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ON SUMS RELATED TO THE NUMERATOR OF GENERATING FUNCTIONS FOR THE kTH POWER OF FIBONACCI NUMBERS

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Number Theory

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ABSTRACT. New results about some sums $s_n(k,l)$ of products of the Lucas numbers, which are of similar type as the sums in [SEIBERT, J.—TROJOVSKÝ, P.: *On multiple sums of products of Lucas numbers*, J. Integer Seq. **10** (2007), Article 07.4.5], and sums $\sigma(k) = \sum\limits_{l=0}^{\frac{k-1}{2}} \binom{k}{l} F_{k-2l} s_n(k,l)$ are derived. These sums are related to the numerator of generating function for the kth powers of the Fibonacci numbers. $s_n(k,l)$ and $\sigma(k)$ are expressed as the sum of the binomial and the Fibonomial coefficients. Proofs of these formulas are based on a special inverse formulas.

1. Introduction

It is well-known that manipulations with generating function of sequences of integers are very helpful in finding some relations for terms of these sequences. In this paper we will use the Fibonacci numbers, which are defined by the recurrence $F_{n+2} = F_n + F_{n+1}$ with $F_0 = 0$, $F_1 = 1$, and the Lucas numbers, which are defined by the same recurrence as the Fibonacci numbers but they have the initial conditions $L_0 = 2$, $L_1 = 1$.

In our previous papers [8, 9] we used manipulations with the denominator of generating functions for the kth power of the Fibonacci numbers to find

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identities for some sums of products of the Lucas numbers, which are a certain generalization of the well-known formula for the Fibonacci and Lucas numbers (see for example [14, p. 179])

$$\sum_{i=0}^{n}(-1)^i L_{n-2i} = 2F_{n+1}.$$

In this paper we will close this line of papers by finding relations for sums which are created from the numerator of generating functions for the kth power of Fibonacci numbers.

Generating function for the Fibonacci sequence $\{F_n\}$ was already known to DeMoivre in 1718 and in 1957 S. W. Golomb [2] found generating function of $\{F_n^2\}$. Golomb's effective proof opened the effort to find a recurrence or a closed form of generating function $f_k(x) = \sum_{n=0}^{\infty} F_n^k x^n$ for the kth powers of the Fibonacci numbers. Riordan [6] found the general recurrence for $f_k(x)$. Carlitz [1], Horadam [3] and Mansour [5] presented some generalizations of Riordan's result and found similar recurrences for the generating functions for powers of any second–order recurrence sequences.

Horadam found some closed forms for the numerator and the denominator of these generating functions, from his general results in [3] it is possible to get specially the following relation for the generating function of the integer powers of the Fibonacci numbers

$$f_k(x) = \frac{\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{\frac{j(j+1)}{2}} \begin{bmatrix} k+1 \\ j \end{bmatrix} F_{i-j}^k\, x^i}{\sum_{i=0}^{k+1}(-1)^{\frac{i(i+1)}{2}} \begin{bmatrix} k+1 \\ j \end{bmatrix} x^i}, \tag{1}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ are the so-called Fibonomial coefficients defined for any nonnegative integers n and k by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k},$$

where $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for $n < k$. In [10] Shannon found by Carlitz' method an expression of the generating function $f_k(x)$, which can be written for any odd integer k as

$$f_k(x) = 5^{-\frac{k-1}{2}} \sum_{j=0}^{\frac{k-1}{2}} \binom{k}{j} \frac{F_{k-2j}\, x}{1-(-1)^j L_{k-2j}x - x^2} \tag{2}$$

and for any even integer k as

$$f_k(x) = 5^{-\frac{k}{2}} \left(\sum_{j=0}^{\frac{k-2}{2}} (-1)^j \binom{k}{j} \frac{2-(-1)^j L_{k-2j}x}{1-(-1)^j L_{k-2j}x + x^2} + \binom{k}{\frac{k}{2}} \frac{(-1)^{\frac{k}{2}}}{1-(-1)^{\frac{k}{2}}x} \right). \quad (3)$$

In 2003 Stănică [11] gave similar results to Horadam but he obtained some new weighted cases and used them to find some combinatorial sums of the type $\sum_{i=0}^{n} \binom{n}{i} F_i^r$. Strazdins [12] considered the polynomials from the denominator of the fraction (1) and he found some factorization of them with the help of the Lucas factor of the type $1 - L_k\, x + (-1)^k x^2$.

Throughout the paper we adopt the conventions that the sum and the product over an empty set is 0 and 1, respectively, and that $\lfloor x \rfloor$ represents the greatest integer less than or equal to x.

In [9] we studied the sequence $\{S_n(k)\}_{n=0}^{\infty}$ where k is a positive integer. The terms of this sequence are defined by

$$S_0(k) = 1, \qquad S_1(k) = \sum_{i_1=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^{i_1} L_{k-2\,i_1}$$

and for any positive integer $n > 1$

$$S_n(k) = \sum_{i_n=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{i_{n-1}=i_n+1}^{\lfloor \frac{k-1}{2} \rfloor} \cdots \sum_{i_1=i_2+1}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^{i_1+i_2+\cdots+i_n} \prod_{j=1}^{n} L_{k-2\,i_j}. \quad (4)$$

If we denote

$$\Theta(i,k,n) = \binom{\lfloor \frac{k+1}{2} \rfloor - n + i}{i} + \binom{\lfloor \frac{k+1}{2} \rfloor - n + i - 1}{i-1} \quad (5)$$

for any positive integers i, k and any nonnegative integer n, we found that (4) can be written as

$$S_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\lfloor \frac{n}{2} \rfloor - i}\, \Theta(i,k,n) \begin{bmatrix} k+1 \\ n-2i \end{bmatrix}$$

for an odd integer k and

$$S_n(k) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2i} (-1)^{i+n(\frac{k}{2}+1)+\frac{j}{2}(j+k+1)}\, \Theta(i,k,n) \begin{bmatrix} k+1 \\ j \end{bmatrix}$$

for an even integer k.

