\ldots,x_r \geq 0} |f(x_1,\ldots,x_r)|, \qquad f \in C_B(K^r).$$

Let $H_{w_r}(K^r)$ denote the space of all real-valued functions defined on K^r such that

$$\begin{aligned} &|f(u_1,\ldots,u_r) - f(x_1,\ldots,x_r)| \\ &\leq w_r\left(f; \left|\frac{u_1}{1+u_1} - \frac{x_1}{1+x_1}\right|,\ldots,\left|\frac{u_r}{1+u_r} - \frac{x_r}{1+x_r}\right|\right) \end{aligned}$$

where $w_r(f;\delta_1,\ldots,\delta_r)$ is the modulus of continuity given by

$$w_r(f;\delta_1,\ldots,\delta_r) = \sup|f(u_1,\ldots,u_r) - f(x_1,\ldots,x_r)|.$$

Here, the supremum is taken over $(u_1,\ldots,u_r),(x_1,\ldots,x_r) \in K^r$ satisfying $|u_i - x_i| \leq \delta_i$ $(i = 1,\ldots,r)$. Then, observe that any function $f \in H_{w_r}(K^r)$ implies $f \in C(K^r)$.

Now we consider the following BBH type operators in several variables.

$$\begin{aligned} &L_n(f;x_1,\ldots,x_r) \\ &= b_n\left\{\prod_{i=1}^{r}\frac{1}{(1+a_{i,n}x_i)^n}\right\}\sum_{k_1=0}^{n}\cdots\sum_{k_r=0}^{n} f\left(\frac{k_1}{n-k_1+1},\ldots,\frac{k_r}{n-k_r+1}\right) \\ &\qquad\times\prod_{m=1}^{r}\binom{n}{k_m}(a_{m,n}x_m)^{k_m} \end{aligned} \tag{2.1}$$

where

$$(x_1,\ldots,x_r) \in K^r, \quad f \in H_{w_r}(K^r), \quad b_n \geq 0,$$
$$a_{i,n} \geq 0 \qquad \text{for all} \quad i = 1,\ldots,r, \quad n \in \mathbb{N}$$

and

$$\mathscr{I}\text{-}\lim_{n\to\infty} b_n = 1, \qquad \mathscr{I}\text{-}\lim_{n\to\infty} a_{i,n} = 1 \quad \text{for } i = 1,\ldots,r \tag{2.2}$$

It should be remarked that letting $r = 1$, $b_n = 1$ and $a_{1,n} = 1$ for all $n \in \mathbb{N}$, in (2.1) we get the well known BBH operators (see [2]) defined on the space $C[0, \infty)$ by;

$$L_n(f; x) = \frac{1}{(1+x)^n} \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right) \binom{n}{k} x^k.$$

3. Approximation properties

Korovkin was the first to notice the idea of approximating to a function by means of linear positive operators ([16], see also [1]). The aim of this section is to prove a Korovkin type theorem via $\mathscr{I}$-convergence according to the test functions

$$\begin{aligned} f_0(u_1, \ldots, u_r) &= 1 \\ f_v(u_1, \ldots, u_r) &= \frac{u_\nu}{1+u_\nu} \qquad (v = 1, 2, \ldots, r) \\ f_{r+1}(u_1, \ldots, u_r) &= \sum_{v=1}^{r} f_v^2(u_1, \ldots, u_r) = \sum_{v=1}^{r} \left(\frac{u_v}{1+u_v}\right)^2. \end{aligned}$$

Our main result is stated in the following theorem.

