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STRONG POINCARÉ RECURRENCE THEOREM IN MV-ALGEBRAS

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Dedicated to Professor Sylvia Pulmannová on the occasion of her 70th birthday

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. The classical Poincaré strong recurrence theorem states that for any probability space $(\Omega, \mathscr{S}, P)$, any P-measure preserving transformation T, and any $A \in \mathscr{S}$, almost all points of A return to A infinitely many times. In the present paper the Poincaré theorem is proved when the σ-algebra $\mathscr{S}$ is substituted by an MV-algebra of a special type. Another approach is used in [RIEČAN, B.: *Poincaré recurrence theorem in MV-algebras*. In: Proc. IFSA-EUSFLAT 2009 (To appear)], where the weak variant of the theorem is proved, of course, for arbitrary MV-algebras. Such generalizations were already done in the literature, e.g. for quantum logic, see [DVUREČENSKIJ, A.: *On some properties of transformations of a logic*, Math. Slovaca **26** (1976), 131–137.

0. Introduction

Let $(\Omega, \mathscr{S}, P)$ be a probability space, i.e. $\Omega \neq \emptyset$, $\mathscr{S}$ is a σ-algebra of subsets of Ω (i.e. $\Omega \in \mathscr{S}$; $A \in \mathscr{S} \implies \Omega \setminus A \in \mathscr{S}$, and $A_n \in \mathscr{S}$ $(n = 1, 2, \dots) \implies \bigcup\limits_{n=1}^{\infty} A_n \in \mathscr{S}$), and $P \colon \mathscr{S} \to [0, 1]$ is such that $P(\Omega) = 1$, and $P\Big(\bigcup\limits_{n=1}^{\infty} A_n\Big) = \sum\limits_{n=1}^{\infty} P(A_n)$, whenever $A_n \in \mathscr{S}$ $(n = 1, 2, \dots)$ and $A_n \cap A_m = \emptyset$ $(n \neq m)$. Let $T \colon \Omega \to \Omega$ be such that $A \in \mathscr{S}$ implies $T^{-1}(A) \in \mathscr{S}$, and

$$P(T^{-1}(A)) = P(A),$$

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whenever $A \in \mathscr{S}$; such transformations T are called measure — preserving. The Poincaré recurrence theorem states that almost every point $x \in A$ will return to A, i.e.

$$P\Big(A \setminus \bigcup_{n=1}^{\infty} T^{-1}(A)\Big) = 0,$$

whenever $A \in \mathscr{S}$. There holds also the stronger variant of the Poincaré theorem: almost all points of A will return to A infinitely many times. It means that to any $x \in A \setminus B$ (where $P(B) = 0$) and any $k \in \mathbb{N}$ there exists $n \geq k$ such that $T^n(x) \in A$:

$$P\Big(A \setminus \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} T^{-n}(A)\Big) = 0.$$

The strong recurrence theorem has been proved in various connections (see [9] for Boolean algebras, [4] for topological spaces, [2] for quantum logics). In the paper the theorem is proved for a type of MV-algebras (σ-complete weakly σ-distributive MV algebras with product).

1. MV-algebras

A prototype of an MV-algebra is the unit interval $[0, 1]$ with two binary operations $\oplus, \odot\colon [0, 1] \times [0, 1] \to [0, 1]$,

$$a \oplus b = \min\{a + b, 1\},$$
$$a \odot b = \max\{a + b - 1, 0\},$$

one unary operation $\neg\colon [0, 1] \to [0, 1]$

$$\neg a = 1 - a,$$

and two fixed elements $0, 1 \in [0, 1]$.