2. The main results

Let us define the sequence $\{s_n(k,l)\}_{n=0}^{\infty}$, where k is any positive integer and $l \le \lfloor \frac{k-1}{2} \rfloor$ is any nonnegative integer, by the following way:

$$s_0(k,l) = 1, \qquad s_1(k,l) = \sum_{\substack{i_1=0 \\ i_1 \ne l}}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^{i_1} L_{k-2i_1}$$

and for $n > 1$

$$s_n(k,l) = \sum_{\substack{i_n=0 \\ i_n \ne l}}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{\substack{i_{n-1}=i_n+1 \\ i_{n-1} \ne l}}^{\lfloor \frac{k-1}{2} \rfloor} \cdots \sum_{\substack{i_1=i_2+1 \\ i_1 \ne l}}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^{i_1+i_2+\dots+i_n} \prod_{j=1}^{n} L_{k-2\,i_j}. \tag{6}$$

Let the sequence $\{\sigma_n(k)\}_{n=0}^{\infty}$, where k is any odd positive integer, be given by

$$\sigma_n(k) = \sum_{l=0}^{\frac{k-1}{2}} \binom{k}{l} F_{k-2l}\, s_n(k,l). \tag{7}$$

Theorem 1. *Let k be any positive integer, n and $l \le \lfloor \frac{k-1}{2} \rfloor$ be any nonnegative integers. Then*

$$s_n(k,l) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2i} (-1)^{n(l+1)+\frac{j}{2}(2l+j+1)} \Theta(i, k-2, n) \begin{bmatrix} k+1 \\ j \end{bmatrix} \frac{F_{(n-2i-j+1)(k-2l)}}{F_{k-2l}} \tag{8}$$

for an odd positive integer k and

$$s_n(k,l) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2m} \sum_{i=0}^{j} (-1)^{\frac{k}{2}(n+j)+n+m+l(j-i)+\frac{i}{2}(i+1)} \left(\binom{\frac{k-2}{2}-n+m}{m} + \binom{\frac{k-4}{2}-n+m}{m-1} \right) \Theta(m, k-2, n) \begin{bmatrix} k+1 \\ i \end{bmatrix} \frac{F_{(j-i+1)(k-2l)}}{F_{k-2l}} \tag{9}$$

for an even positive integer k.

Theorem 2. *Let k be any odd positive integer and n be any positive integer. Then*

$$\sigma_n(k) = 5^{\frac{k-1}{2}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2i} (-1)^{n+\frac{j}{2}(j+1)} \Theta(i, k-2, n) \begin{bmatrix} k+1 \\ j \end{bmatrix} F^{k}_{n-2i+1-j}. \tag{10}$$

Theorem 3. *Let k be any even positive integer, n and $l \le \frac{k-2}{2}$ be any nonnegative integers. Then*

$$\sum_{j=0}^{i}\left(\sum_{l=0}^{\frac{k}{2}-1}\binom{k}{l}(-1)^{l(i+j+1)}L_{(i-j)(k-2l)} + \binom{k}{\frac{k}{2}}(-1)^{\frac{k}{2}(j+i+1)} - 5^{\frac{k}{2}}F_{i-j}^{k}\right)\begin{bmatrix}k+1\\ j\end{bmatrix} = 0.$$

Corollary 4 (Corollary of Theorems 1 and 2)**.** *Let k be any positive integer, $n > \lfloor\frac{k-1}{2}\rfloor$ and $l \le \lfloor\frac{k-1}{2}\rfloor$ be any nonnegative integers. If k is odd positive integer then*

$$\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{j=0}^{n-2i}(-1)^{\frac{j}{2}(2l+j+1)}\,\Theta(i,k-2,n)\begin{bmatrix}k+1\\ j\end{bmatrix}F_{(n-2i-j+1)(k-2l)} = 0,$$

$$\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{j=0}^{n-2i}(-1)^{\frac{j(j+1)}{2}}\,\Theta(i,k-2,n)\begin{bmatrix}k+1\\ j\end{bmatrix}F_{n-2i+1-j}^{k} = 0$$

and if k is even positive integer then

$$\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{j=0}^{n-2m}\sum_{i=0}^{j}(-1)^{(\frac{k}{2}+l)j+m+\frac{i}{2}(2l+i+1)}\,\Theta(i,k-2,n)\begin{bmatrix}k+1\\ i\end{bmatrix}\frac{F_{(j-i+1)(k-2l)}}{F_{k-2l}} = 0\,.$$

Corollary 5 (Corollary of Theorems 1 and 2)**.** *Let k be any odd positive integer. Then*

(i) *Setting $n = 1, 2$ in (8) we have*

$$\sum_{\substack{i_1=0\\ i_1\neq l}}^{\frac{k-1}{2}}(-1)^{i_1}L_{k-2i_1} = F_{k+1} + (-1)^{l+1}L_{k-2l},$$

$$\sum_{\substack{i_2=0\\ i_2\neq l}}^{\frac{k-1}{2}}\sum_{\substack{i_1=i_2+1\\ i_1\neq l}}^{\frac{k-1}{2}}(-1)^{i_1+i_2}L_{k-2i_1}\,L_{k-2i_2} = \frac{k-1}{2} - F_k\,F_{k+1} + (-1)^{l+1}F_{k+1}\,L_{k-2l} + \frac{F_{3(k-2l)}}{F_{k-2l}}.$$

