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SYNAPTIC ALGEBRAS

David J. Foulis

Dedicated to Dr. Sylvia Pulmannová on the occasion of her 70th birthday

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ABSTRACT. A synaptic algebra is both a special Jordan algebra and a spectral order-unit normed space satisfying certain natural conditions suggested by the partially ordered Jordan algebra of bounded Hermitian operators on a Hilbert space. The adjective "synaptic", borrowed from biology, is meant to suggest that such an algebra coherently "ties together" the notions of a Jordan algebra, a spectral order-unit normed space, a convex effect algebra, and an orthomodular lattice.

1. Introduction

Our purpose in this article is introduce and study a class of partially ordered algebraic structures, which we call *synaptic algebras*, that are simultaneously spectral order-unit normed spaces [8] and special Jordan algebras, and that also incorporate convex effect algebras [12] and orthomodular lattices [3, 14]. We have borrowed from biology the adjective 'synaptic', which is derived from the Greek word '*sunaptein*', meaning *to join together*. A synaptic algebra (Definition 1.1 below) is required to satisfy certain natural conditions suggested by an important spacial case, namely the partially ordered Jordan algebra of bounded Hermitian operators on a Hilbert space.

The generalized Hermitian (GH) algebras introduced and studied by Sylvia Pulmannová and the author in [9, 10] are synaptic algebras that satisfy a rather

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strong additional condition on bounded ascending sequences of pairwise commuting elements — see Section 6 below for the details. Example 1.2 below exhibits a commutative synaptic algebra which, in general, fails to be a GH-algebra, showing that synaptic algebras are proper generalizations of GH-algebras. In the sequel, we use the symbols $\mathbb{R}$ and $\mathbb{N}$ for the ordered field of real numbers and the set of positive integers, respectively. Also, we use 'iff' as an abbreviation for 'if and only if', and the symbol ':=' means 'equals by definition'.

Definition 1.1. Let R be a linear associative algebra with unity element 1 over $\mathbb{R}$ and let A be a (real) vector subspace of R. If $a, b \in A$ and $B \subseteq A$, we write $a \, C \, b$ iff a and b commute (i.e. $ab = ba$)[1] and we define

$$C(a) := \{b \in A : \ a \, C \, b\}, \quad C(B) := \bigcap_{b \in B} C(b), \quad \text{and} \quad CC(a) := C(C(a)).$$

The vector space A is a *synaptic algebra* with *enveloping algebra* R iff the following conditions are satisfied:

SA1. A is a partially ordered archimedean real vector space with positive cone $A^+ = \{a \in A : \ 0 \leq a\}$, $1 \in A^+$ is an order unit in A, and $\|\cdot\|$ is the corresponding order-unit norm.[2]

SA2. If $a \in A$ then $a^2 \in A^+$.

SA3. If $a, b \in A^+$, then $aba \in A^+$.

SA4. If $a \in A$ and $b \in A^+$, then $aba = 0 \implies ab = ba = 0$.

SA5. If $a \in A^+$, there exists $b \in A^+ \cap CC(a)$ such that $b^2 = a$.

SA6. If $a \in A$, there exists $p \in A$ such that $p = p^2$ and, for all $b \in A$, $ab = 0 \iff p\,b = 0$.

SA7. If $1 \leq a \in A$, there exists $b \in A$ such that $ab = ba = 1$.

SA8. If $a, b \in A$, $a_1 \leq a_2 \leq a_3 \leq \cdots$ is an ascending sequence of pairwise commuting elements of $C(b)$ and $\lim_{n \to \infty} \|a - a_n\| = 0$, then $a \in C(b)$.

We define $P := \{p \in A : \ p = p^2\}$. Elements $p \in P$ are called *projections*. We define the *unit interval* E in A by $E := \{e \in A : \ 0 \leq e \leq 1\}$. Elements $e \in E$ are called *effects*.[3]

If R is a von Neumann algebra, then the real vector space A of all self-adjoint elements in R is a synaptic algebra. More generally, the self-adjoint elements in a Rickart C*-algebra ([13, §3]), and in particular in an AW*-algebra ([15]), form a synaptic algebra. Additional examples of synaptic algebras are: JW-algebras

[1]We understand that a product of elements of A is the product as calculated in R, which may or may not belong to A.

[2]See Definition 1.6 below.

[3]Actually, E is a so-called *convex effect algebra*([12]).

([17]), AJW-algebras ([17, § 20]), JB-algebras ([2]), and the ordered special Jordan algebras studied by Sarymsakov, *et al.* [16]. All the foregoing examples are norm complete, but the commutative synaptic algebra in the following example need not be norm complete.

Example 1.2. Let $\mathscr{F}$ be a field of subsets of a nonempty set X, let A be the commutative and associative real linear algebra, with pointwise operations, of all functions $f\colon X \to \mathbb{R}$ such that

(i) $\lambda \in \mathbb{R} \implies f^{-1}(\lambda) \in \mathscr{F}$ and

(ii) $\{f(x) : x \in X\}$ is finite.

Then, with the pointwise partial order, A is a synaptic algebra with A as its own enveloping algebra. The projections in A are the characteristic set functions (indicator functions) of sets in $\mathscr{F}$.

Standing Assumptions 1.3. In the sequel, we assume that A is a synaptic algebra with enveloping algebra[4] R, that E is the set of effects in A, and that P is the set of projections in A. We understand that both E and P are partially ordered by the restrictions of the partial order $\leq$ on A. To avoid triviality, we assume that $1 \neq 0$. As is customary, we shall identify each real number $\lambda \in \mathbb{R}$ with the element $\lambda 1 \in A$, so that $\mathbb{R}$ is a one-dimensional linear subspace of A. If n is one of $1, 2, \ldots, 8$, then [SAn] will always refer to the corresponding condition in Definition 1.1.

By [SA2], $a \in A \implies a^2 \in A$, hence A is organized into a special Jordan algebra under the Jordan product $a \circ b := \frac{1}{2}(ab + ba) = \frac{1}{2}[(a+b)^2 - a^2 - b^2] \in A$ for all $a, b \in A$. Clearly, $1 \circ a = a \circ 1 = a$, i.e., A is a *unital* Jordan algebra.

Remarks 1.4. Let $a, b, c \in A$. Then $a\,C\,b \implies ab = ba = a \circ b \in A$. As $a^2 \in A$ and $a\,C\,a^2$, it follows that $a^3 = a \circ a^2 \in A$, and by induction, $a^n \in A$ for all $n \in \mathbb{N}$. Consequently, A is closed under the formation of real polynomials in a. Let $c := 2(a \circ b)$. Then $aba = a \circ c - a^2 \circ b \in A$, hence $aba \in A$. Thus, $acb + bca = (a + b)c(a + b) - aca - bcb \in A$.

Lemma 1.5. *If $a, b \in A^+$ and $a\,C\,b$, then $ab = ba \in A^+$.*

Proof. Assume that $a, b \in A^+$ and $a\,C\,b$. By Remarks 1.4, $ab = ba \in A$. By [SA5], there exist $x \in A^+ \cap CC(a)$ and $y \in A^+ \cap CC(b)$ such that $a = x^2$ and $b = y^2$. As $x \in CC(a)$ and $a\,C\,b$, we have $x\,C\,b$; hence, as $y \in CC(b)$, it follows that $x\,C\,y$. Therefore, $xy = yx \in A$ by Remarks 1.4, and we have $(xy)^2 = x^2y^2 = ab$. Consequently, $ab \in A^+$ by [SA2]. □

[4]We shall not be concerned with the detailed structure of the enveloping algebra R — we regard R merely as an arena in which to study A, E, and P.

By [SA1], A is an *order-unit normed space* according to the following definition (adapted to our present notation).

Definition 1.6. An *order-unit normed space* [1, pp. 67–69] is a partially ordered real vector space A with a distinguished element $1 \in A$, called the *unit*, such that:

(i) A is *archimedean*, i.e., if $a, b \in A$ and $na \leq b$ for all $n \in \mathbb{N}$, then $-a \in A^+$.

(ii) $0 < 1$ and 1 is an *order unit*[5] in A, i.e., for every $a \in A$, there exists $n \in \mathbb{N}$ such that $a \leq n$.[6]

The *order-unit norm* $\|\cdot\|$ on A is defined by

(iii) $\|a\| := \inf\{\lambda \in \mathbb{R} : \ 0 < \lambda \text{ and } -\lambda \leq a \leq \lambda\}$.