Recall that the operation $\oplus$ corresponds to the disjunction of two assertions (a or b), $\odot$ corresponds to the conjunction of assertion (a and b), $\neg$ corresponds to the negation (not a), 1 corresponds to the true assertion, 0 corresponds to the false assertion. From the point of view of set theory (denote by χ_A the characteristic function, $\chi_A(x) = 1$, if $x \in A$, $\chi_A(x) = 0$, if $x \notin A$), $\oplus$ corresponds to the union of sets,

$$\chi_A \oplus \chi_B = \chi_{A \cup B}$$

$\odot$ corresponds to the intersection of sets,

$$\chi_A \odot \chi_B = \chi_{A \cap B},$$

$\neg$ correspons to the complement

$$\neg \chi_A = 1 - \chi_A,$$

1 coresponds to the basic space Ω

$$1 = \chi_\Omega$$

and 0 to the empty set,

$$0 = \chi_\emptyset.$$

MV-algebras play the same role in the multivalued logic as Booloean algebras in two-valued logic. Generally, the notion of an MV-algebra has been introduced by Chang [1]. By the Mundici theorem ([6]), any MV-algebra can be represented as an interval $[0, u]$ in an l-group G, similarly as the MV-algebra $([0,1], \oplus, \odot. \neg, 1, 0)$ in the group $(\mathbb{R}, +, \leq)$. Recall that an l-group is an algebraic system

$$(G, +, \leq),$$

where

(i) $(G, +)$ is an Abelian group,

(ii) $(G, \leq)$ is a lattice,

(iii) $a \leq b \implies a + c \leq b + c$.

By the Mundici theorem ([6]) any MV-algebra

$$(M, \oplus, \odot, \neg, 1, 0)$$

corresponds to a unique unital l-group

$$(G, +, \leq)$$

where

$$M = [0, u] = \{x \in G : \ 0 \leq x \leq u\},$$

$u \in G$ is a strong unit of G (i.e. to any $x \in G$ there exists $n \in \mathbb{N}$ such that $x \leq nu = u + u + \cdots + u$ (n-times)), 0 is the neutral element of $(G, +)$, and

$$\begin{aligned} a \oplus b &= (a + b) \wedge u, \\ a \odot b &= (a + b - u) \vee 0, \\ \neg a &= u - a. \end{aligned}$$

Analogously as in quantum structures (see [2], [3]), instead of a probability, a state $m \colon M \to [0,1]$ can be defined.

1.1. Definition. A state on an MV-algebra $(M, \oplus, \odot, \neg, u, 0)$ is a mapping $m \colon M \to [0,1]$ satisfying the following conditions:

(i) $m(u) = 1$;

(ii) m is additive, i.e. $a \odot b = 0 \implies m(a \oplus b) = m(a) + m(b)$;

(iii) m is continuous, i.e. $a_n \nearrow a \implies m(a_n) \nearrow m(a)$.

The second basic notion in the classical Kolmogorov theory is the notion of a random variable $\xi\colon \Omega \to \mathbb{R}$, i.e. such a mapping that

$$A \in \mathscr{B}(\mathbb{R}) \implies \xi^{-1}(A) \in \mathscr{S}$$

($\mathscr{B}(\mathbb{R})$ is the least σ-algebra obtaining all intervals in $\mathbb{R}$). Therefore in the general theory (similarly as in Q-structures theory ([2])) we can consider a homomorphism

$$x\colon \mathscr{B}(\mathbb{R}) \to M$$

satisfying some conditions. The mapping $x\colon \mathscr{B}(\mathbb{R}) \to M$ is called an observable.

1.2. Definition. Let M be an MV-algebra. By an observable we understand a mapping $x\colon \mathscr{B}(\mathbb{R}) \to M$ satisfying the following conditions:

(i) $x(\mathbb{R}) = u$;

(ii) $A \cap B = \emptyset \implies x(A) \odot x(B) = 0$, $x(A \cup B) = x(A) \oplus x(B)$;

(iii) $A_n \nearrow A \implies x(A_n) \nearrow x(A)$.

1.3. Theorem. *If M is an MV-algebra, $m\colon M \to [0,1]$ is a state, $x\colon \mathscr{B}(\mathbb{R}) \to M$ is an observable, then the composite map $m \circ x\colon \mathscr{B}(\mathbb{R}) \to [0,1]$ is a probability measure.*

Proof. The proof is evident. □

2. MV-algebras with product

Also from the point of view of applications there is useful to have something similar to the notion of a random vector

$$T = (\xi, \eta)\colon \Omega \to \mathbb{R} \times \mathbb{R}.$$

Again it is a possibility to consider the inverse $h\colon \mathscr{B}(\mathbb{R}^2) \to M$,

$$h(A) = T^{-1}(A),$$

particularly

$$h(C \times D) = T^{-1}(C \times D) = \xi^{-1}(C) \cap \eta^{-1}(D).$$

If we consider two observables $x, y\colon \mathscr{B} \to M$, we need something similar to an operation $M \times M \to M$ such that

$$h(C \times D) = x(C) \cdot y(D).$$

It was realized in [10], where the notion of an MV-algebra with product was introduced. Independently, the notion was introduced in [5] (see also [12]).