(ii) *Setting $n = 1, 2$ in (9) we have*

$$\sum_{\substack{i_1=0\\ i_1\neq l}}^{\frac{k-2}{2}}(-1)^{i_1}L_{k-2i_1} = F_{k+1} + (-1)^{l+1}L_{k-2l} + (-1)^{\frac{k}{2}+1},$$

$$\sum_{\substack{i_2=0\\ i_2\neq l}}^{\frac{k-2}{2}} \sum_{\substack{i_1=i_2+1\\ i_1\neq l}}^{\frac{k-2}{2}} (-1)^{i_1+i_2} L_{k-2i_1}\, L_{k-2i_2}$$

$$= \frac{4-k}{2} + (-1)^{\frac{k}{2}+l} L_{k-2l} - F_{k+1}((-1)^{\frac{k}{2}} + (-1)^l L_{k-2l} + F_k) + \frac{F_{3(k-2l)}}{F_{k-2l}}.$$

(iii) *Setting* $n = 1, 2, 3$ *in* (10) *we have*

$$\sum_{l=0}^{\frac{k-1}{2}} \binom{k}{l} F_{k-2l} = 5^{\frac{k-1}{2}}, \qquad \sum_{l=0}^{\frac{k-1}{2}} \sum_{\substack{i_1=0\\ i_1\neq l}}^{\frac{k-1}{2}} (-1)^{i_1} \binom{k}{l} F_{k-2l}\, L_{k-2i_1} = 5^{\frac{k-1}{2}}\, (F_{k+1} - 1)\,,$$

$$\sum_{l=0}^{\frac{k-1}{2}} \sum_{\substack{i_2=0\\ i_2\neq l}}^{\frac{k-1}{2}} \sum_{\substack{i_1=i_2+1\\ i_1\neq l}}^{\frac{k-1}{2}} (-1)^{i_1+i_2} \binom{k}{l} F_{k-2l}\, L_{k-2i_1}\, L_{k-2i_2}$$

$$= 5^{\frac{k-1}{2}} \left(2^k + \frac{k-1}{2} - F_{k+1} - F_k\, F_{k+1} \right)$$

3. The preliminary results

As in [8] we denote the denominator of identity (1) by $D(x)$ and its coefficients $(-1)^{\frac{i}{2}(i+1)} \left[{k+1 \atop i}\right]$ by d_i. The numerator in (1) we denote $N(x)$. Using q-analog of the terminating binomial theorem (see for example [4])

$$\prod_{i=0}^{k} (1 - q^i\, x) = \sum_{i=0}^{k+1} (-1)^i q^{\frac{i}{2}(i-1)} \left\{ {k+1 \atop i} \right\} x^i, \tag{11}$$

where q-binomial coefficients $\left\{ {k+1 \atop i} \right\}$ are defined as

$$\left\{ {k+1 \atop i} \right\} = \frac{(q^{k+1}-1)(q^k-1)\cdots(q^{k-i+2}-1)}{(q-1)(q^2-1)\cdots(q^i-1)}$$

for $i \geq 1$ and any complex numbers q, x and any positive integer k, where $\left\{ {k+1 \atop 0} \right\} = 1$.

Replacing q by β/α and x by $\alpha^k\, x$ in (11) we have

$$D(x) = \sum_{i=0}^{k+1} (-1)^{\frac{i(i+1)}{2}} \begin{bmatrix} k+1 \\ i \end{bmatrix} = \prod_{i=0}^{k} (1 - \alpha^{k-i} \beta^i\, x).$$

Thus for any even integer k we gain

$$\begin{aligned} D(x) &= \prod_{j=0}^{k}(1-\alpha^{k-j}\,\beta^j\,x) \\ &= (1-(\alpha\beta)^{\frac{k}{2}}x)\prod_{\substack{j=0\\ j\neq\frac{k}{2}}}^{k}(1-\alpha^{k-j}\,\beta^j\,x) \\ &= (1-(-1)^{\frac{k}{2}}x)\prod_{j=0}^{\frac{k}{2}-1}\big(1-(-1)^j(\alpha^{k-2j}+\beta^{k-2j})\,x+(\alpha\beta)^{k-2j}x^2\big) \\ &= (1-(-1)^{\frac{k}{2}}x)\prod_{j=0}^{\frac{k}{2}-1}(1-(-1)^jL_{k-2j}x+x^2), \end{aligned} \tag{12}$$

and for any odd integer k

$$\begin{aligned} D(x) &= \prod_{j=0}^{k}(1-\alpha^{k-j}\,\beta^j\,x) \\ &= \prod_{j=0}^{\frac{k-1}{2}}\big(1-(-1)^j\alpha^{k-2j}\,x\big)\,\big(1-(-1)^j\beta^{k-2j}\,x\big) \\ &= \prod_{j=0}^{\frac{k-1}{2}}\big(1-(-1)^j(\alpha^{k-2j}+\beta^{k-2j})\,x+(\alpha\beta)^{k-2j}x^2\big) \\ &= \prod_{j=0}^{\frac{k-1}{2}}(1-(-1)^jL_{k-2j}x-x^2). \end{aligned} \tag{13}$$

Further with respect to (2) and (3) we have for any odd integer k

$$f_k(x) = \frac{5^{\frac{1-k}{2}}\sum_{j=0}^{\frac{k-1}{2}}\left(\binom{k}{j}F_{k-2j}x\prod_{\substack{i=0\\ i\neq j}}^{\frac{k-1}{2}}(1-(-1)^iL_{k-2i}x-x^2)\right)}{\prod_{j=0}^{\frac{k-1}{2}}(1-(-1)^jL_{k-2j}x-x^2)} \tag{14}$$

and for any even integer k

$$f_k(x) = \frac{N(x)}{D(x)},$$

where

$$N(x) = 5^{-\frac{k}{2}}\Bigg((-1)^{\frac{k}{2}}\binom{k}{\frac{k}{2}}\prod_{j=0}^{\frac{k-2}{2}}\big(1-(-1)^j L_{k-2j}x+x^2\big)$$
$$+\sum_{l=0}^{\frac{k-2}{2}}(-1)^l\binom{k}{l}\big(1-(-1)^{\frac{k}{2}}x\big)\big(2-(-1)^l L_{k-2l}x\big)\prod_{\substack{j=0\\ j\neq l}}^{\frac{k-2}{2}}\big(1-(-1)^j L_{k-2j}x+x^2\big)\Bigg). \tag{15}$$