The order-unit norm $\|\cdot\|$ is a *bona fide* norm on A, and it is related to the partial-order structure of A by the following properties,[7] which we shall use routinely in the sequel: For all $a, b \in A$,

$$-\|a\| \leq a \leq \|a\|, \quad \text{and} \quad \text{if } -b \leq a \leq b, \text{ then } \|a\| \leq \|b\|.$$

If $(a_n)_{n\in\mathbb{N}}$ is a sequence in A and $a \in A$, the notation $\lim\limits_{n\to\infty} a_n = a$, or simply $a_n \to a$, will mean that a is the limit of $(a_n)_{n\in\mathbb{N}}$ in the norm topology, i.e., that $\lim\limits_{n\to\infty} \|a - a_n\| = 0$.

Lemma 1.7. *Let $a, b \in A$ and $0 < \lambda \in \mathbb{R}$. Then:*

(i) $-\lambda \leq a \leq \lambda \iff a^2 \leq \lambda^2$.

(ii) $\|a^2\| = \|a\|^2$.

(iii) $0 \leq a, b \implies \|a - b\| \leq \max\{\|a\|, \|b\|\}$.

(iv) $\|a \circ b\| \leq \|a\|\|b\|$.

(v) *If $a\, C\, b$, then $\|ab\| \leq \|a\|\|b\|$.*

Proof. If $-\lambda \leq a \leq \lambda$, then $0 \leq \lambda - a, \lambda + a$, and as $(\lambda - a)\, C\, (\lambda + a)$, Lemma 1.5 implies that $0 \leq (\lambda - a)(\lambda + a) = \lambda^2 - a^2$. Conversely, suppose that $a^2 \leq \lambda^2$. Then $0 \leq (\lambda - a)^2$ by [SA2], whence $0 \leq (\lambda^2 - a^2) + (\lambda - a)^2 = 2(\lambda^2 - \lambda a)$ and since $0 < \lambda$, it follows that $a \leq \lambda$. As $a^2 \leq \lambda^2$, we also have $(-a)^2 \leq \lambda^2$, whence $-a \leq \lambda$, i.e., $-\lambda \leq a$, proving (i).

Part (ii) follows from (i).

To prove (iii), we can assume that $\|a\| \leq \|b\|$. As $0 \leq b$, we have $a \leq \|a\| \leq \|b\| \leq \|b\| + b$, whence $a - b \leq \|b\|$. Also, as $0 \leq a$, we have $b \leq \|b\| \leq \|b\| + a$, whence $b - a \leq \|b\|$, and therefore $-\|b\| \leq a - b \leq \|b\|$. Consequently, $\|a - b\| \leq \|b\| = \max\{\|a\|, \|b\|\}$. To prove (iv), it will be sufficient by normalization to

[5] Some authors use the terminology "strong order unit".

[6] Recall that we are identifying $n \in \mathbb{N} \subseteq \mathbb{R}$ with $n1$.

[7] See [1, Proposition II.1.2] and [11, Proposition 7.12 (c)]

prove that $\|a\| = \|b\| = 1 \implies \|a \circ b\| \leq 1$. Thus, we assume $\|a\| = \|b\| = 1$, so that $\|a \pm b\| \leq 2$, and therefore by (ii), $\|(a \pm b)^2\| \leq 4$. Consequently, by (iii),

$$\|a \circ b\| = \frac{1}{4}\|(a+b)^2 - (a-b)^2\| \leq \frac{1}{4}\max\{\|(a+b)^2\|, \|(a-b)^2\|\} \leq 1.$$

If $a\, C\, b$, then $ab = a \circ b$, so (v) follows immediately from (iv). $\square$

2. Square roots, projections, and carriers

Remarks 2.1. Let $a, b \in A$. Then:

(i) By [SA4] with $b = 1$, we have $a^2 = 0 \implies a = 0$.

(ii) If $0 \leq a, b$ and $a + b = 0$, then $0 \leq a = -b \leq 0$, whence $a = b = 0$.

Theorem 2.2. *Let $0 \leq a \in A$. Then there exists a unique $r \in A$ such that $0 \leq r$ and $r^2 = a$; moreover, $r \in CC(a)$.*

Proof. Suppose that $0 \leq a \in A$. By [SA5], there exists $b \in CC(a)$ such that $0 \leq b$ and $b^2 = a$. As $a \in C(a)$, we have $a\, C\, b$. Suppose also that $r \in A$ with $0 \leq r$, $r^2 = a$. Obviously, $r\, C\, a$, whence $b\, C\, r$. It will be sufficient to prove that $r = b$.

By [SA5], there exists $s \in CC(b)$ such that $0 \leq s$ and $s^2 = b$. As $b, r \in C(b)$, we have $s\, C\, b$ and $s\, C\, r$. By [SA5] again, there exists $t \in CC(r)$ such that $0 \leq t$ and $t^2 = r$. As $b, r \in C(r)$, we have $t\, C\, b$ and $t\, C\, r$.

Since $s\, C\, b$ and $s\, C\, r$, it follows that $s\, C\, (b-r)$, hence $s(b-r) = s \circ (b-r) \in A$. Likewise, since $t\, C\, b$ and $t\, C\, r$, we have $t(b-r) \in A$. Moreover, as $b^2 = r^2 = a$, it follows that

$$(s(b-r))^2 + (t(b-r))^2 = (s^2+t^2)(b-r)^2 = (b+r)(b-r)^2 = (b^2-r^2)(b-r) = 0.$$

But $0 \leq (s(b-r))^2$ and $0 \leq (t(b-r))^2$ by [SA2], whence $(s(b-r))^2 = (t(b-r))^2 = 0$, so $s(b-r) = t(b-r) = 0$ by Remarks 2.1.

As $s(b-r) = 0$, it follows that $b(b-r) = s^2(b-r) = 0$. Likewise, $r(b-r) = t^2(b-r) = 0$, whence $(b-r)^2 = b(b-r) - r(b-r) = 0$, and by Remarks 2.1.(i), $r = b$. $\square$

If $0 \leq a \in A$, then of course, the unique element r in Theorem 2.2 is called the *square root* of a, and in what follows *we denote it in the usual way as* $a^{1/2}$.

Remarks 2.3. Let $p \in P$. Then, as $p = p^2$, [SA2] implies that $0 \leq p$. Also, $(1-p)^2 = 1 - 2p + p^2 = 1 - p$, so $1 - p \in P$, and therefore $0 \leq 1 - p$, i.e., $p \leq 1$. Consequently, $0 \leq p \leq 1$, and it follows that $P \subseteq E$.

Theorem 2.4. *Let $e \in E$ and $p \in P$. Then the following conditions are mutually equivalent:*

(i) $e \leq p$.

(ii) $e = ep = pe$.

(iii) $e = pep$.

(iv) $e = ep$.

(v) $e = pe$.

Proof.

(i) $\Longrightarrow$ (ii). Assume that $e \leq p$ and let $d := p - e$. Then $0 \leq e, d, 1-p$, $e + d = p$, and

$$(1-p)e(1-p) + (1-p)d(1-p) = (1-p)p(1-p) = 0.$$

By [SA3], $0 \leq (1-p)e(1-p)$, $(1-p)d(1-p)$, and it follows from Remarks 2.1.(ii) that $(1-p)e(1-p) = (1-p)d(1-p) = 0$. Therefore, by [SA4], $(1-p)e = e(1-p) = 0$, i.e., $e = pe = ep$.

(ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv). Follows from $p = p^2$.

(iv) $\Longleftrightarrow$ (v). By [SA4], $e = ep \implies e(1-p) = 0 \implies (1-p)e(1-p) = 0 \implies (1-p)e = 0 \implies e = pe$, and the converse implication follows by symmetry.

(v) $\Longrightarrow$ (i). Assume (v). Since (iv) $\Longleftrightarrow$ (v), we have $pe = ep = e$, so $(1-e)p = p(1-e) = p - e$, whence $0 \leq p - e$ by Lemma 1.5, and therefore $e \leq p$. □

Lemma 2.5. *Let $e \in E$. Then:*

(i) $e^2 \in E$ *with* $0 \leq e^2 \leq e$.

(ii) $2e - e^2 \in E$.

(iii) $e - e^2 \in E$ *with* $e - e^2 \leq e,\ 1 - e$.

Proof. By [SA2], $0 \leq e^2$, and as $eC(1-e)$ with $0 \leq e, 1-e$, Lemma 1.5 implies that $0 \leq e(1-e)$, whence $0 \leq e^2 \leq e \leq 1$, proving (i). Also, $0 \leq (1-e)^2 = 1 - 2e + e^2$, so by (i), $0 \leq e + (e - e^2) = 2e - e^2 \leq 1$, proving (ii). Part (iii) follows from (i) and (ii). □

Obviously, E is a convex set, and by Remarks 2.3, $P \subseteq E$. The following theorem characterizes, in various ways, those effects $p \in E$ that are projections.