Theorem 3.1. *Let $\mathscr{I}$ be an admisible ideal in $\mathbb{N}$ and let $\{L_n\}$ be a sequence of positive linear operators from $H_{w_r}(K^r)$ into $C_B(K^r)$. If, for each $v = 0, 1, \ldots, r+1$,*

$$\mathscr{I}\text{-}\lim_n \|L_n(f_v) - f_v\| = 0 \tag{3.1}$$

holds, then for all $f \in H_{w_r}(K^r)$ we have

$$\mathscr{I}\text{-}\lim_n \|L_n(f) - f\| = 0. \tag{3.2}$$

Proof. Assume that (3.1) holds and $f \in H_{w_r}(K^r)$. By the definition of $H_{w_r}(K^r)$, for every $\varepsilon > 0$, there exists $\delta_v > 0$, $v = 1, \ldots, r$, such that

$$|f(u_1, \ldots, u_r) - f(x_1, \ldots, x_r)| < \varepsilon$$

for all $(u_1, \ldots, u_r), (x_1, \ldots, x_r) \in K^r$ satisfying $\left|\frac{u_v}{1+u_v} - \frac{x_v}{1+x_v}\right| \le \delta_v$ $(v = 1, \ldots, r)$. Now setting

$$K^r_{\delta_1, \ldots, \delta_r} = \left\{(u_1, \ldots, u_r) \in K^r : \left|\frac{u_v}{1+u_v} - \frac{x_v}{1+x_v}\right| \le \delta_v, \ v = 1, \ldots, r\right\},$$

one can write that

$$\begin{aligned}&\left|f(u_1,\dots,u_r)-f(x_1,\dots,x_r)\right|\\&\quad\le |f(u_1,\dots,u_r)-f(x_1,\dots,x_r)|\,\chi_{K^r_{\delta_1,\dots,\delta_r}}(u_1,\dots,u_r)\\&\qquad+|f(u_1,\dots,u_r)-f(x_1,\dots,x_r)|\,\chi_{K^r\setminus K^r_{\delta_1,\dots,\delta_r}}(u_1,\dots,u_r),\end{aligned}$$

which yields

$$|f(u_1,\dots,u_r)-f(x_1,\dots,x_r)|\le\varepsilon+2M\chi_{K^r\setminus K^r_{\delta_1,\dots,\delta_r}}(u_1,\dots,u_r), \tag{3.3}$$

where χ_S denotes the characteristic function of the set S and $M:=\|f\|$. Let $\delta:=\min\{\delta_i:\ i=1,\dots,r\}$. Since

$$\chi_{K^r\setminus K^r_{\delta_1,\dots,\delta_r}}(u_1,\dots,u_r)\le\frac{1}{\delta^2}\sum_{v=1}^{r}\left(\frac{u_v}{1+u_v}-\frac{x_v}{1+x_v}\right)^2,$$

it follows from (3.3) that

$$|f(u_1,\dots,u_r)-f(x_1,\dots,x_r)|\le\varepsilon+\frac{2M}{\delta^2}\sum_{v=1}^{r}\left(\frac{u_v}{1+u_v}-\frac{x_v}{1+x_v}\right)^2. \tag{3.4}$$

By linearity and positivity of the operators we get

$$\begin{aligned}&|L_n(f;x_1,\dots,x_r)-f(x_1,\dots,x_r)|\\&\le L_n\big(|f(u_1,\dots,u_r)-f(x_1,\dots,x_r)|;x_1,\dots,x_r\big)\\&\quad+M\,|L_n(f_0;x_1,\dots,x_r)-f_0(x_1,\dots,x_r)|\\&\le\varepsilon+(\varepsilon+M)\,|L_n(f_0;x_1,\dots,x_r)-f_0(x_1,\dots,x_r)|\\&\quad+\frac{2M}{\delta^2}\sum_{v=1}^{r}L_n\left(\left(\frac{u_v}{1+u_v}-\frac{x_v}{1+x_v}\right)^2;\,x_1,\dots,x_r\right)\\&\le\varepsilon+(\varepsilon+M)\,|L_n(f_0;x_1,\dots,x_r)-f_0(x_1,\dots,x_r)|\\&\quad+\frac{2M}{\delta^2}L_n\left(\sum_{v=1}^{r}\left(\frac{u_v}{1+u_v}\right)^2;\,x_1,\dots,x_r\right)\\&\quad-\frac{4M}{\delta^2}\sum_{v=1}^{r}\left(\frac{x_v}{1+x_v}\right)L_n\left(\frac{u_v}{1+u_v};\,x_1,\dots,x_r\right)\\&\quad+\frac{2M}{\delta^2}\sum_{i=1}^{r}\left(\frac{x_v}{1+x_v}\right)^2L_n(f_0;x_1,\dots,x_r).\end{aligned}$$