Lemma 6. *Let k be any positive integer and let n and $l \le \lfloor\frac{k-1}{2}\rfloor$ be any non-negative integers. Then $s_n(k,l)=0$ for each $\lfloor\frac{k-1}{2}\rfloor < n$.*

P r o o f. Rewriting relation (6) in the form

$$s_n(k,l) = \sum_{\substack{0\le i_n<i_{n-1}<\dots<i_1\le\lfloor\frac{k-1}{2}\rfloor\\ i_1\neq l,\, i_2\neq l,\dots i_n\neq l}} (-1)^{i_1+i_2+\dots+i_n}\prod_{j=1}^{n} L_{k-2\,i_j}$$

the assertion easily follows from the condition

$$0\le i_n<i_{n-1}<\dots<i_1\le\left\lfloor\frac{k-1}{2}\right\rfloor,$$

which are not valid for any values $i_1\neq l$, $i_2\neq l$, …, $i_n\neq l$ if $\lfloor\frac{k-1}{2}\rfloor<n$. □

Lemma 7. *Let k and n be any positive integers. Then*

(i) $\sum_{i=0}^{n}\binom{n+i-\frac{k+1}{2}}{n-i}s_{2i}(k,l)=0$ *for* $n\ge\frac{k+1}{2}$,

(ii) $\sum_{i=0}^{n}\binom{n+i-\frac{k-1}{2}}{n-i}s_{2i+1}(k,l)=0$ *for* $n\ge\frac{k-1}{2}$

for any odd positive integer k and

(iii) $\sum_{i=0}^{n}\binom{\frac{k-2}{2}-2i}{n-i}s_{2i}(k,l)=0$ *for* $n\ge\frac{k}{2}$,

(iv) $\sum_{i=0}^{n}\binom{\frac{k-2}{2}-2i-1}{n-i}s_{2i+1}(k,l)=0$ *for* $n\ge\frac{k}{2}-1$

for any even positive integer k.

P r o o f. It can be done analogously as in [9] using properties of the involved binomial coefficients and Lemma 7. □

Lemma 8. *Let n be any positive integer and let p be any integer. Then the following inverse law for any sequences $\{a_n\}$, $\{b_n\}$ holds:*

$$a_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^n \binom{n-i+p}{i} b_{n-2i}$$

if and only if

$$b_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^n \left(\binom{i-n-p-1}{i} + \binom{i-n-p-2}{i-1} \right) a_{n-2i}. \tag{16}$$

P r o o f. First replacing $\{a_n\}$ by $\{a_{2n}\}$, $\{b_i\}$ by $\{b_{2i}\}$, n by $\frac{n}{2}$ and i by $\frac{n}{2} - i$ and then $\{a_n\}$ by $\{a_{2n+1}\}$, $\{b_i\}$ by $\{-b_{2i+1}\}$, n by $\frac{n-1}{2}$, i by $\frac{n-1}{2} - i$ and p by $p+1$ in the following inversion formula from [7, p. 74, identity (23)]

$$a_n = \sum_{i=0}^{n} \binom{n+i+p}{n-i} b_i$$

if and only if

$$b_n = \sum_{i=0}^{n} (-1)^{n+i} \left(\binom{2n+p}{n-i} - \binom{2n+p}{n-i-1} \right) a_i,$$

we get two inversion formulas which can be joined to the assertion. □

Lemma 9. ([9, Lemma 11]) *Let n be any positive integer and let q be any integer. Then the following inverse formula holds:*

$$a_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^n \binom{t-n+2i}{i} b_{n-2i}$$

if and only if

$$b_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+i} \left(\binom{t-n+i}{i} + \binom{t-n+i-1}{i-1} \right) a_{n-2i}.$$

Let us consider the general second order recurrence relation

$$y_{n+2} = g\, y_{n+1} - h\, y_n, \tag{17}$$

where g and $h \neq 0$ are arbitrary integers. Let its characteristic equation has different roots ε and ω. If the initial conditions are $y_0 = 0$ and $y_1 = 1$ then the solution of (17) is $y_n = \frac{\varepsilon^n - \omega^n}{\varepsilon - \omega}$, $\varepsilon \neq \omega$.

Let k be any nonnegative integer. Suppose $\{x_n\}$ is any sequence of real numbers satisfying the recurrence relation

$$x_{n+2} - \lambda x_{n+1} + (-1)^k x_n = 0, \quad x_0 = 0, \quad x_1 = 1, \tag{18}$$

where λ is a real number. As (18) is a special case of (17) it is evident that for any nonnegative integer n

$$x_n = \frac{1}{2^n D}\left((\lambda + D)^n - (\lambda - D)^n\right), \tag{19}$$

where $D = \sqrt{\lambda^2 - 4(-1)^k}$.