Theorem 2.6. *If $p \in E$, then the following conditions are mutually equivalent:*

(i) $p \in P$.

(ii) *If $\lambda \in \mathbb{R}$, $0 < \lambda < 1$, and $e \in E$, then $\lambda e \leq p \iff e \leq p$.*

(iii) *p is an extreme point of the convex set E.*

(iv) *If $e, f, e+f \in E$, then $e, f \leq p \implies e+f \leq p$.*

(v) *If $e \in E$ and $e \leq p, 1-p$, then $e = 0$.*

P r o o f.

(i) $\Longrightarrow$ (ii). Suppose $p \in P$, $e \in E$, and $0 < \lambda < 1$. Then $0 \leq \lambda e \leq e \leq 1$, so $\lambda e \in E$. Therefore, by Theorem 2.4, $\lambda e \leq p \iff \lambda ep = \lambda e \iff ep = e \iff e \leq p$.

(ii) $\Longrightarrow$ (iii) Assume (ii) and suppose that $p = \lambda e + (1-\lambda)f$ with $0 < \lambda < 1$ and $e, f \in E$. Then $\lambda e \leq p$, whence $e \leq p = \lambda e + (1-\lambda)f$, therefore $(1-\lambda)e \leq (1-\lambda)f$, and it follows that $e \leq f$. Similarly, $f \leq e$, so $e = f = p$.

(iii) $\Longrightarrow$ (i) Assume (iii). By parts (i) and (ii) of Lemma 2.5, p^2, $2p - p^2 \in E$, and since $p = \frac{1}{2}p^2 + \frac{1}{2}(2p - p^2)$, (iii) implies that $p = p^2 = 2p - p^2$, whence $p \in P$.

(i) $\Longrightarrow$ (iv) Assume that $p \in P$, $e, f, e+f \in E$, and $e, f \leq p$. Then by Theorem 2.4, $e = pep$ and $f = pfp$. As $e + f \in E$, we have $0 \leq 1 - (e+f)$, whence by [SA3], $0 \leq p(1 - e - f)p$, i.e., $e + f = pep + pfp \leq p^2 = p$.

(iv) $\Longrightarrow$ (v) Assume (iv) and suppose that $e \in E$ with $e \leq p, 1-p$. Then $e, p \in E$, $0 \leq e + p \leq 1$, and $e, p \leq p$, whence $e + p \leq p$ by (iv), and therefore $e \leq 0$. But $0 \leq e$, so $e = 0$.

(v) $\Longrightarrow$ (i) Assume (v). By Lemma 2.5.(iii), $0 \leq p - p^2 \leq p, 1-p$, whence $p = p^2$ by (v). □

Theorem 2.7. *Let $a \in A$. Then there exists a unique projection $p \in P$ such that, for all $b \in A$, $ab = 0 \iff pb = 0$.*

P r o o f. By [SA6], there exists $p \in P$ such that, for all $b \in A$, $ab = 0 \iff pb = 0$. Suppose $q \in P$ and, for all $b \in A$, $ab = 0 \iff qb = 0$. Putting $b = 1 - p$, we find that $a(1-p) = 0$, whence $q(1-p) = 0$, i.e., $q = qp$, and therefore $q \leq p$ by Theorem 2.4. By symmetry, $p \leq q$, so $p = q$, proving the uniqueness of p. □

Definition 2.8. If $a \in A$, then the unique projection p in Theorem 2.7 is called the *carrier projection* of (or for) a and is denoted by a^{o}. Thus, $a^{\mathrm{o}} \in P$ and, for all $b \in A$, $ab = 0 \iff a^{\mathrm{o}}b = 0$.

Lemma 2.9. *Let $a, b \in A$ and $p \in P$. Then:*

(i) $pb = 0 \iff bp = 0$.

(ii) $pa = a \iff ap = a$.

(iii) $aa^{\mathrm{o}} = a^{\mathrm{o}}a = a$.

(iv) $ab = 0 \iff ba = 0$.

Proof. By [SA4] and the fact that $0 \le p$, we have $p\,b = 0 \implies bp\,b = 0 \implies bp = 0$, whence $p\,b = 0 \implies bp = 0$. A similar argument yields the converse, proving (i). By (i), $pa = a \iff (1-p)a = 0 \iff a(1-p) = 0 \iff ap = a$, proving (ii). As $a^{\circ} \in P$, we have $a^{\circ}(1-a^{\circ}) = 0$, so $a(1-a^{\circ}) = 0$, i.e., $aa^{\circ} = a$, whence $a^{\circ}a = a$ by (ii), proving (iii). To prove (iv), assume that $ab = 0$. Then $a^{\circ}b = 0$, so $ba^{\circ} = 0$ by (i). Also, $a = a^{\circ}a$ by (iii), whereupon $ba = ba^{\circ}a = 0$. Thus, $ab = 0 \implies ba = 0$, and the converse follows by symmetry. □

Theorem 2.10. *Let $a, b \in A$. Then:*

(i) $a = 0 \iff a^{\circ} = 0$.

(ii) $a \in P \iff a = a^{\circ}$.

(iii) *a° is the smallest projection $p \in P$ such that $a = ap$.*

(iv) *If $e \in E$, then e° is the smallest projection $p \in P$ such that $e \le p$.*

(v) $ab = 0 \iff ab^{\circ} = 0 \iff a^{\circ}b^{\circ} = 0$.

(vi) $a^{\circ} \in CC(a)$.

(vii) *If $n \in \mathbb{N}$, then $(a^n)^{\circ} = a^{\circ}$.*

(viii) *If $0 \le a \le b$, then $a^{\circ} \le b^{\circ}$.*

Proof.

(i) and (ii) are obvious from the definition of a°.

(iii) We have $aa^{\circ} = a$ by Lemma 2.9.(iii). Suppose that $p \in P$ and $a = ap$. Then $a(1-p) = 0$, whence $a^{\circ}(1-p) = 0$, so $a^{\circ} = a^{\circ}p$, and therefore $a^{\circ} \le p$ by Theorem 2.4.

(iv) Part (iv) is a consequence of (iii) and Theorem 2.4.

(v) By Lemma 2.9.(iv),

$$ab = 0 \iff ba = 0 \iff b^{\circ}a = 0 \iff ab^{\circ} = 0 \iff a^{\circ}b^{\circ} = 0.$$

(vi) Suppose that $c \in C(a)$ and let $d := (1-a^{\circ})ca^{\circ} + a^{\circ}c(1-a^{\circ})$. Thus, $d \in A$ (see Remarks 1.4), and as $aa^{\circ} = a$, we have

$$ad = a(1-a^{\circ})ca^{\circ} + aa^{\circ}c(1-a^{\circ}) = 0 + ac(1-a^{\circ}) = ca(1-a^{\circ}) = 0,$$

and therefore

$$0 = a^{\circ}d = 0 + a^{\circ}c(1-a^{\circ}) = a^{\circ}c - a^{\circ}ca^{\circ}, \qquad \text{i.e.,} \quad a^{\circ}c = a^{\circ}ca^{\circ}.$$

Also, as $a^{\circ}d = 0$, Lemma 2.9 implies that $0 = da^{\circ} = (1-a^{\circ})ca^{\circ}$, i.e., $ca^{\circ} = a^{\circ}ca^{\circ}$. Therefore $ca^{\circ} = a^{\circ}ca^{\circ} = a^{\circ}c$, so $c \in C(a^{\circ})$.

(vii) Let $n \in \mathbb{N}$. As $aa^{\circ} = a$, we have $a^n a^{\circ} = a^n$, whence $(a^n)^{\circ} \le a^{\circ}$ by (iii). We have to prove that $a^{\circ} \le (a^n)^{\circ}$. Put $q := 1 - (a^n)^{\circ}$. By (vi), $C(a^n) \subseteq C((a^n)^{\circ})$, whence $a\,C\,q$. Evidently, $a^n q = 0$, so there is a smallest positive integer k such that $a^k q = 0$. If k is even, then $a^{k/2}qa^{k/2} = 0$, so $a^{k/2}q = 0$ by [SA4], contradicting the minimality of k. Therefore, k is odd and $a^{k+1}q = 0$, whence $a^{(k+1)/2}qa^{(k+1)/2} = 0$, so $a^{(k+1)/2}q = 0$ by [SA4] again, whereupon $k \leq (k+1)/2$, i.e., $k = 1$. Therefore, $aq = 0$, whence $a = a(a^n)^{\text{o}}$, and again by (iii), $a^{\text{o}} \leq (a^n)^{\text{o}}$.