and hence

$$\begin{aligned}
&|L_n(f;x_1,\dots,x_r)-f(x_1,\dots,x_r)|\\
&\le \varepsilon+\left\{\varepsilon+M+\frac{2M}{\delta^2}\sum_{v=1}^{r}\left(\frac{x_v}{1+x_v}\right)^2\right\}|L_n(f_0;x_1,\dots,x_r)-f_0(x_1,\dots,x_r)|\\
&\quad+\frac{4M}{\delta^2}\sum_{v=1}^{r}\left(\frac{x_v}{1+x_v}\right)|L_n\left(f_v;x_1,\dots,x_r\right)-f_v(x_1,\dots,x_r)|\\
&\quad+\frac{2M}{\delta^2}\left|L_n\left(f_{r+1};x_1,\dots,x_r\right)-f_{r+1}(x_1,\dots,x_r)\right|.
\end{aligned}$$

Since $(x_1,\dots,x_r)\in K^r$, we obtain that

$$\begin{aligned}
&|L_n(f;x_1,\dots,x_r)-f(x_1,\dots,x_r)|\\
&\le \varepsilon+\left\{\varepsilon+M+\frac{2rM}{\delta^2}\right\}|L_n(f_0;x_1,\dots,x_r)-f_0(x_1,\dots,x_r)|\\
&\quad+\frac{4M}{\delta^2}\sum_{v=1}^{r}|L_n\left(f_v;x_1,\dots,x_r\right)-f_v(x_1,\dots,x_r)|\\
&\quad+\frac{2M}{\delta^2}\left|L_n\left(f_{r+1};x_1,\dots,x_r\right)-f_{r+1}(x_1,\dots,x_r)\right|.
\end{aligned}$$

Now setting

$$C:=\varepsilon+M+\frac{2M}{\delta^2}\{r+2\}$$

we conclude that

$$\begin{aligned}
&\left|L_n(f;x_1,\dots,x_r)-f(x_1,\dots,x_r)\right|\\
&\qquad\le \varepsilon+C\sum_{v=0}^{r+1}\left|L_n\left(f_v;x_1,\dots,x_r\right)-f_v(x_1,\dots,x_r)\right|.
\end{aligned}$$

By taking supremum over $(x_1,\dots,x_r)\in K^r$ we have the following

$$\|L_n(f)-f\|\le\varepsilon+C\sum_{v=0}^{r+1}\|L_n(f_v)-f_v\|. \tag{3.5}$$

Now for a given $s>0$ choose $\varepsilon>0$ such that $\varepsilon<s$. Then define the following sets:

$$\begin{aligned}
D&:=\left\{n:\ \|L_n(f)-f\|\ge s\right\}, \qquad\qquad (3.6)\\
D_v&:=\left\{n:\ \|L_n(f_v)-f_v\|\ge \tfrac{s-\varepsilon}{C(r+2)}\right\}\qquad (v=0,1,\dots,r+1).
\end{aligned}$$

It follows from (3.6) that

$$D\subset\bigcup_{v=0}^{r+1}D_v.$$

By (3.1), $D_v \in \mathscr{I}$ for $v = 0, 1, 2, \ldots, r+1$. Therefore by the definition of $\mathscr{I}$, we have $D \in \mathscr{I}$.

Whence the result. $\square$

We now turn to our operators defined by (2.1). To get a Korovkin type result for these operators we first need the following auxiliary results.

