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ON DIRECT LIMITS OF MV-ALGEBRAS

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ABSTRACT. We prove that the correspondence between MV-algebras and abelian ℓ-groups with the strong unit is preserved in the direct limit construction. Further, several classes of MV-algebras which are closed under formation of direct limits will be distinguished.

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The direct limit construction is well known method to build up algebras from families of algebras. It is often used in the algebraic research, recently see e.g. [6] for lattices.

We are interested in direct limits of MV-algebras.

We deal with the correspondence between MV-algebras and abelian ℓ-groups with strong unit in the direct limit construction. We will prove that this correspondence is preserved.

Every variety is closed under formation of direct limits. The class of all retracts of a finite algebra is closed under formation of direct limits, cf. [4]. The second aim of this paper is to identify some direct limit closed classes of MV-algebras.

We will show that MV-chains, hyperarchimedean MV-algebras, simple MV-algebras, directly indecomposable MV-algebras and a subclass of atomless MV-algebras are closed under formation of direct limits. Further we will see that finite MV-algebras, infinite MV-algebras, directly decomposable MV-algebras, complete MV-algebras, atomic MV-algebras are not closed under formation of direct limits.

For direct limit closed classes of multialgebras, cyclically ordered groups, monounary algebras resp. see [7], [5], [3] resp.

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Preliminaries

Denote by $\mathbb{N}$ the set of all positive integers.

For the notion of a direct limit, cf. e.g. Grätzer [2, §21].

Let $\langle P, \leq \rangle$ be a directed partially ordered set. For each $p \in P$, let $\mathscr{A}_p = (A_p, F)$ be an algebra of some fixed type. Assume that if $p, q \in P$, $p \neq q$, then $A_p \cap A_q = \emptyset$. Suppose that for each pair of elements p and q in P with $p < q$, we have a homomorphism φ_{pq} of $\mathscr{A}_p$ into $\mathscr{A}_q$ such that $p < q < s$ implies that $\varphi_{ps} = \varphi_{pq} \circ \varphi_{qs}$. For each $p \in P$, suppose that φ_{pp} is the identity on A_p. The family $\{P, \mathscr{A}_p, \varphi_{pq}\}$ is said to be direct.

Assume that $p, q \in P$ and $x \in A_p$, $y \in A_q$. Put $x \equiv y$ if there exists $s \in P$ with $p \leq s$, $q \leq s$ such that $\varphi_{ps}(x) = \varphi_{qs}(y)$. For each $z \in \bigcup\limits_{p \in P} A_p$ put $\overline{z} = \Big\{t \in \bigcup\limits_{p \in P} A_p : \ z \equiv t\Big\}$. Denote $\overline{A} = \Big\{\overline{z} : \ z \in \bigcup\limits_{p \in P} A_p\Big\}$.

Let $f \in F$ be an n-ary operation. Let $x_j \in A_{p_j}$, $1 \leq j \leq n$, and let s be an upper bound of p_j. Define $f(\overline{x_1}, \ldots, \overline{x_n}) = \overline{f(\varphi_{p_1 s}(x_1), \ldots, \varphi_{p_n s}(x_n))}$. Then the algebra $\overline{\mathscr{A}} = (\overline{A}, F)$ is said to be the direct limit of the direct family $\{P, \mathscr{A}_p, \varphi_{pq}\}$. We express this situation as follows

$$\{P, \mathscr{A}_p, \varphi_{pq}\} \longrightarrow \overline{\mathscr{A}}. \tag{1}$$

Let us define the mapping $\varphi_{p\infty} \colon x \to \overline{x}$ for $x \in A_p$.

Then $\varphi_{p\infty}$ is a homomorphism of $\mathscr{A}_p$ into $\overline{\mathscr{A}}$.

Let $\mathscr{A}$ be an algebra. Then every retract of $\mathscr{A}$ can be constructed by a direct limit construction from $\mathscr{A}$, cf. e.g. [3]. We will often use this property.

We will deal with direct limits of MV-algebras and with direct limits of (abelian) ℓ-groups.

Example 1. Denote by R the set of all real numbers and Z the set of all integers.

Let M be the ℓ-group of all functions $f \colon R \to Z$ with the coordinate order and the coordinate addition. Let L be the ℓ-subgroup of M such that L contains all constant functions from M. Then L is a retract of M. That means L can be obtained by direct limit construction from M.

We will apply the definition of an MV-algebra by [1].