(viii) Suppose that $0 \leq a \leq b$. The case $b = 0$ is trivial, so we assume that $b \neq 0$. Let $\lambda := \|b\|^{-1}$, $e := \lambda a$, and $f := \lambda b$. Clearly, $e, f \in E$, $e \leq f$, $e^{\text{o}} = a^{\text{o}}$, and $f^{\text{o}} = b^{\text{o}}$. By (iv), $e \leq f \leq f^{\text{o}} \in P$, whence $e^{\text{o}} \leq f^{\text{o}}$, i.e., $a^{\text{o}} \leq b^{\text{o}}$. □

3. Absolute value and polar decomposition

If $a \in A$, then by [SA2], $0 \leq a^2$, so we can formulate the following definition.

Definition 3.1. If $a \in A$, then the *absolute value* of a is defined and denoted by $|a| := (a^2)^{1/2}$. Also we define $a^+ := \frac{1}{2}(|a| + a)$ and $a^- := \frac{1}{2}(|a| - a)$.

Remarks 3.2. Let $a \in A$. Obviously, $0 \leq |a| = |-a|$ and $|a|^2 = a^2$. Also, $C(a) \subseteq C(a^2) \subseteq C(|a|)$, and therefore $|a|, a^+, a^- \in CC(a)$. Moreover, $a = a^+ - a^-$, $|a| = a^+ + a^-$, $a^+a^- = a^-a^+ = 0$, and $a^- = (-a)^+$.

Theorem 3.3. *Let $a \in G$, $p := (a^+)^{\text{o}}$, and $q := (a^-)^{\text{o}}$. Then:*

(i)	$p, q \in CC(a)$	(ii)	$pC\|a\|$ *and* $qC\|a\|$
(iii)	$pa = ap = a^+$.	(iv)	$qa = aq = -a^-$.
(v)	$0 \leq p\|a\| = \|a\|p = a^+$.	(vi)	$0 \leq q\|a\| = \|a\|q = a^-$.
(vii)	$pq = qp = 0$.	(viii)	$p + q = a^{\text{o}}$.

P r o o f.

(i) As $C(a) \subseteq C(a^+)$ and $C(a^+) \subseteq C((a^+)^{\text{o}})$, we have $C(a) \subseteq C(p)$. Likewise, as $a^- = (-a)^+$ and $C(a) = C(-a)$, we have $C(a) \subseteq C(q)$.

(ii) As $|a| \in C(a)$, (ii) follows from (i).

(iii) By (i), $pa = ap$. Also, $a^+ = (a^+)^{\text{o}}a^+ = pa^+$, and since $a^+a^- = 0$, it follows that $pa^- = 0$, whence $pa = p(a^+ - a^-) = a^+$.

(iv) By (iii), $-qa = q(-a) = (-a)^+ = a^-$.

(v) By (ii), $p\,C\,|a|$, and as in the proof of (iii), $p|a| = p(a^+ + a^-) = a^+$. As $0 \leq p$ and $0 \leq |a|$, we have $0 \leq p|a|$ by Lemma 1.5.

(vi) The proof of (vi) is similar to the proof of (v).

(vii) As $a^+a^- = 0$, we have $pa^- = 0$, whence $pq = 0$.

(viii) By (vii), $(p+q)^2 = p^2 + q^2 = p + q$, so $p + q \in P$. By (iii) and (iv), $a(p+q) = a^+ - a^- = a$, whence $a^{\text{o}} \leq p + q$ by Theorem 2.10.(iii). Let $r := 1 - a^{\text{o}}$. Then $r \in P$ and $0 = ar = a^+r - a^-r$, i.e., $a^+r = a^-r$. Consequently, $a^+r = pa^+r = pa^-r = pqa^-r = 0$ by (vii), and it follows that $pr = 0$. Likewise, $qr = 0$, so $(p+q)r = 0$, and therefore $(p+q)a^{\text{o}} = (p+q)(1-r) = p + q$, i.e., $p + q \leq a^{\text{o}}$ by Theorem 2.4; hence $p + q = a^{\text{o}}$. □

Corollary 3.4. *If $0 \le a, b \in A$ and $a\, C\, b$, then $a^2 \le b^2 \iff a \le b$.*

Proof. Assume the hypotheses and suppose that $a^2 \le b^2$. As $0 \le (b-a)^2$, we have

$$0 \le (b-a)^2 + b^2 - a^2 = 2(b^2 - ab), \qquad \text{whence} \quad 0 \le (b-a)b. \tag{1}$$

Also, by parts (vii), (viii), and (iii) of Theorem 2.10,

$$a^{\mathrm{o}} = (a^2)^{\mathrm{o}} \le (b^2)^{\mathrm{o}} = b^{\mathrm{o}}, \qquad \text{whence} \quad ab^{\mathrm{o}} = a. \tag{2}$$

Let $c := (b-a)^+$ and $d := (b-a)^-$. Then by Remarks 3.2 and parts (v) and (vi) of Theorem 3.3, $b \in C(b-a) \subseteq C(c) \cap C(d)$, and we have

$$b\, C\, c,\ b\, C\, d,\ c\, C\, d,\ 0 \le c,\ 0 \le d,\ dc = 0, \text{ and } b - a = c - d. \tag{3}$$

By (1) and (3),

$$0 \le (b-a)b = (c-d)b = cb - db. \tag{4}$$

Since $d\, C\, (cb - db)$ and $0 \le d$, it follows from (4), (3), and Lemma 1.5 that $0 \le d(cb - db) = -d^2 b$, i.e., $d^2 b \le 0$. Likewise, as $0 \le d^2$, $0 \le b$, and $b\, C\, d^2$, we also have $0 \le d^2 b$; hence $d^2 b = 0$, and consequently

$$d^{\mathrm{o}} b = (d^2)^{\mathrm{o}} b = 0, \quad \text{so } db = 0, \quad \text{whence } db^{\mathrm{o}} = 0. \tag{5}$$

As $c \in C(b) \subseteq C(b^{\mathrm{o}})$, $0 \le c$, and $0 \le b^{\mathrm{o}}$, we have $0 \le cb^{\mathrm{o}}$ by Lemma 1.5, whence by (5), (3), and (2),

$$0 \le cb^{\mathrm{o}} = (c-d)b^{\mathrm{o}} = (b-a)b^{\mathrm{o}} = bb^{\mathrm{o}} - ab^{\mathrm{o}} = b - a.$$

Conversely, suppose that $a \le b$, i.e., $0 \le b - a$. As $a\, C\, b$, we have $a\, C\, (b-a)$, and it follows from Lemma 1.5 that $0 \le a(b-a) = ab - a^2$, i.e., $a^2 \le ab$. Similarly, $0 \le (b-a)b = b^2 - ab$, whence $ab \le b^2$, and it follows that $a^2 \le b^2$. □

Definition 3.5. If $a \in A$, then the *signum* of a is defined and denoted by $\operatorname{sgn}(a) := (a^+)^{\mathrm{o}} - (a^-)^{\mathrm{o}}$.

Theorem 3.6. *Let $a \in A$. Then:*

(i) $\operatorname{sgn}(a) \in CC(a)$.

(ii) $\operatorname{sgn}(a)^2 = a^{\mathrm{o}}$.

(iii) $\operatorname{sgn}(a)a = a\operatorname{sgn}(a) = |a|$.

(iv) $\operatorname{sgn}(a)|a| = |a|\operatorname{sgn} a = a$.

Proof. By Theorem 3.3.(i), $C(a) \subseteq C((a^+)^{\mathrm{o}}) \cap C((a^-)^{\mathrm{o}})$, from which (i) follows. Part (ii) follows from parts (vii) and (viii) of Theorem 3.3, and parts (iii) and (iv) are consequences of parts (iii) and (iv) of Theorem 3.3. □

The formula $a = \operatorname{sgn}(a)|a| = |a|\operatorname{sgn}(a)$ in Theorem 3.6 is called the *polar decomposition* of a.

Corollary 3.7. *Let $a, b \in A$. Then:*

(i) $ab = 0 \iff |a||b| = 0$.

(ii) $|a|^{\mathrm{o}} = a^{\mathrm{o}}$.