Lemma 3.2. *For all* $n \in \mathbb{N}$, $(x_1, \ldots, x_r) \in K^r$ *and* $\nu = 1, 2, \ldots, r$, *we have,*

(i) $L_n(f_0; x_1, \ldots, x_r) = b_n$,

(ii) $L_n(f_v; x_1, \ldots, x_r) = b_n \frac{n}{n+1} \frac{a_{\nu,n} x_\nu}{1+a_{\nu,n} x_\nu}$,

(iii) $L_n(f_{r+1}; x_1, \ldots, x_r) = \sum\limits_{v=1}^{r} b_n \left\{ \frac{n(n-1)}{(n+1)^2} \left(\frac{a_{\nu,n} x_\nu}{1+a_{v,n} x_\nu} \right)^2 + \frac{n}{(n+1)^2} \frac{a_{v,n} x_\nu}{1+a_{\nu,n} x_\nu} \right\}$.

P r o o f.

(i) It immediately follows from the definition of the operators.

(ii) For $v = 1, \ldots, r$, observe that

$$
\begin{aligned}
L_n(f_\nu; x_1, \ldots, x_r) &= \frac{b_n}{(1+a_{\nu,n} x_\nu)^n} \sum_{k_\nu=1}^{n} \frac{k_\nu}{n+1} \binom{n}{k_\nu} (a_{\nu,n} x_\nu)^{k_\nu} \\
&= b_n \frac{a_{\nu,n}}{(1+a_{\nu,n} x_\nu)^n} \sum_{k_\nu=1}^{n} \frac{n}{n+1} \binom{n-1}{2} (a_{\nu,n} x_\nu)^{k_\nu - 1} \\
&= b_n \frac{n}{n+1} \frac{a_{\nu,n} x_\nu}{(1+a_{\nu,n} x_\nu)^n} \sum_{k_\nu=0}^{n-1} \binom{n-1}{2} (a_{\nu,n} x_\nu)^{k_\nu} \\
&= b_n \frac{n}{n+1} \frac{a_{\nu,n} x_\nu}{(1+a_{\nu,n} x_\nu)}
\end{aligned}
$$

which gives (ii).

(iii) Now we first compute the value $L_n(f_v^2; x_1, \ldots, x_r)$ for each $v = 1, \ldots, r$.

$$
\begin{aligned}
L_n(f_v^2; x_1, \ldots, x_r) &= \frac{b_n}{(1+a_{\nu,n} x_\nu)^n} \sum_{k_\nu=1}^{n} \left(\frac{k_\nu}{n+1} \right)^2 \binom{n}{2} (a_{\nu,n} x_\nu)^{k_\nu} \\
&= \frac{b_n}{(1+a_{\nu,n} x_\nu)^n} \sum_{k_\nu=1}^{n} \frac{k_\nu}{(n+1)^2} \frac{n!}{(k_\nu - 1)!\,(n-k_\nu)!} (a_{\nu,n} x_\nu)^{k_\nu} \\
&= \frac{b_n}{(1+a_{v,n} x_\nu)^n} \left\{ \sum_{k_\nu=2}^{n} \frac{n\,(n-1)}{(n+1)^2} \frac{(n-2)!}{(k_\nu - 2)!\,(n-k_\nu)!} (a_{\nu,n} x_\nu)^{k_v} \right. \\
&\qquad \left. + \sum_{k_\nu=1}^{n} \frac{n}{(n+1)^2} \frac{(n-1)!}{(k_\nu - 1)!\,(n-k_\nu)!} (a_{\nu,n} x_\nu)^{k_v} \right\}
\end{aligned}
$$

$$= b_n \frac{n\,(n-1)}{(n+1)^2} \frac{a_{\nu,n}^2 x_\nu^2}{(1+a_{v,n}x_\nu)^n} \sum_{k_\nu=0}^{n-2} \binom{n-2}{2} (a_{\nu,n}x_\nu)^{k_v}$$

$$+ b_n \frac{n}{(n+1)^2} \frac{a_{\nu,n}x_\nu}{(1+a_{\nu,n}x_\nu)^n} \sum_{k_\nu=0}^{n-1} \binom{n-1}{2} (a_{\nu,n}x_\nu)^{k_v}$$

$$= b_n \frac{n\,(n-1)}{(n+1)^2} \left(\frac{a_{\nu,n}x_\nu}{1+a_{v,n}x_\nu}\right)^2 + b_n \frac{n}{(n+1)^2} \left(\frac{a_{\nu,n}x_\nu}{1+a_{v,n}x_\nu}\right).$$