2.1. Definition. An MV-algebra with product is an algebraic system $(M, \oplus, \odot, \cdot, \neg, u, 0)$, where $(M, \oplus, \odot, \neg, u, 0)$ is an MV-algebra, and $\cdot\colon M \times M \to M$ is a binary operation satisfying the following identities:

(i) $a \cdot u = a$,

(ii) $a \cdot (b \oplus c) = (a \cdot b) \oplus (a \cdot c)$,

(iii) $a_n \nearrow a \implies a_n \cdot b \nearrow a \cdot b$.

2.2. Definition. Let M be an MV-algebra with product, $x, y\colon \mathscr{B}(\mathbb{R}) \to M$ be two observables. By a joint observable of x, y we mean a mapping $h\colon \mathscr{B}(\mathbb{R}^2) \to M$ satisfying the following conditions:

(i) $h(\mathbb{R} \times \mathbb{R}) = u$;

(ii) $A \cap B = \emptyset \implies h(A) \odot h(B) = 0$, $h(A \cup B) = h(A) \oplus h(B)$;

(iii) $A_n \nearrow A \implies h(A_n) \nearrow h(B)$;

(iv) $h(C \times D) = x(C) \cdot y(D)$ for any $C, D \in \mathscr{B}(\mathbb{R})$.

The main result stated in [10] ([12, Theorem 3.6]) implies the existence of the joint observable. Of course, the corresponding theorem holds only under the following additional condition. It corresponds to the classical ε-technique:

$a_n \to 0 \iff \forall \varepsilon > 0\ \exists n_0\ \forall n \geq n_0 :\ |a_n| < \varepsilon.$

In the general l-group the technique is impossible, it was substituted by the condition of weak regularity what is not only sufficient but also necessary condition for a posibility of the extension of G-valued measure from an algebra to the generated σ-algebra.

2.3. Definition. An l-group G is said to be a weakly σ-distributive l-group, if

$$a_{ij} \searrow 0 \ \ (j \to \infty,\ \ i = 1, 2, 3, \dots) \implies \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \bigvee_{i=1}^{\infty} a_{i\varphi(i)} = 0.$$

The term σ-distributive can be justified by the following change of operations:

$$\bigwedge_{\varphi} \bigvee_{i} a_{i\varphi(i)} = \bigvee_{i} \bigwedge_{j} a_{ij} = \bigwedge 0 = 0.$$

If $G = \mathbb{R}$, then $a_{ij} \searrow 0$ $(j \to \infty)$ implies

$$\forall \varepsilon > 0\ \exists \varphi(i)\ \forall j \geq \varphi(i) :\ a_{ij} < \varepsilon,$$

hence also

$$a_{i\varphi(i)} < \varepsilon \qquad \text{for all} \quad \varphi,$$

and therefore

$$\sup_{i} a_{i\varphi(i)} \leq \varepsilon \qquad \text{for all} \quad \varepsilon > 0.$$

It follows that

$$\inf_{\varphi} \sup_{i} a_{i\varphi(i)} = 0$$

what justifies the previous definition.

2.4. Proposition. *Let (M, m) be a probability MV-algebra, i.e. $m\colon M \to [0,1]$ is such a state that $m(a) = 0 \implies a = 0$.*

Then M is weakly σ-distributive.