Proof. We have $ab = 0 \implies |a||b| = \mathrm{sgn}(a)ab\,\mathrm{sgn}(b) = 0$, and conversely, $|a||b| = 0 \implies ab = \mathrm{sgn}(a)|a||b|\,\mathrm{sgn}(b) = 0$, proving (i). Arguing as above, we find that $|a|b = 0 \iff ab = 0$, whence $|a|^{\mathrm{o}} = a^{\mathrm{o}}$, proving (ii). $\square$

4. Quadratic, compression, and Sasaki mappings

Definition 4.1. If $a \in A$, the mapping $J_a\colon A \to A$ defined by $J_a(b) := aba$ for all $b \in A$ is called the *quadratic mapping* determined by a. If $p \in P$, the quadratic mapping J_p is called the *compression* on A with *focus* p.

Theorem 4.2. *If $a \in A$, then the quadratic mapping $J_a\colon \to A$ is both linear and order preserving.*

Proof. Obviously, J_a is linear. Suppose that $0 \le h \in A$. By [SA3], $0 \le |a|h|a|$, and we define $k := (|a|h|a|)^{1/2}$. Thus, $k^2|a|^{\mathrm{o}} = |a|h|a||a|^{\mathrm{o}} = |a|h|a| = k^2$, so by (ii) and parts (vii) and (iii) of Theorem 2.10,

$$k^{\mathrm{o}} = (k^2)^{\mathrm{o}} \le |a|^{\mathrm{o}} = a^{\mathrm{o}}, \qquad \text{whence} \quad ka^{\mathrm{o}} = k. \tag{1}$$

Let $w := \mathrm{sgn}(a)$. Then by parts (ii) and (iv) of Theorem 3.6, $w^2 = a^{\mathrm{o}}$ and $a = w|a| = |a|w$; hence by (1)

$$0 \le (wkw)^2 = wkw^2kw = wka^{\mathrm{o}}kw = wk^2w = w|a|h|a|w = aha = J_a(h).$$

Suppose $b, c \in A$ with $b \le c$, and put $h := c - b$. Then $0 \le h$, therefore $0 \le J_a(h) = J_a(c) - J_a(b)$, whence $J_a(b) \le J_a(c)$, i.e., J_a is order preserving. $\square$

Remark 4.3. Condition [SA3] requires that $a, b \in A^+ \implies aba \in A^+$; however, by Theorem 4.2, we now have the stronger result $b \in A^+ \implies aba \in A^+$ *for all* $a \in A$.

Lemma 4.4. *Let $a, b \in A$ and $p \in P$. Then:*

(i) $\|J_a(b)\| \le \|a^2\|\|b\| = \|a\|^2\|b\|$.

(ii) *$J_a\colon A \to A$ is norm continuous.*

(iii) *If $p \ne 0$, then $\|p\| = 1$.*

(iv) $\|J_p(a)\| \le \|a\|$.

Proof. As $-\|b\| \leq b \leq \|b\|$, we have

$$-\|b\|a^2 = a(-\|b\|)a \leq aba \leq a\|b\|a = \|b\|a^2,$$

whence $\|aba\| \leq \|(\|b\|a^2)\| = \|a^2\|\|b\|$. By Lemma 1.7.(ii), $\|a^2\| = \|a\|^2$, proving (i), and (ii) follows from (i). Also by Lemma 1.7.(ii), $\|p\|^2 = \|p^2\| = \|p\|$, from which (iii) follows, and (iv) is a consequence of (i) and (iii). □

Let $p \in P$ and $e \in E$. By Theorem 4.2, J_p is linear and order preserving. Clearly, $J_p(1) = p \in P \subseteq E$. By Theorem 2.4, $e \leq p \implies J_p(e) = e$. Also, if $J_p(e) = 0$, then $pep = 0$, whence $pe = ep = 0$, so $e \leq 1 - p$. Conversely, as a consequence of [4, Corollary 4.6], compressions on A are characterized as in the following theorem.

Theorem 4.5. *Let $J\colon A \to A$ be a linear and order-preserving mapping such that $J(1) \leq 1$ and, for every $e \in E$, $e \leq J(1) \implies J(e) = e$. Then $p := J(1) \in P$ and $J = J_p$.*

Lemma 4.6. *Let $a \in A$ and $p \in P$. Then:*

(i) *$a \in C(p) \iff a = J_p(a) + J_{1-p}(a)$.*

(ii) *$C(p)$ is norm closed in A.*

Proof. If aCp, it is clear that $a = pap+(1-p)a(1-p)$. Conversely, if $a = pap+ (1-p)a(1-p)$ then $pa = pap = ap$, proving (i). Define the mapping $c_p\colon A \to A$ by $c_p(a) := J_p(a) + J_{1-p}(a)$. By Lemma 4.4.(ii), c_p is norm continuous, and by (i), $C(p)$ is its set of fixed points, proving (ii). □

Theorem 4.7. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in A and suppose that $\lim\limits_{n\to\infty} a_n = a \in A$. Then:*

(i) *If $a_n \leq b \in A$ for all $n \in \mathbb{N}$, than $a \leq b$.*

(ii) *If $a_1 \leq a_2 \leq \cdots$, then a is the supremum (least upper bound) of $(a_n)_{n\in\mathbb{N}}$ in A.*

(iii) *The positive cone A^+ is norm closed in A.*

Proof. By hypothesis, for each $m \in \mathbb{N}$, there exists $N_m \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$,

$$N_m \leq n \implies a_n - a \leq \|a_n - a\| \leq 1/m \implies a_n \leq a + 1/m. \tag{1}$$

(i) Assume the hypothesis of (i). Then, for all $m \in \mathbb{N}$, $a - b \leq a - a_{N_m}$. Let $p := ((a-b)^+)^{\mathrm{o}} \in CC(a-b)$. Then, $(a-b)^+ = p(a-b) = p(a-b)p = J_p(a-b)$, so by Lemma 4.4.(iv) and (1), for every $m \in \mathbb{N}$,

$$(a-b)^+ = J_p(a-b) \leq J_p(a - a_{N_m}) \leq \|J_p(a - a_{N_m})\| \leq \|a - a_{N_m}\| \leq 1/m,$$

whence $m(a-b)^+ \leq 1$, and since A is archimedean, it follows that $(a-b)^+ \leq 0$. But $0 \leq (a-b)^+$, so $(a-b)^+ = 0$, and consequently, $a-b = -(a-b)^- \leq 0$, i.e., $a \leq b$.

(ii) By (1), for each $m \in \mathbb{N}$,

$$a_1 \leq a_2 \leq \cdots \leq a_{N_m} \leq a + 1/m;$$

hence $a_n \leq a + 1/m$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we have $m(a_n - a) \leq 1$ for all $m \in \mathbb{N}$, and since A is archimedean, it follows that $a_n - a \leq 0$, i.e., $a_n \leq a$. If $a_n \leq b \in A$ for all $n \in \mathbb{N}$, then by (i), $a \leq b$; hence a is the least upper bound of $(a_n)_{n\in\mathbb{N}}$.

(iii) Let $(c_n)_{n\in\mathbb{N}}$ be a sequence in A^+ and suppose that $c_n \to c$. Then $-c_n \to -c$, and as $-c_n \leq 0$ for all $n \in \mathbb{N}$, (i) implies that $-c \leq 0$, i.e., $c \in A^+$. $\square$

By combining the quadratic mapping J_a with the carrier, we obtain the *Sasaki mapping* on A as per the following definition.

Definition 4.8. For each $a \in A$, the *Sasaki mapping*[8] $\phi_a \colon A \to P$ is defined by $\phi_a(b) := (J_a(b))^{\mathrm{o}} = (aba)^{\mathrm{o}}$ for all $b \in B$.

Theorem 4.9. *Let $a, b, c \in A$. Then:*

(i) $\phi_a(b) \leq \phi_a(1) = a^{\mathrm{o}}$.

(ii) $0 \leq b \leq c \implies \phi_a(b) \leq \phi_a(c)$.

(iii) *If* $0 \leq b$, *then* $\phi_a(b)c = 0 \implies \phi_a(c)b = 0$.

(iv) *If* $0 \leq b, c$, *then* $\phi_a(b)c = 0 \iff \phi_a(c)b = 0$.

(v) *If* $0 \leq b$, *then* $\phi_a(b) = \phi_a(b^{\mathrm{o}})$.

Proof.

(i) As $abaa^{\mathrm{o}} = aba$, Theorem 2.10.(iii) implies that $\phi_a(b) = (aba)^{\mathrm{o}} \leq a^{\mathrm{o}}$. Also, $\phi_a(1) = (a^2)^{\mathrm{o}} = a^{\mathrm{o}}$ by Theorem 2.10.(vii).

(ii) Assume that $0 \leq b \leq c$. Then $0 \leq J_a(b) \leq J_a(c)$, so $\phi_a(b) \leq \phi_a(c)$ by Theorem 2.10.(viii).