Since $f_{r+1} = \sum\limits_{v=1}^{r} f_v^2$ and $L_n(f;x_1,\dots,x_r)$ is a linear operator, we obtain that

$$L_n(f_{r+1};x_1,\dots,x_r)$$
$$= \sum_{v=1}^{r} b_n \left\{ \frac{n\,(n-1)}{(n+1)^2} \left(\frac{a_{v,n}x_\nu}{1+a_{v,n}x_\nu}\right)^2 + \frac{n}{(n+1)^2} \frac{a_{v,n}x_\nu}{1+a_{v,n}x_\nu} \right\}.$$

Whence the result. □

Lemma 3.3. *For all $n \in \mathbb{N}$, we have,*

(i) $\|L_n(f_0;\cdot) - f_0\| \le |b_n - 1|$,

(ii) $\|L_n(f_v;\cdot) - f_v\| \le b_n\,|(1-a_{\nu,n})| + |b_n a_{\nu,n} - 1| \quad (v = 1,\dots,r)$,

(iii) $\|L_n(f_{r+1};\cdot) - f_{r+1}\|$

$$= \sum_{v=1}^{r} \left\{ \frac{b_n}{2}\,|1-a_{\nu,n}| + \left|1-a_{\nu,n}^2\right| \left|b_n a_{\nu,n}^2 - 1\right| + \frac{nb_n}{(n+1)^2} \right\}.$$

P r o o f.

(i) It is obvious.

(ii) By Lemma 3.2.(ii), one can get, for each $v = 1,\dots,r$ and $(x_1,\dots,x_r) \in K^r$, that

$$|L_n(f_v;x_1,\dots,x_r) - f_v(x_1,\dots,x_r)|$$
$$\le \left| b_n \frac{n}{n+1} \frac{a_{\nu,n}x_\nu}{1+a_{\nu,n}x_\nu} - \frac{x_\nu}{1+x_\nu} \right|$$
$$\le \left| b_n \frac{a_{\nu,n}x_\nu}{1+a_{\nu,n}x_\nu} - b_n \frac{a_{\nu,n}x_\nu}{1+x_\nu} + b_n \frac{a_{\nu,n}x_\nu}{1+x_\nu} - \frac{x_\nu}{1+x_\nu} \right|$$
$$\le \left| b_n a_{\nu,n}x_\nu \left(\frac{1}{1+a_{\nu,n}x_\nu} - \frac{1}{1+x_\nu} \right) \right| + \left| \frac{x_\nu}{1+x_\nu} (b_n a_{\nu,n} - 1) \right|$$
$$\le \left| b_n\,(1-a_{\nu,n}) \left(\frac{x_\nu}{1+x_\nu} \right) \left(\frac{a_{\nu,n}x_\nu}{1+a_{\nu,n}x_\nu} \right) \right| + \left| \frac{x_\nu}{1+x_\nu} (b_n a_{\nu,n} - 1) \right|$$

Now taking supremum over $(x_1,\dots,x_r)\in K^r$ on the both sides of the above inequality, the proof of (ii) is completed.