Lemma 10. *Let a, b be any integers. Then*

$$L_a + (-1)^b\sqrt{5}F_a = 2\left(\frac{1+(-1)^b\sqrt{5}}{2}\right)^a. \tag{20}$$

P r o o f. By Binet formulas for F_n and L_n we have

$$L_a - \sqrt{5}F_a = (\alpha^a + \beta^a) - (\alpha^a - \beta^a) = 2\beta^a$$

for any odd integer b and

$$L_a + \sqrt{5}F_a = (\alpha^a + \beta^a) + (\alpha^a - \beta^a) = 2\alpha^a$$

for any even integer b. Hence the assertion follows. □

Lemma 11. *Let $l \le \frac{k-1}{2}$ be any nonnegative integer. Let $\{x_n\}$ be any sequence of real numbers defined by the recurrence $x_{n+2} = (-1)^l L_{k-2l}\, x_{n+1} - (-1)^k x_n$ for $n \ge 0$, with $x_0 = 0$, $x_1 = 1$. Then*

$$x_n = (-1)^{l(n+1)}\frac{F_{n(k-2l)}}{F_{k-2l}}.$$

P r o o f. Using (19), the well-known formula $L_n^2 - 4\,(-1)^n = 5\,F_n^2$ (cf. [14, p. 177]) and (20) we have

$$\begin{aligned}
x_n &= \frac{1}{2^n\sqrt{5}F_{k-2l}}\left(((-1)^l L_{k-2l} + \sqrt{5}F_{k-2l})^n - ((-1)^l L_{k-2l} - \sqrt{5}F_{k-2l})^n\right)\\
&= \frac{(-1)^{ln}}{2^n\sqrt{5}F_{k-2l}}\left((L_{k-2l} + (-1)^l\sqrt{5}F_{k-2l})^n - (L_{k-2l} - (-1)^l\sqrt{5}F_{k-2l})^n\right)\\
&= \frac{(-1)^{ln}}{\sqrt{5}F_{k-2l}}\left(\left(\left(\frac{1+(-1)^l\sqrt{5}}{2}\right)^{k-2l}\right)^n - \left(\left(\frac{1-(-1)^l\sqrt{5}}{2}\right)^{k-2l}\right)^n\right)\\
&= (-1)^{ln}\frac{1}{\sqrt{5}F_{k-2l}}(-1)^l(\alpha^{n(k-2l)} - \beta^{n(k-2l)})\\
&= (-1)^{l(n+1)}\frac{F_{n(k-2l)}}{F_{k-2l}}.
\end{aligned}$$

□

Lemma 12. ([13, Lemma 4]) *Let $m \geq 1$ be any integer, $\{a_n\}$, $\{b_n\}$ be any sequences of real numbers and A_i, $i = 1, \ldots, m$, be any real numbers. Let $a_{-i} = 0$ for $i = 1, 2, \ldots, m$ and $\{z_n\}$ be any sequence of real numbers defined by $z_n = \sum_{i=1}^{m} A_i \, z_{n-i}$ with $z_j = 0$ for each nonnegative integer $j \leq m-2$ and $z_{m-1} = 1$.*

Then for $n \geq 0$

$$b_n = a_n - \sum_{j=1}^{m} A_j \, a_{n-j} \tag{21}$$

if and only if

$$a_n = \sum_{i=0}^{n} x_{i+m-1} \, b_{n-i}. \tag{22}$$

Using latter Lemma 12 we immediately obtain the following two results.

Lemma 13. *Let $\{a_n\}$, $\{b_n\}$ be any sequences of real numbers, let k be any nonnegative integer and let $\{x_n\}$ be any sequence (18). Then for any $n > 1$*

$$b_n = a_n - \lambda \, a_{n-1} + (-1)^k a_{n-2} \tag{23}$$

if and only if

$$a_n = \sum_{i=0}^{n-2} x_{i+1} \, b_{n-i} + x_n \, a_1 - (-1)^k x_{n-1} \, a_0. \tag{24}$$

Lemma 14. *Let $\{a_n\}$, $\{b_n\}$ be any sequences of real numbers and let m be any nonnegative integer. Let $a_0 = b_0$ and $a_{-1} = 0$. Then for $n \geq 0$*

$$a_n - (-1)^m \, a_{n-1} = b_n \tag{25}$$

if and only if

$$a_n = \sum_{i=0}^{n} (-1)^{m(n+i)} b_i. \tag{26}$$

Lemma 15. *Let k be any even positive integer and let $l \leq \frac{k-2}{2}$ be any nonnegative integer. Then*

$$(1 - (-1)^{\frac{k}{2}} x) \prod_{j=0}^{\frac{k-2}{2}} (1 - (-1)^j L_{k-2j} x + x^2) = \sum_{i=0}^{k+1} (-1)^{\frac{i}{2}(i+1)} \begin{bmatrix} k+1 \\ i \end{bmatrix} x^i, \tag{27}$$

$$(1-(-1)^{\frac{k}{2}}x)\prod_{\substack{j=0\\ j\neq l}}^{\frac{k-2}{2}}(1-(-1)^{j}L_{k-2j}x+x^{2}) \\ = \sum_{i=0}^{k-1}\sum_{j=0}^{i}(-1)^{l(i-j)}\,\frac{F_{(i-j+1)(k-2l)}}{F_{k-2l}}\,d_j\,x^{i}, \tag{28}$$

$$\prod_{\substack{j=0\\ j\neq l}}^{\frac{k-2}{2}}(1-(-1)^{j}L_{k-2j}x+x^{2}) \\ = \sum_{m=0}^{k-2}\sum_{i=0}^{m}\sum_{j=0}^{i}(-1)^{\frac{k}{2}(m+i)+l(i-j)}\,\frac{F_{(i-j+1)(k-2l)}}{F_{k-2l}}\,d_j\,x^{m}, \tag{29}$$

$$\begin{aligned}(2-(-1)^{l}L_{k-2l}x)(1-(-1)^{\frac{k}{2}}x)&\prod_{\substack{j=0\\ j\neq l}}^{\frac{k-2}{2}}(1-(-1)^{j}L_{k-2j}x+x^{2})\\ &=\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{l(j+i)}L_{(i-j)(k-2l)}d_j\,x^{i}\\ &=\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{l(j+i)+\frac{j}{2}(j+1)}\begin{bmatrix}k+1\\ j\end{bmatrix}L_{(i-j)(k-2l)}\,x^{i}.\end{aligned} \tag{30}$$

P r o o f. Identity (27) immediately follows from (12). Now we will deal with (28). Define polynomial $P_{k-1}(x)=\sum\limits_{i=0}^{k-1}p_i(k,l)\,x^{i}$ by

$$\begin{aligned}P_{k-1}(x)&=(1-(-1)^{\frac{k}{2}}x)\prod_{\substack{j=0\\ j\neq l}}^{\frac{k-2}{2}}(1-(-1)^{j}L_{k-2j}x+x^{2})\\ &=\frac{D(x)}{1-(-1)^{l}L_{k-2l}x+x^{2}},\end{aligned} \tag{31}$$

where $l\le\frac{k-2}{2}$ is any nonnegative integer.