(iii) Suppose that $0 \leq b$ and $\phi_a(b)c = 0$. Then $(aba)^{\mathrm{o}}c = 0$, whence $abac = 0$, and therefore $(aca)b(aca) = ac(abac)a = 0$, whereupon $acab = 0$ by [SA4], and it follows that $(aca)^{\mathrm{o}}b = 0$, i.e., $\phi_a(c)b = 0$.

(iv) Follows from (iii).

(v) Suppose that $0 \leq c$. We have $\phi_a(c)b = 0 \iff \phi_a(c)b^{\mathrm{o}} = 0$, and as $0 \leq b^{\mathrm{o}}$, it follows from (iv) that $\phi_a(c)b^{\mathrm{o}} = 0 \iff \phi_a(b^{\mathrm{o}})c = 0$. Consequently, $\phi_a(b)c = 0 \iff \phi_a(b^{\mathrm{o}})c = 0$. Putting $c = 1 - \phi_a(b^{\mathrm{o}})$, we find that $\phi_a(b) = \phi_a(b)\phi_a(b^{\mathrm{o}})$, hence $\phi_a(b) \leq \phi_a(b^{\mathrm{o}})$. Similarly, putting $c = 1 - \phi_a(b)$, we obtain $\phi_a(b^{\mathrm{o}}) \leq \phi_a(b)$. $\square$

[8]The terminology derives from the fact that, for $p \in P$, the restriction of ϕ_p to P is a so-called *Sasaki projection* on P [14, p. 99]. See Theorem 5.6 below.

THEOREM 4.10. *Let $0 \neq v \in P$ and define $vAv := J_v(A) = \{vav : a \in A\} = \{b \in A : b = bv = vb\}$. Then vAv is norm-closed in A and, with the partial order inherited from A, vAv is a synaptic algebra with unit v and enveloping algebra vRv.*[9] *Moreover, the order-unit norm on vAv is the restriction to vAv of the order-unit norm on A, and for all $a, b \in vAv$, we have: $a \circ b$, a^{o}, $|a|$, a^+, a^-, $\operatorname{sgn}(a) \in vAv$; $J_a(A) \subseteq vAv$; $\phi_a(A) \subseteq vAv$; and $0 \leq a \implies a^{1/2} \in vAv$.*

P r o o f. By Lemma 4.4.(ii), $J_v \colon A \to A$ is norm continuous, and since vAv is the set of fixed points of J_v, it follows that vAv is a norm-closed linear subspace of A. Let $b \in vAv$. Then there exists $n \in \mathbb{N}$ such that $b \leq n = n1$; hence $b = J_v(b) \leq nJ_v(1) = nv$, so v is an order unit in vAv. By a similar argument, if $0 \leq \lambda \in \mathbb{R}$, then $-\lambda \leq b \leq \lambda \implies -\lambda v \leq b \leq \lambda v$; conversely, $-\lambda v \leq b \leq \lambda v \implies -\lambda \leq b \leq \lambda$ follows from the fact that $0 \leq v \leq 1$; hence $\|b\| = \inf\{0 < \lambda \in \mathbb{R} : -\lambda v \leq b \leq \lambda v\}$. Thus, [SA1] holds for vAv.

That vAv satisfies [SA2]–[SA4] is obvious. If $0 \leq b \in vAv$, then, since $b = bv = vb$ and $b^{1/2} \in CC(b)$, we have $0 \leq vb^{1/2}v = vb^{1/2} = b^{1/2}v$ with $(vb^{1/2})^2 = b$; hence $b^{1/2} = vb^{1/2}$ by the uniqueness of square roots (Theorem 2.2), and it follows that $b^{1/2} \in vAv$. Thus, vAv satisfies [SA5]. If $b \in vAv$, we again have $b = bv = vb$, whence $b^{\mathrm{o}} \leq v$, and since $b^{\mathrm{o}} \in CC(b)$, it follows easily that $b^{\mathrm{o}} \in vAv$. Thus, vAv satisfies [SA6].

To show that vAv satisfies [SA7], suppose that $v \leq b \in vAv$. Then $1 = v + (1 - v) \leq b + 1 - v$ with $b = vb = bv$. By [SA7], there exists $c \in A$ such that $1 = c(b + 1 - v) = (b + 1 - v)c$. Applying J_v to both sides of the latter equation, we find that $v = vcbv = vbcv = vcvb = bvcv$, and since $vcv \in vAv$, it follows that vAv satisfies [SA7]. Obviously, vAv inherits condition [SA8] from A. We omit the completely straightforward proofs of the remaining assertions of the theorem. □

5. Orthomodularity of the projection lattice

DEFINITION 5.1. The mapping ${}^{\perp} \colon P \to P$ is defined by $p^{\perp} := 1 - p$ for all $p \in P$. If $p, q \in P$, we say that p is *orthogonal* to q, in symbols $p \perp q$, iff $p \leq q^{\perp}$.

We note that $p \perp q \implies q \perp p$ and that $p \perp p \iff p = 0$. In this section we are going to prove that, with $p \mapsto p^{\perp} := 1 - p$ as the *orthocomplementation*, P is a *orthomodular lattice* as per the following definition ([3, 14]).

[9] In dealing with the synaptic algebra vAv in the presence of the synaptic algebra A, we cannot follow the convention (previously adopted for A) of identifying real numbers λ with multiples λv of the unit element v.

Definition 5.2. Let X be a partially ordered set (poset). A mapping $x \mapsto x^\perp$ from X to X is called an *involution* iff it is order reversing ($x \le y \implies y^\perp \le x^\perp$) and of period 2 ($(x^\perp)^\perp = x$) for all $x, y \in X$. An *orthomodular poset* (OMP) is a partially ordered set X with a smallest element 0, a largest element 1, and an involution $^\perp \colon X \to X$, called the *orthocomplementation*, such that, for all $x, y \in X$:

(i) The infimum (greatest lower bound) $x \wedge x^\perp$ of x and $x^\perp$ exists in X and $x \wedge x^\perp = 0$.

(ii) If $x \le y^\perp$, then the supremum (least upper bound) $x \vee y$ exists in X.

(iii) If $x \le y$, then $y = x \vee (x^\perp \wedge y)$.

An *orthomodular lattice* (OML) is an OMP X that is a lattice (i.e., every pair $x, y \in X$ has an infimum $x \wedge y$ and a supremum $x \vee y$ in X.)

Let X be a poset and let $a, b, x, y \in X$. If we write $a = x \wedge y$, or $x \wedge y = a$, we mean that the infimum (greatest lower bound) $x \wedge y$ of x and y exists in X and that it equals a. A similar convention applies to an existing supremum (least upper bound) $b = x \vee y$ of x and y in X. An involution $x \mapsto x^\perp$ on X gives rise to a *De Morgan duality* on X whereby existing infima are converted to suprema and *vice versa*. For instance, if $a = x \wedge y$, then $a^\perp = x^\perp \vee y^\perp$. Also, if X has a smallest element 0 and a largest element 1, then $0^\perp = 1$ and $1^\perp = 0$. Obviously, the mapping $p \mapsto p^\perp = 1 - p$ (respectively, $e \mapsto 1 - e$) is an involution on the poset P (respectively, on the poset E), and $a \mapsto -a$ is an involution on A.

Suppose that X is an OMP with $x \mapsto x^\perp$ as the orthocomplementation. Then by Definition 5.2.(i) and De Morgan duality, we have both $x \wedge x^\perp = 0$ and $x \vee x^\perp = 1$, i.e., $x^\perp$ is an *orthogonal complement*, or for short, an *orthocomplement* of x in X. Let $x, y \in X$ with $x \le y$. Then $x \le (y^\perp)^\perp$, whence $x \vee y^\perp$ exists in X by Definition 5.2.(ii), and therefore $x^\perp \wedge y = (x \vee y^\perp)^\perp$ exists in X by De Morgan duality. Since $x \le x \vee y^\perp = (x^\perp \wedge y)^\perp$, it also follows from Definition 5.2.(ii) that the supremum $x \vee (x^\perp \wedge y)$ exists in X. The condition $x \le y \implies y = x \vee (x^\perp \wedge y)$ in Definition 5.2.(iii) is called the *orthomodular law*.

Lemma 5.3. *For all $p, q \in P$:*

(i) $p\, C\, q \implies pq = p \wedge q$.

(ii) $p \perp q \iff pq = 0$.

(iii) $p \perp q \implies p \vee q = p + q$.

(iv) $p \le q \implies q - p = p^\perp \wedge q \in P$.