An MV-algebra is an algebra $\mathscr{A} = (A, \oplus, \neg, 0,)$ with a binary operation $\oplus$, an unary operation $\neg$ and a constant 0 satisfying the following identities:

(1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(2) $x \oplus 0 = x$,

(3) $x \oplus y = y \oplus x$,

(4) $\neg\neg x = x$,

(5) $x \oplus \neg 0 = \neg 0$,

(6) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$.

Let $\mathscr{A} = (A, \oplus, \neg, 0)$ be an MV-algebra.

We define the constant $1 = \neg 0$.

We have $\neg x \oplus x = 1$ for every $x \in A$ by setting $y = \neg 0$ into (6).

If $x, y \in A$ and $\neg x \oplus y = 1$, then we put $x \leq y$. If $x \leq y$ and $x \neq y$, then $x < y$. If $\leq$ is total, then we will say that A is an MV-chain.

Example 2. Let $\langle \mathbb{N}, \leq \rangle$ be the linearly ordered set of all positive integers. For each $i \in \mathbb{N}$ let $A_i = \{a/2^i : a \in \{0, \dots, 2^i\}\}$ and $\mathscr{A}_i = (A_i, \oplus, \neg, 0)$ be MV-chain, cf. [1, p. 72]. Let $\varphi_{ij}(a/2^i) = 2^{j-i}a/2^j$ for every $i \leq j$ and $a \in \{0, \dots, 2^i\}$. Then $(\mathbb{N}, \mathscr{A}_i, \varphi_{ij})$ is a direct family of MV-algebras with the direct limit isomorphic to the MV-chain on $\{a/2^k : k \in \mathbb{N}, a \in \{0, \dots, 2^k\}\}$.

The functor Γ in the direct limit

There is a natural correspondence between abelian lattice ordered groups with strong unit and MV-algebras, see [1, Corollary 7.1.8].

Let $G = (G, +, \vee, \wedge, -, 0, u)$ be an abelian lattice ordered group with the strong unit u. Let A be the interval $[0, u]$ in G. For each $a, b \in A$ we put $a \oplus b = (a + b) \wedge u$, $\neg a = u - a$. Then $(A, \oplus, \neg, 0)$ is the MV-algebra. We denote this algebra by $\mathbf{\Gamma}(G, u)$, cf. [1, Proposition 2.1.2].

Let $\mathscr{A}$ be an MV-algebra. Then there exists an abelian ℓ-group G with the strong unit 1 such that $\mathscr{A} = \mathbf{\Gamma}(G, 1)$. This group is uniquely determined up to isomorphism.

The functor Γ together with its opposite functor establish a categorial equivalence. The results of this section can be obtained from categorial equivalence too, cf. e.g. [8].

Lemma 1. *Let $\{P, G_p, \varphi_{pq}\}$ be a direct family of abelian ℓ-groups with strong unit, $G_p = (G_p, +, \vee, \wedge, -, 0_p, u_p)$ for every $p \in P$.*

Let $\{P, G_p, \varphi_{pq}\} \to \overline{G}$ and $\overline{G} = (\overline{G}, +, \vee, \wedge, -, \overline{0}, \overline{u})$. Then $\overline{G}$ is the abelian ℓ-group with the strong unit $\overline{u}$.

P r o o f. We have $\{u_p : p \in P\} \subseteq \overline{u}$ and $\{0_p : p \in P\} \subseteq \overline{0}$. The statement follows from the definition. $\square$

Theorem 1. *Let $\langle P, \leq \rangle$ be a directed partially ordered set.*

Suppose that $G_p = (G_p, +, \vee, \wedge, -, 0_p, u_p)$ is an abelian ℓ-groups with strong unit for every $p \in P$. Let $\{P, G_p, \varphi_{pq}\}$ be a direct family with the direct limit $\overline{G}$.

Then $\{P, \mathbf{\Gamma}(G_p, u_p), \varphi_{pq} \mid [0_p, u_p]\}$ is the direct family of MV-algebras and if $\tilde{\mathscr{A}}$ is the direct limit of this family, then $\tilde{\mathscr{A}} \cong \mathbf{\Gamma}(\overline{G}, \overline{u})$.

Proof. Let $p, q \in P$. If φ is a homomorphism from G_p into G_q, then $\varphi \mid [0_p, u_p]$ is the homomorphism from $\mathbf{\Gamma}(G_p, u_p)$ into $\mathbf{\Gamma}(G_q, u_q)$.