Proof. See [12, Theorem 2.1.4]. □

2.5. Theorem. *Let M be a σ-complete, weakly σ-distributive MV-algebra, $x, y\colon \mathscr{B}(\mathbb{R}) \to M$ be two observables. Then there exists a unique joint observable of x, y.*

Proof. See [10, Theorem 2]. □

2.6. Corollary. *Let M be a σ-complete probability MV-algebra, $x, y\colon \mathscr{B}(\mathbb{R}) \to M$ be two observables. Then there exists the unique joint observable of x, y.*

3. Main result

3.1. Definition. Let M be a σ-complete MV-algebra, $m\colon M \to [0,1]$ be a state. By an m-preserving transformation of M we understand a mapping $\tau\colon M \to M$ satisfying the following conditions:

(i) $\tau(u) = u, \tau(0) = 0$;

(ii) $\tau(a \odot b) = \tau(a) \odot \tau(b)$;

(iii) $\tau(a \oplus b) = \tau(a) \oplus \tau(b)$;

(iv) $\tau(a \cdot b) = \tau(a) \cdot \tau(b)$;

(v) $\tau(a \vee b) = \tau(a) \vee \tau(b)$;

(vi) $\tau(a \wedge b) = \tau(a) \wedge \tau(b)$;

(vii) $a_n \nearrow a \implies \tau(a_n) \nearrow \tau(a)$;

(viii) $m(\tau(a)) = m(a)$.

3.2. Theorem. *Let M be a σ-complete, a weakly σ-distributive MV-algebra with product, $x, y\colon \mathscr{B}(\mathbb{R}) \to M$ be two observables. Then*

$$m\Big(\bigwedge_{k=1}^{\infty} a \cdot \Big(\prod_{n=k}^{\infty} \tau^n(\neg a) \Big)\Big) = 0.$$

for any $a \in M$.

Here

$$\prod_{n=k}^{\infty} \tau^n(b) = \bigwedge_{i=1}^{\infty} \prod_{n=k}^{k+i} \tau^n(b),$$

where

$$\prod_{j=l}^{m} b_j = b_l \cdot b_{l+1} \cdots b_m.$$

4. Proof: Kolmogorov construction

Let $a \in M$. First we consider an observable $x\colon \mathscr{B}(\mathbb{R}) \to M$ by the following prescription:

$$x(B) = \begin{cases} 0, & \text{if} \quad 0 \notin B,\ 1 \notin B \\ a, & \text{if} \quad 0 \notin B,\ 1 \in B \\ \neg a, & \text{if} \quad 0 \in B,\ 1 \notin B \\ u, & \text{if} \quad 0 \in B,\ 1 \in B. \end{cases}$$

It is not difficult to see that x is an observable. Moreover, also the mappings $\tau \circ x$, $\tau^2 \circ x$, $\tau^3 \circ x$, $\ldots$ are observables. Applying the natural extension of Theorem 2.5, there exists the joint observable $h_n\colon \mathscr{B}(\mathbb{R}^n) \to M$ of observables x, $\tau \circ x$, $\tau^2 \circ x$, $\ldots$, $\tau^{n-1} \circ x$, hence

$$m(h_n(C_0 \times C_1 \times \cdots \times C_{n-1})) = m(x(C_0) \cdot \tau(x(C_1)) \cdots \tau^{n-1}(x(C_{n-1}))$$

for any $C_0, C_1, \ldots, C_{n-1} \in \mathscr{B}(\mathbb{R})$, $n \in \mathbb{N}$. Evidently

$$m \circ h_{n+1}\big|_{\mathscr{B}(\mathbb{R}^n)} = m \circ h_n.$$

Denote by $\mathscr{C}$ the family of cylinders in $\mathbb{R}^{\mathbb{N}}$,

$$\mathscr{C} = \big\{A \subset \mathbb{R}^{\mathbb{N}} :\ A = \pi_n^{-1}(B),\ B \in \mathscr{B}(\mathbb{R}^n),\ n \in \mathbb{N}\big\}$$

(here $\pi_n\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^n$ is the projection, $\pi((u_i)_i) = (u_1, \ldots, u_n)$). Then by the Kolmogorov consistency theorem there exists exactly one measure on the σ-algebra $\sigma(\mathscr{C})$ generated by $\mathscr{C}$,

$$P\colon \sigma(\mathscr{C}) \to [0,1],$$

such that

$$P \circ \pi_n^{-1} = m \circ h_n, \qquad n = 1, 2, \ldots$$

Define the shift $T\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by the formula

$$T((u_n)_{n=1}^{\infty}) = (v_n)_{n=1}^{\infty}, \qquad v_n = u_{n+1},\ \ n = 1, 2, \ldots$$