(v) *With $p \mapsto p^\perp := 1 - p$ as the orthocomplementation, P is an OMP.*

Proof.

(i) Assume that $pq = qp$. Obviously, $(pq)^2 = pq$, so $pq \in P$. Also $p(pq) = pq$ and $q(pq) = pq$, so $pq \le p, q$ by Theorem 2.4. Suppose that $r \in P$ and $r \le p, q$.

Again by Theorem 2.4, $rp = pr = r$ and $rq = qr = r$, whence $rpq = r$, i.e., $r \leq pq$. Therefore $pq = p \wedge q$.

(ii) By Theorem 2.4, $p \perp q \iff p \leq 1 - q \iff p = p(1-q) = p - pq \iff pq = 0$.

(iii) Suppose that $p \perp q$ Then $pq = 0$ by (ii), so $qp = 0$ by Lemma 2.9.(iv), and it follows that $(p+q)^2 = p^2 + q^2 = p + q$, i.e., $p+q \in P$. As $0 \leq p, q$, we have $p, q \leq p + q$. Suppose that $r \in P$ with $p, q \leq r$. Then, by Theorem 2.4, $p = pr$ and $q = qr$, whereupon $p + q = (p+q)r$, i.e., $p + q \leq r$. Therefore, $p + q = p \vee q$.

(iv) Suppose that $p \leq q = (q^{\perp})^{\perp}$. Then by (iii), $p + q^{\perp} = p \vee q^{\perp} \in P$, whence $(p + q^{\perp})^{\perp} = p^{\perp} \wedge q \in P$. But $(p + q^{\perp})^{\perp} = 1 - (p + 1 - q) = q - p$.

(v) Obviously, 0 is the smallest element and 1 is the largest element in the poset P. In view of (ii), it remains only to show that the orthomodular law holds in P. But, if $p, q \in P$ with $p \leq q$, then by (iv), $q - p = p^{\perp} \wedge q$ and by (iii), $q = p + (q - p) = p + (p^{\perp} \wedge q) = p \vee (p^{\perp} \wedge q)$. □

Theorem 5.4. *Let $a \in A$. Then:*

(i) *If $p, q \in P$, then $\phi_a(p) \perp q \iff p \perp \phi_a(q)$.*

(ii) *ϕ_a preserves all existing suprema in P, i.e., if $Q \subseteq P$ and $r = \bigvee Q$, then $\phi_a(r) = \bigvee\{\phi_a(q) :\ q \in Q\}$.*

Proof. Part (i) follows from Theorem 4.9.(iv) and Lemma 5.3.(ii). To prove part (ii), suppose that $Q \subseteq P$ and $r = \bigvee Q$. Then, for all $q \in Q$, $0 \leq r \leq q$, whence $\phi_a(q) \leq \phi_a(r)$ by Theorem 4.9.(ii). Suppose that $t \in P$ and $\phi_a(q) \leq t$ for all $q \in Q$. Then, for all $q \in Q$, $\phi_a(q) \perp t^{\perp}$, whence, by (i), $q \perp \phi_a(t^{\perp})$, i.e., $q \leq (\phi_a(t^{\perp}))^{\perp}$, and it follows that $r \leq (\phi_a(t^{\perp}))^{\perp}$. Consequently, by (i) again, $\phi_a(r) \perp t^{\perp}$, i.e., $\phi_a(r) \leq t$, and therefore $\phi_a(r) = \bigvee\{\phi_a(q) :\ q \in Q\}$. □

Lemma 5.5. *Let $p, q, r \in P$. Then:*

(i) *$\phi_p(r) \leq p$.*

(ii) *$r \leq p \iff \phi_p(r) = r$.*

(iii) *$r \perp p \iff \phi_p(r) = 0$.*

(iv) *$p \wedge q$ exists in P and $p \wedge q = p - \phi_p(q^{\perp})$.*

Proof.

(i) By Theorems 4.9.(i) and 2.10.(ii), $\phi_p(r) \leq p^{\mathrm{o}} = p$.

(ii) If $r \leq p$, then $r = pr = rp$, so $\phi_p(r) = (prp)^{\mathrm{o}} = r^{\mathrm{o}} = r$. The converse implication follows from (i).

(iii) By Lemma 5.3.(ii), [SA4], and Theorem 2.10.(i), $p \perp r \iff pr = 0 \iff prp = 0 \iff (prp)^{\mathrm{o}} = 0 \iff \phi_p(r) = 0$.

(iv) Let $t := (\phi_p(q^\perp))^\perp = 1 - \phi_p(q^\perp) \in P$. By (i), $\phi_p(q^\perp) \leq p$, whence $pC\phi_p(q^\perp)$ and $p\phi_p(q^\perp) = \phi_p(q^\perp)$. Therefore, by parts (i) and (iv) of Lemma 5.3,

$$p \wedge t = pt = tp = p - \phi_p(q^\perp) = p \wedge (\phi_p(q^\perp))^\perp \in P. \tag{1}$$

By (1), $p \wedge t \perp \phi_p(q^\perp)$, whence by Theorem 5.4.(i), $\phi_p(p \wedge t) \perp q^\perp$, i.e. $\phi_p(p \wedge t) \leq q$. But, $p \wedge t \leq p$, whence by (ii), $\phi_p(p \wedge t) = p \wedge t$, and we have $p \wedge t \leq q$. Thus $p \wedge t \leq p, q$. Suppose $r \in P$ and $r \leq p, q$. By (ii), $\phi_p(r) = r \leq q$, so $\phi_p(r) \perp q^\perp$, and therefore $r \perp \phi_p(q^\perp)$ by Theorem 5.4.(i); hence, $r \leq (\phi_p(q^\perp))^\perp = t$. But $r \leq p$; hence $r \leq p \wedge t$ by (1), and it follows that $p \wedge t = p \wedge q$. □

Theorem 5.6. *P is an OML and, for all $p, q \in P$, $\phi_p(q) = p \wedge (p^\perp \vee q)$.*

Proof. Let $p, q \in P$. Then by Lemma 5.5.(iv), $p \wedge q$ exists in P, so by De Morgan duality, $p \vee q = (p^\perp \wedge q^\perp)^\perp$ also exists in P. Therefore, P is an OML. Also, as $p^\perp \leq p^\perp \vee q$, we have $p^\perp \vee q = p^\perp \vee (p \wedge (p^\perp \vee q))$ by the orthomodular law; hence, by Theorem 5.4.(ii) and parts (iii) and (ii) of Lemma 5.5,

$$\begin{aligned}\phi_p(q) &= \phi_p(p^\perp) \vee \phi_p(q) = \phi_p(p^\perp \vee q)\\ &= \phi_p(p^\perp \vee (p \wedge (p^\perp \vee q)))\\ &= \phi_p(p^\perp) \vee \phi_p(p \wedge (p^\perp \vee q)) = p \wedge (p^\perp \vee q).\end{aligned}$$ □

Two elements p and q of an orthomodular lattice are said to be *compatible* (or to *commute*) iff $p = (p \wedge q) \vee (p \wedge q^\perp)$ [14, p. 20]. By a standard argument (e.g., [7, Theorem 3.11]), if $p, q \in P$, then p and q are compatible in the foregoing sense iff $p\,C\,q$.

6. Synaptic versus GH-algebras

Every generalized Hermitian (GH) algebra G [9, Definition 2.1] is a synaptic algebra. Indeed, [SA1] follows from [9, Theorem 4.2] and parts (ii), (iii), and (iv) of [9, Definition 2.1] imply [SA2]–[SA4]. Also, [SA5] follows from [9, Theorem 4.5], [9, Theorem 5.2] implies [SA6], and [SA7] is a consequence of [10, Lemma 4.1]. Finally, by [9, Lemma 6.6.(iii)], G satisfies [SA8]; hence G is a synaptic algebra.

By [9, Definition 2.1.(vii)], a generalized Hermitian algebra G has the following *commutative Vigier*[10] *property*:

[CV] *Every bounded ascending sequence $g_1 \leq g_2 \leq \cdots$ of pairwise commuting elements in G has a supremum g in G and $g \in CC(\{g_n :\ n \in \mathbb{N}\})$.*

[10] See [6, Section 5] for the origin of the terminology

Clearly, a synaptic algebra A is a GH-algebra iff it satisfies [CV]. The condition [CV] is quite strong[11] (see [9, Section 4]), and the main impetus for the formulation in Definition 1.1 is to replace [CV] by some of its algebraic consequences [SA5], [SA6], and [SA7], accompanied by the considerably weaker condition [SA8].