Put $\psi(\tilde{x}) = \overline{x}$ for every $x \in \bigcup_{p \in P} [0_p, u_p]$. We will show that ψ is an isomorphism between $\tilde{\mathscr{A}}$ and $\mathbf{\Gamma}(\overline{G}, \overline{u})$.

Suppose that $p, q \in P$, $x \in [0_p, u_p]$, $y \in [0_q, u_q]$ and $\psi(\tilde{x}) = \psi(\tilde{y})$. Then $\overline{x} = \overline{y}$ and there exists $r \geq p, q$ such that $\varphi_{pr}(x) = \varphi_{qr}(y)$. We have $\varphi_{pr}(x) \in [0_r, u_r]$. This imply $\varphi_{pr}(x) \in \tilde{x}$, $\varphi_{qr}(y) \in \tilde{y}$. We obtain $\tilde{x} = \widetilde{\varphi_{pr}(x)} = \widetilde{\varphi_{qr}(y)} = \tilde{y}$.

Now let $p \in P$, $x \in G_p$ be such that $\overline{x} \in [\overline{0}, \overline{u}]$. We need to find $q \in P$ and $a \in [0_q, u_q]$ such that $a \in \overline{x}$. We have $\overline{x \vee 0_p} = \overline{x}$ in view of $\overline{x} \vee \overline{0} = \overline{x}$. That means there exists $r \in P$ such that $p \leq r$ and $\varphi_{pr}(x) = \varphi_{pr}(x \vee 0_p)$. Similarly there exists $s \in P$ such that $p \leq s$ and $\varphi_{ps}(x) = \varphi_{ps}(x \wedge u_p)$. Choose $q \in P$ such that $q \geq r, s$ and put $a = \varphi_{pq}(x)$.

It is obvious that ψ is a homomorphism. □

Let $\langle P, \leq \rangle$ be a directed partially ordered set, $\mathscr{A}_p = (A_p, \oplus, \neg, 0_p)$ be an MV-algebra for each $p \in P$. Let $\{P, \mathscr{A}_p, \varphi_{pq}\}$ be a direct family with the limit $\overline{\mathscr{A}} = (\overline{A}, \oplus, \neg, \overline{0})$. For $p \in P$ denote $1_p = \neg 0_p$ and G_p the ℓ-group with the strong unit 1_p such that $\mathscr{A}_p = \mathbf{\Gamma}(G_p, 1_p)$. We have $G_p = (G_p, +, \vee, \wedge, -, 0_p, 1_p)$. For $p, q \in P$ let ψ_{pq} be the homomorphism from G_p into G_q such that $\varphi_{pq} = \psi_{pq} \mid [0_p, 1_p]$. This homomorphism ψ_{pq} exists and is uniquely determined, see [1, Lemma 7.2.1].

Lemma 2. *The family $\{P, G_p, \psi_{pq}\}$ is direct.*

Proof. We need to show that homomorphisms ψ_{pq} are closed according to composition.

Let $p, q, s \in P$, $p \leq q \leq s$. Suppose that $x \in G_p - A_p$. Take $m, n \in \mathbb{N}$, $x_1, \dots, x_n$, $y_1, \dots y_m \in A_p$ such that $x^+ = x_1 + \dots + x_n$ and $x^- = y_1 + \dots + y_m$. (Note that $x^+ = x \vee 0$, $x^- = -(x \wedge 0)$ and $x = x^+ - x^-$.) We have $\psi_{pq} \circ \psi_{qs}(x) = \psi_{qs}(\psi_{pq}(x)) = \psi_{qs}(\psi_{pq}(x^+ - x^-)) = \varphi_{ps}(x_1) + \dots \varphi_{ps}(x_n) - (\varphi_{ps}(y_1) + \dots \varphi_{ps}(y_m)) = \psi_{ps}(x^+) - \psi_{ps}(x^-) = \psi_{ps}(x^+ - x^-) = \psi_{ps}(x)$. □

Theorem 2. *Let $\{P, G_p, \psi_{pq}\}$ be the direct family from the previous lemma. Let $\tilde{G} = (\tilde{G}, +, \vee, \wedge, -, \tilde{0}, \tilde{u})$ be the direct limit of this family. Then $\overline{\mathscr{A}} \cong \Gamma(\tilde{G}, \tilde{u})$.*

P r o o f. Let $p \in P$ and $x \in A_p$. We define $\Phi(\overline{x}) = \tilde{x}$.