(iii) Let $(x_1,\dots,x_r)\in K^r$. Then it follows from Lemma 3.2.(iii) that

$$\begin{aligned}
&|L_n(f_{r+1};x_1,\dots,x_r)-f_{r+1}(x_1,\dots,x_r)|\\
&\leq \sum_{\upsilon=1}^{r}\left| b_n\frac{n\,(n-1)}{(n+1)^2}\left(\frac{a_{\nu,n}x_\nu}{1+a_{\upsilon,n}x_\nu}\right)^2\right.\\
&\qquad\left.+\,b_n\frac{n}{(n+1)^2}\left(\frac{a_{\upsilon,n}x_\nu}{1+a_{\nu,n}x_\nu}\right)-\left(\frac{x_\nu}{1+x_\nu}\right)^2\right|\\
&\leq \sum_{\upsilon=1}^{r}\left\{\left| b_n\left(\frac{a_{\nu,n}x_\nu}{1+a_{\upsilon,n}x_\nu}\right)^2-\left(\frac{x_\nu}{1+x_\nu}\right)^2\right|\right.\\
&\qquad\left.+\left| b_n\frac{n}{(n+1)^2}\left(\frac{a_{\upsilon,n}x_\nu}{1+a_{\nu,n}x_\nu}\right)\right|\right\}\\
&\leq \sum_{\upsilon=1}^{r}\left\{\left| b_n\left(\frac{a_{\nu,n}x_\nu}{1+a_{\upsilon,n}x_\nu}\right)^2-b_n\left(\frac{a_{\nu,n}x_\nu}{1+x_\nu}\right)^2\right.\right.\\
&\qquad\left.\left.+b_n\left(\frac{a_{\nu,n}x_\nu}{1+x_\nu}\right)^2-\left(\frac{x_\nu}{1+x_\nu}\right)^2\right|+\left| b_n\frac{n}{(n+1)^2}\left(\frac{a_{\upsilon,n}x_\nu}{1+a_{\nu,n}x_\nu}\right)\right|\right\}\\
&\leq \sum_{\upsilon=1}^{r}\left\{\left| b_n\left(a_{\nu,n}x_\nu\right)^2\left(\frac{1}{(1+a_{\upsilon,n}x_\nu)^2}-\frac{1}{(1+x_\nu)^2}\right)\right|\right.\\
&\qquad\left.+\left|\left(\frac{x_\nu}{1+x_\nu}\right)^2\left(b_na_{\nu,n}^2-1\right)\right|+\left| b_n\frac{n}{(n+1)^2}\left(\frac{a_{\upsilon,n}x_\nu}{1+a_{\nu,n}x_\nu}\right)\right|\right\}\\
&\leq \sum_{\upsilon=1}^{r}\left\{\left| b_n\left(1-a_{\nu,n}\right)\frac{2x_\nu}{(1+x_\nu)^2}\right|+\left| b_n\left(1-a_{\nu,n}^2\right)\left(\frac{x_\nu}{1+x_\nu}\right)^2\right|\right.\\
&\qquad\left.+\left|\left(\frac{x_\nu}{1+x_\nu}\right)^2\left(b_na_{\nu,n}^2-1\right)\right|+\frac{n}{(n+1)^2}b_n\left|\frac{a_{\upsilon,n}x_\nu}{1+a_{\nu,n}x_\nu}\right|\right\}
\end{aligned}$$

so the proof is completed if we take supremum over $(x_1,\dots,x_r)\in K^r$ on the both sides of the above inequality. □

Theorem 3.4. *Let $\mathscr{I}$ be an admissible ideal of $\mathbb{N}$. Assume that, b_n and $a_{i,n}$ satisfies* (2.2)*, for each $i=1,2,\dots,r$. Then, for a $f\in H_{w_r}(K^r)$, we have*

$$\mathscr{I}\text{-}\lim_n \|L_n(f;\cdot)-f\|=0.$$

Proof. By Lemma 3.3.i) and the assumption that, $\mathscr{I}\text{-}\lim_n b_n = 1$, we have $\mathscr{I}\text{-}\lim_n \|L_n(f_0,\cdot) - f_0\| = 0$.