Multiplying $P_{k-1}(x)$ by $1-(-1)^l L_{k-2l}x+x^2$ and comparing with $D(x)$ we have

$$\begin{aligned} p_0(k,l) &= d_0 = 1, \\ p_1(k,l) - (-1)^l L_{k-2l}\, p_0(k,l) &= d_1 = -F_{k+1}, \\ p_i(k,l) - (-1)^l L_{k-2l}\, p_{i-1}(k,l) + p_{i-2}(k,l) &= d_i, \qquad i = 2,\, 3, \dots, k-1, \\ -(-1)^l L_{k-2l}\, p_{k-1}(k,l) + p_{k-2}(k,l) &= d_k = (-1)^{\frac{k+1}{2}}\, F_{k+1}, \\ p_{k-1}(k,l) &= d_{k+1} = -(-1)^{\frac{k-1}{2}}. \end{aligned}$$

Putting $p_i(k,l)=0$ for $i<0$ or $i>k-1$ we can rewrite the previous relations into the recurrence

$$p_i(k,l) - (-1)^l L_{k-2l}\, p_{i-1}(k,l) + p_{i-2}(k,l) = d_i,$$

which holds for any integer i. Hence using Lemma 13 we get

$$\begin{aligned} p_0(k,l) =& 1, \qquad p_1(k,l) = (-1)^l L_{k-2l} + d_1\,, \\ p_i(k,l) =& \sum_{r=0}^{i-2} x_{r+1}\, d_{i-r} + \big((-1)^l L_{k-2l} + d_1\big)\; x_n - x_{n-1}\,, \qquad i>0. \end{aligned}$$

Further using Lemma 11 and the formula $L_p\, F_{pq} = F_{p(q+1)} + (-1)^p\, F_{p(q-1)}$, cf. [14, p. 177], we gain

$$\begin{aligned} p_i(k,l) &= \sum_{r=0}^{i-1} x_{r+1} d_{i-r} + (-1)^l L_{k-2l}\, (-1)^{l(i+1)} \frac{F_{i(k-2l)}}{F_{k-2l}} - (-1)^{li} \frac{F_{(i-1)(k-2l)}}{F_{k-2l}} \\ &= \sum_{r=0}^{i-1} x_{r+1} d_{i-r} + \frac{(-1)^{li}}{F_{k-2l}} \left(L_{k-2l} F_{i(k-2l)} - F_{(i-1)(k-2l)}\right) \\ &= \sum_{r=0}^{i} x_{r+1} d_{i-r} = \sum_{r=0}^{i} x_{i-r+1} d_r \\ &= \sum_{r=0}^{i} (-1)^{l\,(i-r)}\, \frac{F_{(i-r+1)(k-2l)}}{F_{k-2l}}\, d_r \end{aligned} \tag{32}$$

hence (28) follows. Let us deal with relation (29). We define polynomials $Q_{k-2}(x) = \sum\limits_{i=0}^{k-2} q_i(k,l)\, x^i$ by

$$Q_{k-2}(x) = \prod_{\substack{j=0 \\ j\neq l}}^{\frac{k-2}{2}} (1 - (-1)^j L_{k-2j} x + x^2). \tag{33}$$

With respect to (31) we obtain $(1 - (-1)^{\frac{k}{2}} x)\, Q_{k-2}(x) = P_{k-1}(x)$ and after multipling of the left-hand side we get

$$\begin{gathered} q_0(k,l) = p_0(k,l) = 1, \\ q_i(k,l) + (-1)^{\frac{k}{2}+1}\, q_{i-1}(k,l) = p_i(k,l), \qquad i = 1,2,\dots,k, \\ (-1)^{\frac{k}{2}+1} q_k(k,l) = p_{k+1}(k,l). \end{gathered}$$

Putting $q_i(k,l) = 0$ for $i < 0$ or $i > k-2$ we obtain the general recurrence

$$q_i(k,l) + (-1)^{\frac{k}{2}+1} q_{i-1}(k,l) = p_i(k,l),$$

for any integer i, between coefficients of polynomials P_{k-1} and Q_{k-2}. Using Lemma 14 we obtain

$$q_i(k,l) = \sum_{j=0}^{i} (-1)^{\frac{k}{2}(i+j)}\, p_j(k,l).$$

and using (32) we obtain (29).