There is $\tilde{x} \in [\tilde{0}, \tilde{u}]$ according to $A_p = [0_p, 1_p]$, $\tilde{0_p} = \tilde{0}$, $\tilde{1_p} = \tilde{u}$.

It is easy to see that Φ is a homomorphism.

The mapping Φ is injective because $\overline{x} \subseteq \tilde{x}$.

Let $p \in P$ and $b \in G_p$ be such that $\tilde{b} \in \Gamma(\tilde{G}, \tilde{u})$. That means $\tilde{0} \leq \tilde{b} \leq \tilde{u}$. The definition of a direct limit yields that there exist $q \in P$ and $y \in \tilde{b} \cap G_q$ such that $0_q \leq y \leq 1_q$. Then $y \in A_q$ and $\Phi(\overline{y}) = \tilde{y} = \tilde{b}$. □

Direct limit classes of MV-algebras

We are interested whether some well known classes of MV-algebras which are not varieties are closed under formation of direct limits.

First we remind some definitions, cf. [1].

Let $\mathscr{A} = (A, \oplus, \neg, 0)$ be an MV-algebra.

An element $a \in A$ is called an atom of $\mathscr{A}$ if $a > 0$ and whenever $x \in A$ and $x \leq a$ then either $x = 0$ or $x = a$. We say that $\mathscr{A}$ is atomic if for every $x \in A - \{0\}$ there exists an atom a of $\mathscr{A}$ such that $a \leq x$. We say that $\mathscr{A}$ is atomless if no element of A is an atom of $\mathscr{A}$.

We shall write $n \cdot x$ as an abbreviation of $x \oplus \cdots \oplus x$ (n-times) for $n \in \mathbb{N}$, $x \in A$.

An element $a \in A$ is called archimedean if there is $n \in \mathbb{N}$ such that $n \cdot a = (n+1) \cdot a$. An MV-algebra is called hyperarchimedean if all its elements are archimedean.

We will use the following segment of [1, Thm. 3.5.1]:

Lemma 3. *An MV-algebra $\mathscr{A} = (A, \oplus, \neg, 0)$ is simple if and only if A is nontrivial and for every nonzero element $x \in A$ there is $n \in \mathbb{N}$ such that $1 = n \cdot x$.*

As an easy consequence of this assertion we obtain that every simple MV-algebra is hyperarchimedean. On the other hand for example the direct product of two 2-element MV-chains is hyperarchimedean MV-algebra which is not simple.

Lemma 4. *Let $\mathscr{A}$, $\mathscr{A}'$ be simple MV-algebras and φ be a homomorphism from $\mathscr{A}$ into $\mathscr{A}'$. Let $a, b \in A$. If $a < b$ then $\varphi(a) < \varphi(b)$.*

Proof. Let $a < b$. That means is $\neg a \oplus b = 1$. Therefore $\neg\varphi(a) \oplus \varphi(b) = 1$. We have $\varphi(a) \leq \varphi(b)$.

Assume that $\varphi(a) = \varphi(b)$.

Put $c = \neg(\neg b \oplus a)$. Then $c > 0$. Take $n \in \mathbb{N}$ such that $n \cdot c = 1$ according to simplicity of $\mathscr{A}$. We obtain $1 = \varphi(1) = \varphi(n.c)) = n \cdot \varphi(\neg(\neg b \oplus a)) = n \cdot \neg(\neg\varphi(b) \oplus \varphi(a)) = n \cdot \neg(\neg\varphi(a) \oplus \varphi(a)) = n \cdot \neg 1 = n \cdot 0 = 0$, a contradiction. We conclude $\varphi(a) < \varphi(b)$. □

Proposition 1. *Let $\mathscr{K}$ be the class of all*

(i) *MV-chains.*

(ii) *hyperarchimedean MV-algebras.*

(iii) *directly indecomposable MV-algebras.*

(iv) *simple MV-algebras.*

(v) *atomless simple MV-algebras.*

Then $\mathscr{K}$ is closed under formation of direct limits.

Proof.

(i) Suppose that $\{P, \mathscr{A}_p, \varphi_{pq}\}$ is a direct family of MV-chains. Let $\overline{\mathscr{A}}$ be the direct limit of this family. We prove that $\overline{\mathscr{A}}$ is an MV-chain.

Assume $a, b \in \overline{A}$. We choose $p \in P$ such that there exist $x \in a \cap A_p$ and $y \in b \cap A_p$. Since A_p is an MV-chain, we have $x \leq y$ or $y \leq x$.