Now given $\varepsilon > 0$, define the following sets;

$$U_\nu = \Big\{n \in \mathbb{N} : \ \|L_n(f_\nu;\cdot) - f_\nu\| \geq \varepsilon\Big\},$$
$$U_\nu^1 = \Big\{n \in \mathbb{N} : \ \beta_{n,\nu}^{(1)} = b_n\,|1 - a_{\nu,n}| \geq \frac{\varepsilon}{2}\Big\},$$
$$U_\nu^2 = \Big\{n \in \mathbb{N} : \ \beta_{n,\nu}^{(2)} = |\,b_n a_{\nu,n} - 1| \geq \frac{\varepsilon}{2}\Big\}.$$

By Lemma 3.3.ii) it is obvious that, $U_\nu \subset U_\nu^1 \cup U_\nu^2$, $\nu = 1,2,\dots,r$. Since

$$\mathscr{I}\text{-}\lim_n \beta_{n,\nu}^{(i)} = 0, \qquad i = 1,2, \ \ \nu = 1,2,\dots,r,$$

we have, $U_\nu^1, U_\nu^2 \in \mathscr{I}$, $\nu = 1,2,\dots,r$. By using the definition of ideal and $\mathscr{I}$-convergence we get

$$\mathscr{I}\text{-}\lim_n \|L_n(f_\nu,\cdot) - f_\nu\| = 0, \qquad \nu = 1,2,\dots,r.$$

Finally, given $\varepsilon > 0$, define the following sets;

$$V_{r+1} = \Big\{n \in \mathbb{N} : \ \|L_n(f_{r+1};\cdot) - f_{r+1}\| \geq \varepsilon\Big\},$$
$$V_{r+1,\nu}^1 = \Big\{n \in \mathbb{N} : \ \alpha_{\nu,n}^{(1)} = \big|1 - a_{\nu,n}^2\big| \geq \frac{\varepsilon}{4r}\Big\},$$
$$V_{r+1,\nu}^2 = \Big\{n \in \mathbb{N} : \ \alpha_{\nu,n}^{(2)} = \big|b_n a_{\nu,n}^2 - 1\big| \geq \frac{\varepsilon}{4r}\Big\},$$
$$V_{r+1,\nu}^3 = \Big\{n \in \mathbb{N} : \ \alpha_{\nu,n}^{(3)} = \frac{b_n}{2}\,|1 - a_{\nu,n}| \geq \frac{\varepsilon}{4r}\Big\},$$
$$V_{r+1,\nu}^4 = \Big\{n \in \mathbb{N} : \ \alpha_{\nu,n}^{(4)} = \frac{b_n n}{(n+1)^2} \geq \frac{\varepsilon}{4r}\Big\},$$

$\nu = 1,2,\dots,r$. Lemma 3.3.iii) yields,

$$V_{r+1} \subset \bigcup_{\nu=1}^{r} \left(V_{r+1,\nu}^1 \cup V_{r+1,\nu}^2 \cup V_{r+1,\nu}^3 \cup V_{r+1,\nu}^4\right).$$

Using,

$$\mathscr{I}\text{-}\lim_{n\to\infty} \alpha_{\nu,n}^{(i)} = 0, \qquad i = 1,2,3,4, \ \ \nu = 1,2,\dots,r;$$

we have

$$\mathscr{I}\text{-}\lim_n \|L_n(f_{r+1},\cdot) - f_{r+1}\| = 0,$$

which completes the proof. □

4. Concluding remarks

It is known that, if $\mathscr{I}_1$ is the class of all finite subsets of $\mathbb{N}$ then $\mathscr{I}$-convergence coincides with the classical convergence (see [4], [17]). Therefore Theorem 3.1 reduces to the result obtained by Çakar and Gadjiev (see [3]). On the other hand choosing $\mathscr{I}_2 = \{E \subset \mathbb{N} : \ \delta_A(E) = 0\}$ where A is a non-negative, regular summability matrix, Theorem 3.1 coincides with [8, Theorem 2.2, p. 503]. Theorem 3.1 can be restated for statistical convergence by choosing $A = C_1$, the Cesaro matrix of order one. Similar remarks are also valid for Theorem 3.4.

It should be noted that $\mathscr{I}_1$ and $\mathscr{I}_2$ both have AP property and for these type of ideals $\mathscr{I}$-convergence and $\mathscr{I}^*$-convergence coincides. Therefore for such ideals, proofs of Theorem 3.1 and Theorem 3.4 can be given on $\mathscr{I}^*$-convergence, which is more simpler. But in this paper we prefer to give these proofs in more general case.

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