We shall show that T is a P-measure preserving map, i.e.

$$P(T^{-1}(A)) = P(A), \qquad A \in \sigma(\mathscr{C}). \tag{4.1}$$

Since $\mu\colon A \mapsto P(T^{-1}(A))$ is a measure, it suffices to prove (4.1) for the sets of the form

$$A = \pi_n^{-1}(C_0 \times C_1 \times \cdots \times C_{n-1}), \qquad n \in \mathbb{N}.$$

We have

$$\begin{aligned} P(A) &= P(\pi_n^{-1}(C_0 \times C_1 \times \cdots \times C_{n-1})) \\ &= m(h_n(C_0 \times C_1 \times \cdots \times C_{n-1})) \\ &= m\big(x(C_0) \cdot \tau(x(C_1)) \cdots \tau^{n-1}(x(C_{n-1}))\big). \end{aligned}$$

On the other hand

$$\begin{aligned} P(T^{-1}(A)) &= P(T^{-1}(C_0 \times C_1 \times \cdots \times C_{n-1} \times \mathbb{R} \times \mathbb{R} \times \cdots)) \\ &= P(\mathbb{R} \times C_0 \times C_1 \times C_{n-1} \times \mathbb{R} \times \cdots) \\ &= P(\pi_{n+1}^{-1}(\mathbb{R} \times C_0 \times C_1 \times \cdots \times C_{n-1})) \\ &= m(h_{n+1}(\mathbb{R} \times C_0 \times C_1 \times \cdots \times C_{n-1})) \\ &= m(x(\mathbb{R}) \cdot \tau(x(C_0)) \cdot \tau^2(x(C_1)) \cdots \tau^n(x(C_{n-1}))) \\ &= m(\tau(x(C_0)) \cdot \tau^2(x(C_1)) \cdots \tau^n(x(C_{n-1}))) \\ &= m\big(x(C_0) \cdot \tau(x(C_1)) \cdots \tau^{n-1}(x(C_{n-1})\big) = P(A). \end{aligned}$$

By the Poincaré recurrence theorem we obtain:

$$\begin{aligned} 0 &= P\bigg(A \setminus \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} T^{-n}(A)\bigg) \\ &= \lim_{k\to\infty} \lim_{i\to\infty} P\bigg(\bigcap_{n=k}^{k+i} A \cap T^{-n}(A^c)\bigg). \end{aligned} \tag{4.2}$$

On the other hand (if $\xi_j\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is the projection to the ith coordinate)

$$\begin{aligned} & m\bigg(a \cdot \prod_{n=k}^{k+i} \tau^n(\neg a)\bigg) \\ &= m(a \cdot u \cdot u \cdots u \cdot \tau_k(\neg a) \cdot \tau^{k+1}(\neg a) \cdots \tau^{k+i}(\neg a)) \\ &= m\big(h_{k+i}\big(\{1\} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \{0\} \times \{0\} \times \cdots \times \{0\}\big)\big) \\ &= P\big(\xi_1^{-1}(\{1\}) \cap \xi_2^{-1}(\mathbb{R}) \cap \cdots \cap \xi_{k-1}^{-1}(\mathbb{R}) \cap \xi_k^{-1}(\{0\}) \cap \cdots \cap \xi_{k+i}^{-1}(\{0\})\big) \\ &= P\big(A \cap \Omega \cap \cdots \cap \Omega \cap T^{-1}(A^c) \cap \cdots \cap T^{-(k+i)}(A^c)\big) \\ &= P\bigg(A \cap \bigcap_{n=k}^{k+i} T^{-n}(A^c)\bigg). \end{aligned}$$