Let e.g. $x \leq y$. We obtain $\neg a \oplus b = \neg\varphi_{p\infty}(x) \oplus \varphi_{p\infty}(y) = \varphi_{p\infty}(\neg x \oplus y) = \varphi_{p\infty}(1) = \overline{1}$. Thus $a \leq b$.

(ii) Let $\{P, \mathscr{A}_p, \varphi_{pq}\} \longrightarrow \overline{\mathscr{A}}$ and $\mathscr{A}_p$ be a hyperarchimedean MV-algebra for every $p \in P$. Take $a \in \overline{A}$. If $x \in a \cap A_p$, then there is $n \in \mathbb{N}$ such that $n \cdot x = (n+1) \cdot x$. We have $n \cdot a = n \cdot \varphi_{p\infty}(x) = \varphi_{p\infty}(n \cdot x) = \varphi_{p\infty}((n+1) \cdot x) = (n+1) \cdot \varphi_{p\infty}(x) = (n+1) \cdot a$.

(iii) Let $\mathscr{A} = (A, \oplus, \neg, 0)$ is an MV-algebra. Denote $B(\mathscr{A}) = \{x \in A : x \oplus x = x\}$. [1, Thm. 6.4.7] says that $\mathscr{A}$ is directly indecomposable if and only if $B(\mathscr{A}) = \{0, 1\}$. We will use this characterization.

Let $\{P, \mathscr{A}_p, \varphi_{pq}\}$ be a direct family of directly indecomposable MV-algebras with the direct limit $\overline{\mathscr{A}}$. Let $x \in \overline{A}$ be such that $x \oplus x = x$.

Suppose that $p \in P$ and $a \in x \cap A_p$. We have $\overline{a \oplus a} = \overline{a} \oplus \overline{a} = x \oplus x = x = \overline{a}$. Thus there exists $q \in P$ such that $q \geq p$ and $\varphi_{pq}(a \oplus a) = \varphi_{pq}(a)$. Denote $b = \varphi_{pq}(a)$. We obtain $b \oplus b = \varphi_{pq}(a \oplus a) = \varphi_{pq}(a) = b$. Therefore $b \in \{0_q, 1_q\}$ in view of indecomposability of $\mathscr{A}_q$. Conclude $x = \overline{b} \in \{\overline{0}, \overline{1}\}$.

(iv) Let (1) be satisfied and $\mathscr{A}_p$ be simple for each $p \in P$.

Suppose that b is a nonzero element of $\overline{A}$. Take $p \in P$ and $a \in A_p \cap b$. Then $a \neq 0$. Thus we can choose $n \in \mathbb{N}$ such that $n \cdot a = 1$ in view of simplicity of $\mathscr{A}_p$. We obtain $n \cdot \overline{a} = \overline{n \cdot a} = \overline{1}$.

We conclude $\overline{\mathscr{A}}$ is a simple MV-algebra by Lemma 3.

(v) Let (1) be valid and $\mathscr{A}_p$ be an atomless simple MV-algebra for every $p \in P$. Suppose that $p \in P$ and $a \in A_p - \{0\}$. Take $b \in A_p$ such that $0 < b < a$. We have $0 < \varphi_{p\infty}(b) < \varphi_{p\infty}(a)$ according to Lemma 4. □

Proposition 2. *Let $\mathscr{K}$ be the class of all*

(i) *directly decomposable MV-algebras.*

(ii) *complete MV-algebras.*

(iii) *atomic MV-algebras.*

(iv) *finite MV-algebras.*

(v) *infinite MV-algebras.*

Then $\mathscr{K}$ is not closed under formation of direct limits.

Proof.

(i) Let B be a two-element boolean MV-algebra. Then B is isomorphic to a retract of $B \times B$. That means an MV-algebra isomorphic to B can be obtained as the direct limit of a direct family which possesses $B \times B$ only.

(ii)–(iv) See Example 2.

(v) We use Example 1.

Let $\mathbf{1}$ assign the integer 1 to every $x \in R$. According to Thm. 1 we can construct $\mathbf{\Gamma}(L, \mathbf{1})$ by a direct limit from $\mathbf{\Gamma}(M, \mathbf{1})$. The MV-algebra $\mathbf{\Gamma}(M, \mathbf{1})$ has the cardinality of continuum and $\mathbf{\Gamma}(L, \mathbf{1})$ is the two-element MV-algebra. □

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