Finally we will consider relation (28). The polynomial

$$(2 - (-1)^l L_{k-2l} x)(1 - (-1)^{\frac{k}{2}} x) \prod_{\substack{j=0 \\ j\neq l}}^{\frac{k-2}{2}} (1 - (-1)^j L_{k-2j} x + x^2)$$

we denote by $R_k(x) = \sum\limits_{i=0}^{k} r_i(k,l)\, x^i$. Hence with respect to (29) the identity $R_k(x) = (2 - (-1)^l L_{k-2l} x)\, P_{k-1}(x)$ is valid and after multipling of the right-hand side we get $r_i = 2p_i + (-1)^{l+1} L_{k-2l}\, p_{i-1}$, for $i = 0,1,\dots,k$. Therefore using (32) we have

$$
\begin{aligned}
r_i &= 2\sum_{j=0}^{i}(-1)^{l(i-j)}\frac{F_{(i-j+1)(k-2l)}}{F_{k-2l}}\,d_j \\
&\quad + (-1)^{l+1}L_{k-2l}\sum_{j=0}^{i-1}(-1)^{l(i-j-1)}\frac{F_{(i-j)(k-2l)}}{F_{k-2l}}\,d_j \\
&= 2\,d_i + \sum_{j=0}^{i-1}(-1)^{l(i+j)}\frac{2F_{(i-j+1)(k-2l)} - L_{k-2l}\,F_{(i-j)(k-2l)}}{F_{k-2l}}\,d_j .
\end{aligned}
$$

With respect to well-known identity (see [14, p. 177]), $F_m\,F_n + L_m\,F_n = 2\,F_{m+n}$ we get

$$
r_i = 2\,d_i + \sum_{j=0}^{i-1}(-1)^{l(i+j)}L_{(k-2l)(i-j)}\,d_j
$$

and finally $r_i = \sum\limits_{j=0}^{i}(-1)^{l(i+j)}L_{(k-2l)(i-j)}\,d_j$, which leads to (30). $\square$

4. Proofs of main theorems

Proof of Theorem 1. First we prove relation (8). Analogously as in the proof of (28) we can define, with respect to identity (13), polynomials $D_l(x) = \sum\limits_{i=0}^{k-1} p_i(k,l)\,x^i$, where $l \le \frac{k-1}{2}$ is any nonnegative integer, by

$$
D_l(x) = \frac{D(x)}{1-(-1)^l L_{k-2l}x - x^2} = \prod_{\substack{j=0\\ j\neq l}}^{\frac{k-1}{2}}(1-(-1)^j L_{k-2j}x - x^2). \tag{34}
$$

Expanding $(1-(-1)^l L_{k-2l}x - x^2)\sum\limits_{i=0}^{k-1} p_i(k,l)\,x^i$ and comparing with $D(x)$ we get some relations, which could be rewritten to the recurrence

$$
p_n(k,l) - (-1)^l L_{k-2l}\,p_{n-1}(k,l) - p_{n-2}(k,l) = d_n
$$

for any nonnegative integer n, when it is set $p_n(k,l) = 0$ for $n < 0$ or $n > k-1$. Hence we obtain as in (32) that for any integer n

$$
p_n(k,l) = \sum_{j=0}^{n}(-1)^{l\,(n-j)}\,\frac{F_{(n-j+1)(k-2l)}}{F_{k-2l}}\,d_j. \tag{35}
$$

By the direct multiplication of the product in identity (34) we get for $n = 0, 1, 2, \dots, \frac{k-1}{2}$, respectively for $n = 0, 1, 2, \dots, \frac{k-3}{2}$,

$$p_{2n}(k,l) = \sum_{i=0}^{n} \binom{n+i-\frac{k+1}{2}}{n-i} s_{2i}(k,l),$$

$$p_{2n+1}(k,l) = -\sum_{i=0}^{n} \binom{n+i-\frac{k-1}{2}}{n-i} s_{2i+1}(k,l).$$

When indexes of the sums are shifted similarly as in the proof of Lemma 8 and with respect to Lemma 7, then these two identities can be joined into the following formula

$$p_n(k,l) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^n \binom{n-i-\frac{k+1}{2}}{i} s_{n-2i}(k,l) \tag{36}$$

for any nonnegative integer n. Now we have to invert (36) to obtain explicit formula for the sums $s_n(k,l)$. To get the assertion we set $a_n = p_n(k,l)$, $b_n = s_n(k,l)$ and $p = -\frac{k+1}{2}$ in Lemma 8. Thus we have

$$s_n(k,l) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^n \left(\binom{\frac{k-1}{2}-n+i}{i} + \binom{\frac{k-3}{2}-n+i}{i-1} \right) p_{n-2i}(k,l).$$

From this relation and (35) we get

$$s_n(k,l) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2i} (-1)^{n+l\,(n-j)}\, \Theta(i,k-2,n)\, \frac{F_{(n-2i-j+1)(k-2l)}}{F_{k-2l}}\, d_j.$$

Finally putting $d_j = (-1)^{\frac{j}{2}(j+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix}$ and after simplifications we have the assertion.

Now we prove relation (9). By the direct multiplication of the factors in (33) we get the identities

$$q_{2n+1}(k,l) = -\sum_{i=0}^{n} \binom{\frac{k-2}{2}-(2i+1)}{n-i} s_{2i+1}(k,l)$$

for $n = 0, 1, 2, \dots, \frac{k-4}{2}$, and

$$q_{2n}(k,l) = \sum_{i=0}^{n} \binom{\frac{k-2}{2}-2i}{n-i} s_{2i}(k,l)$$

for $n = 0, 1, 2, \dots, \frac{k-2}{2}$.

By shifting indexes of summation it is possible to join the previous two identities into the relation

$$q_n(k,l) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^n \binom{\frac{k-2}{2} - n + 2m}{m} s_{n-2m}(k,l)$$

for $n = 0, 1, 2, \ldots, k-2$.

To complete the proof of (9) we have to invert the previous identity. Setting $a_n = q_n(k,l)$, $b_n = s_n(k,l)$ and $t = \frac{k-2}{2}$ in the inverse formula in Lemma 9 we obtain

$$s_n(k,l) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^{n+m} \left(\binom{\frac{k-2}{2} - n + m}{m} + \binom{\frac{k-4}{2} - n + m}{m-1} \right) q_{n-2m}(k,l).$$

From the last identity and identities (32) and (33) we get

$$s_n(k,l) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-2m} \sum_{r=0}^{j} (-1)^{n+m} \Theta(i, k-2, n)$$

$$(-1)^{\frac{k}{2}(n-2m+j)} (-1)^{l(j-r)} \frac{F_{(j-r+1)(k-2l)}}{F_{k-2l}} d_r.$$

Putting $d_r = (-1)^{\frac{r}{2}(r+1)} \left[{k+1 \atop r} \right]$ and after simplifications we finally gain the assertion.