Therefore

$$\begin{aligned} & m\Big(\bigwedge_{k=1}^{\infty} a \cdot \Big(\prod_{n=k}^{\infty} \tau^n(\neg a)\Big)\Big) \\ &= \lim_{k\to\infty} m\Big(a \cdot \Big(\prod_{n=k}^{\infty} \tau^n(\neg a)\Big)\Big) \\ &= \lim_{k\to\infty}\lim_{i\to\infty} m\Big(a \cdot \Big(\prod_{n=k}^{k+i} \tau^n(\neg a)\Big)\Big) \\ &= \lim_{k\to\infty}\lim_{i\to\infty} P\Big(A \cap \bigcap_{n=k}^{k+i} T^{-n}(A^c)\Big) \\ &= P\Big(A \setminus \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} T^{-n}(A)\Big) = 0. \end{aligned}$$

5. Example

Consider a probability space $(\Omega, \mathscr{S}, \mu)$ and the MV-algebra

$$\mathscr{M} = \big\{f\colon \Omega \to [0,1] : \ f \ \text{ is } \mathscr{S}\text{-measurable}\big\},$$

where

$$\begin{aligned} f \oplus g &= \min\{f+g, 1\}, \\ f \odot g &= \max\{f+g-1, 0\}, \\ \neg f &= 1 - f, \end{aligned}$$

$u = 1_\Omega$, $0 = 0_\Omega$. $\mathscr{M}$ is a weakly σ-distributive MV-algebra, since an MV-algebra has the property if and only if the corresponding l-group has the property, and $(\mathbb{R}, +, \leq)$ is a weakly σ-distributive l-group. Let $T\colon \Omega \to \Omega$ be an $\mathscr{S}$-measurable map. Moreover, define $m\colon \mathscr{M} \to [0,1]$ by the formula

$$m(f) = \int_\Omega f \,\mathrm{d}\mu,$$

and

$$\tau(f) = f \circ T.$$

By Theorem 3.2 we obtain that

$$\int_\Omega \inf_{k\geq 1}\Big(f \cdot \Big(\prod_{n=k}^{\infty}(1 - f\circ T^n)\Big)\Big)\,\mathrm{d}\mu = 0,$$

hence μ-almost everywhere

$$\inf_{k\geq 1}\left(f\cdot\left(\prod_{n=k}^{\infty}(1-f\circ T^n)\right)\right)=0.$$

REFERENCES

[1] CHANG, C. C.: *Algebraic analysis of multivalued logics*, Trans. Amer. Math. Soc. **88** (1958), 467–490.

[2] DVUREČENSKIJ, A.: *On some properties of transformations of a logic*, Math. Slovaca **26** (1976), 131–137.

[3] DVUREČENSKIJ, A.—PULMANNOVÁ, S.: *New Trends in Quantum Structures*, Kluwer, Dordrecht, 2000.

[4] MALIČKÝ, P.: *Category version of the Poincaré recurrence theorem*, Topology Appl. **154** (2007), 2709–2713.

[5] MONTAGNA, F.: *An algebraic approach to propositional fuzzy logic*, J. Logic Lang. Inform. **9** (2000), 91–124.

[6] MUNDICI, D.: *Interpretation of AFC^*-algebras in Lukasiewicz sentential calculus*, J. Funct. Anal. **65** (1986), 15–63.

[7] NADKARNI, M. G.: *Basic Ergodic Theory*, Birkhauser Verlag, Basel, 1998.

[8] POINCARÉ, H.: *Les methodes nouvelles de la mecanique classique celeste. Vol. 3*, Gauthiers-Villars, Paris, 1899.

[9] RIEČAN, B.: *A note on the Poincaré recurrence theorem on Boolean rings*, Mat.-Fyz. Časopis **15** (1965), 13–22 (Russian).

[10] RIEČAN, B.: *On the product MV-algebras*, Tatra Mt. Math. Publ. **16** (1999), 143–149.

[11] RIEČAN, B.: *Poincaré recurrence theorem in MV-algebras*. In: Proc. IFSA-EUSFLAT 2009 (To appear).

[12] RIEČAN, B.—MUNDICI, D.: *Probability on MV-algebras*. In: Handbook of Measure Theory. Vol. I, II (E. Pap, ed.), North-Holland, Amsterdam, 2002, pp. 869–909.

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