Proof of Theorem 2. The numerator of (14) is a polynomial $N(x)$ of odd degree k. If we use the notation from the beginning of the proof of Theorem 1 particularly identity (34) then the following relation holds for it

$$\begin{aligned} N(x) &= 5^{\frac{1-k}{2}} \sum_{j=0}^{\frac{k-1}{2}} \binom{k}{j} F_{k-2j}\, x\, D_j(x) \\ &= 5^{\frac{1-k}{2}} \sum_{j=0}^{\frac{k-1}{2}} \binom{k}{j} F_{k-2j}\, x \sum_{i=0}^{k-1} p_i(k,j)\, x^i \\ &= \sum_{i=1}^{k} \left(5^{\frac{1-k}{2}} \sum_{j=0}^{\frac{k-1}{2}} \binom{k}{j} F_{k-2j}\, p_{i-1}(k,j) \right) x^i. \end{aligned}$$

Therefore, with respect to (1), for $i = 1, 2, \ldots, k$ we get

$$5^{\frac{1-k}{2}} \sum_{j=0}^{\frac{k-1}{2}} \binom{k}{j} F_{k-2j}\, p_{i-1}(k,j) = \sum_{j=0}^{i-1} d_j\, F_{i-j}^{k}. \tag{37}$$

Using the following identity for the Fibonomial coefficients, see for example [15],

$$\sum_{j=0}^{k+1}(-1)^{\frac{j}{2}(j+1)}\begin{bmatrix}k+1\\ j\end{bmatrix}F^k_{n-j}=0,$$

where n, k are any positive integers such that $n\geq k+1$, it is possible to generalize (37) for any positive integer i. Hence from (37), using (36), we have

$$5^{\frac{1-k}{2}}\sum_{j=0}^{\frac{k-1}{2}}\binom{k}{j}F_{k-2j}\sum_{m=0}^{\lfloor\frac{i-1}{2}\rfloor}(-1)^{i-1}\binom{i-m-\frac{k+3}{2}}{m}s_{i-1-2m}(k,j)=\sum_{j=0}^{i-1}d_j\,F^k_{i-j}$$

and, with respect to (7),

$$\sum_{m=0}^{\lfloor\frac{i-1}{2}\rfloor}(-1)^{i-1}\binom{i-m-\frac{k+3}{2}}{m}\sigma_{i-1-2m}(k)=5^{\frac{k-1}{2}}\sum_{j=0}^{i-1}d_j\,F^k_{i-j}.$$

Setting $n=i-1$ and replacing m by i the latter identity can be rewritten for any nonnegative integer n as

$$h_n(k)=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^n\binom{n-i-\frac{k+1}{2}}{i}\sigma_{n-2i}(k),\tag{38}$$

where we denoted $h_n(k)=5^{\frac{k-1}{2}}\sum\limits_{j=0}^{n}d_j\,F^k_{n+1-j}$. To obtain the assertion we invert (38) using Lemma 8 (similarly as in Theorem 1):

$$\begin{aligned}\sigma_n(k)&=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^n\left(\binom{\frac{k-1}{2}-n+i}{i}+\binom{\frac{k-3}{2}-n+i}{n-i-1}\right)h_{n-2i}(k)\\&=5^{\frac{k-1}{2}}\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{j=0}^{n-2i}(-1)^n\,\Theta(i,k-2,n)\,d_j\,F^k_{n+1-2i-j},\end{aligned}$$

where we put $d_j=(-1)^{\frac{j}{2}(j+1)}\left[{k+1\atop j}\right]j$.

Proof of Theorem 3. As polynomial $N(x)$ in (15) is numerator in (1) we have

$$\begin{aligned}&\sum_{l=0}^{\frac{k}{2}-1}(-1)^l\binom{k}{l}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{l(i+j)}L_{(i-j)(k-2l)}\,d_j\,x^i\\&\qquad+(-1)^{\frac{k}{2}}\binom{k}{\frac{k}{2}}\sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{\frac{k}{2}(j+i)}\,d_j\,x^i=5^{\frac{k}{2}}\sum_{i=0}^{k}\sum_{j=0}^{i}F^k_{i-j}\,d_j\,x^i\end{aligned}$$

for $i = 0, 1, \dots, k$. Using Lemma 15 we get

$$\sum_{j=0}^{i}\Bigg(\sum_{l=0}^{\frac{k}{2}-1}\binom{k}{l}(-1)^{l(i+j+1)}L_{(i-j)(k-2l)} + \binom{k}{\frac{k}{2}}(-1)^{\frac{k}{2}(j+i+1)} - 5^{\frac{k}{2}}\,F_{i-j}^{k}\Bigg)d_j = 0$$

for $i = 0, 1, \dots, k$.

Proof of Corollary 4. Both identities can be obtained directly from the identities in Theorem 1 and in Theorem 2 with respect to $s_n(k,l) = 0$ for each even $k \le 2n-1$ (see Lemma 6).

5. Concluding remark

It is interesting to compare the effectiveness of formulas (8), (9) and (10) in contrast to defining formulas (6) and (7) for computation of $s_n(k,l)$ and $\sigma_n(k)$. Therefore, we found the CPU time (in seconds) required for computation of sums $s_3(k,l)$ for some values of k using the system Mathematica on a standard PC. The measured time is given in Table 1.

TABLE 1. CPU TIME IN SECONDS FOR $s_3(k, \frac{k-1}{4})$

k	101	201	301	401	501
(6)	0.55	4.45	15.28	36.83	73.42
(8)	0.02	0.02	0.05	0.10	0.16

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