As an indication of the extent to which synaptic algebras generalize GH-algebras, we may consider the commutative case. The projections in a commutative GH-algebra form a σ-complete Boolean algebra; moreover, every σ-complete Boolean algebra can be realized as the (Boolean) lattice of projections in a commutative GH-algebra [10, Theorem 5.7]. On the other hand, the projections in a commutative synaptic algebra form a Boolean algebra, which need not be σ-complete; moreover, *every Boolean algebra B can be realized as the (Boolean) lattice of projections in a commutative synaptic algebra.* Indeed, by Stone's theorem, B can be represented as the field $\mathscr{F}$ of compact open subsets of a totally disconnected Hausdorff space X, and the projection lattice of the commutative synaptic algebra A in Example 1.2 is isomorphic to B.

7. Invertible and regular elements

As we now show, the results in [10, Section 4] pertaining to invertible and von Neumann regular elements of a GH-algebra G go through for our synaptic algebra A, although we must be a little careful since the proof of [10, Lemma 4.1] depends on the property [CV]. As usual, an element $a \in A$ is *invertible* iff there exists a (necessarily unique) element $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = 1$. If a is invertible, it is clear that $a^{-1} \in CC(a)$ and that $a^{\mathrm{o}} = 1$.

Lemma 7.1. *Let $a \in A$. Then:*

(i) *If $0 \leq a$ and a is invertible, then $0 \leq a^{-1}$.*

(ii) *a is invertible iff $|a|$ is invertible, and if a is invertible, then $|a|^{-1} = |a^{-1}|$.*

Proof.

(i) Suppose $0 \leq a$ and a is invertible. As $aC(a^{-1})^2$ and $0 \leq (a^{-1})^2$, Lemma 1.5 implies that $0 \leq a(a^{-1})^2 = a^{-1}$.

(ii) Let $s := \mathrm{sgn}(a)$. By Theorem 3.6, $s \in CC(a)$, $s^2 = a^{\mathrm{o}}$, $sa = as = |a|$, and $s|a| = |a|s = a$. Suppose a is invertible. As $s \in CC(a)$, we have $s\,C\,a^{-1}$ and $|a|(sa^{-1}) = (sa^{-1})|a| = 1$; hence $|a|$ is invertible and $|a|^{-1} = sa^{-1}$. Also, $s^2 = a^{\mathrm{o}} = 1$, and by (i), $0 \leq sa^{-1}$. But, $(sa^{-1})^2 = s^2(a^{-1})^2 = (a^{-1})^2$, whence

[11] For instance, as a consequence of [CV], the orthomodular lattice of projections in a GH-algebra is necessarily σ-complete [9, Theorem 5.4].

$|a|^{-1} = sa^{-1} = |sa^{-1}| = |a^{-1}|$. Conversely, if $|a|$ is invertible, it is clear that a is invertible with $a^{-1} = s|a|^{-1}$. $\square$

Theorem 7.2. *If $a \in A$, then a is invertible iff there exists $0 < \epsilon \in \mathbb{R}$ such that $\epsilon \leq |a|$.*

Proof. Suppose first that a is invertible. Then, by Lemma 7.1, $|a|$ is invertible. As 1 is an order unit, there exists $n \in \mathbb{N}$ such that $|a|^{-1} \leq n$, and since $|a|$ commutes with $n - |a|^{-1}$, Lemma 1.5 implies that $0 \leq (n - |a|^{-1})|a|$, i.e., $1 \leq n|a|$. Consequently, with $0 < \epsilon := 1/n$, we have $\epsilon \leq |a|$.

Conversely, suppose $0 < \epsilon \leq |a|$. Then $1 \leq \epsilon^{-1}|a|$; hence by [SA7], $\epsilon^{-1}|a|$ is invertible, and it follows that $|a|$ is invertible with $|a|^{-1} = \epsilon^{-1}(\epsilon^{-1}|a|)^{-1}$. Thus a is invertible by Lemma 7.1. $\square$

Definition 7.3. Let $a \in A$.

(i) a is *von Neumann regular* iff there exists $b \in A$ such that $ab, ba \in A$ and $aba = a$.

(ii) a is *regular* iff there exists $0 < \epsilon \in \mathbb{R}$ such that $\epsilon a^{\mathrm{o}} \leq |a|$.

Obviously, 0 is both von Neumann regular and regular. The proof of the following theorem is virtually identical[12] to the proof of [10, Theorem 4.5].

Theorem 7.4. *If $0 \neq a \in A$, then the following conditions are mutually equivalent:*

(i) *a is von Neumann regular.*

(ii) *There exists $r \in a^{\mathrm{o}}Aa^{\mathrm{o}}$ such that $ar = ra = a^{\mathrm{o}}$.*

(iii) *a is invertible in the synaptic algebra $a^{\mathrm{o}}Aa^{\mathrm{o}}$.*

(iv) *a is regular.*

Corollary 7.5. *If $a \in A$, then a is invertible iff a is regular and $a^{\mathrm{o}} = 1$.*

If $0 \neq a \in A$ and a is regular, then the (necessarily unique) inverse of a in $a^{\mathrm{o}}Aa^{\mathrm{o}}$ (Theorem 7.4) is called the *pseudo-inverse* of a in A, and by definition, the pseudo-inverse of 0 is 0. If a is regular, it is not difficult to show that the pseudo-inverse of a belongs to $CC(a)$.

Theorem 7.6. *If $a \in A$, then a is regular iff both a^+ and a^- are regular.*

Proof. Let $p := (a^+)^{\mathrm{o}}$ and $q := (a^-)^{\mathrm{o}}$. Then by Theorem 3.3, $p, q \in CC(a)$, $p + q = a^{\mathrm{o}}$, $pq = pa^- = 0$, $qp = qa^+ = 0$, $pa = pa^+ = a^+$, and $qa = qa^- = a^-$.

Suppose that a is regular. Then there exists $0 < \epsilon \in \mathbb{R}$ with $\epsilon(p+q) = \epsilon a^{\mathrm{o}} \leq |a| = a^+ + a^-$, so $\epsilon p = p(\epsilon(p+q)) \leq p(a^+ + a^-) = a^+ = |a^+|$, whence a^+ is regular. Likewise, $\epsilon q \leq a^-$, so a^- is regular. Conversely, if both a^+ and a^-

[12]Note that $a^{\mathrm{o}}Aa^{\mathrm{o}}$ is a synaptic algebra by Theorem 4.10.

are regular, there exist $0 < \alpha, \beta$ such that $\alpha p \leq a^{+}$ and $\beta q \leq a^{-}$; hence with $\epsilon := \min\{\alpha, \beta\}$, we have $\epsilon a^{\mathrm{o}} = \epsilon(p+q) \leq a^{+} + a^{-} = |a|$, and it follows that a is regular. □

Corollary 7.7. *$a \in A$ is invertible iff $a^{\mathrm{o}} = 1$ and both a^{+} and a^{-} are regular.*

8. Spectral resolution

In this section, we show that the synaptic algebra A is a so-called *spectral order-unit normed space*; hence the results of [8] are at our disposal. In particular, every element in A both determines and is determined by a family of projections — its *spectral resolution.*

As per [8, Definition 1.5 (i)], an element $a \in A$ is *compatible* with a projection $p \in P$ iff $a = J_p(a) + J_{1-p}(a)$. Thus, by Lemma 4.6.(i), $C(p)$ is the set of all elements of A that are compatible with p; hence, the notation used in [8, Definition 1.5 (i) and ff.] is consistent with our notation in this article.

Theorem 8.1. *The family $(J_p)_{p \in P}$ is a spectral compression base* [8, Definition 1.7] *for the order-unit space A.*

Proof. To begin with, we have to show that P is a normal sub-effect algebra of E ([5, Definition 1]). Of course, $0, 1 \in P$, and $p \in P \implies 1 - p = p^{\perp} \in P$. Also, if $p, q \in P$ with $p + q \leq 1$, then $p + q = p \vee q \in P$ by Lemma 5.3.(iii). Therefore P is a sub-effect algebra of E. Suppose that $d, e, f, d+e+f \in E$ with $p := d + e \in P$ and $q := d + f \in P$. Then $e + q = d + e + f \leq 1$, so $e \leq 1 - q$, and therefore by Theorem 2.4, $e = e(1-q)$, i.e., $eq = 0$. Also, $d \leq d + f = q$, so $dq = d$ by Theorem 2.4, and it follows that $pq = (d+e)q = dq = d$. By symmetry, $qp = d$; hence by Lemma 5.3.(i), $d = p \wedge q \in P$, and it follows that P is a normal sub-effect algebra of E.

Now let $p, q, r \in P$ with $p+q+r \leq 1$. Then $pq = pr = qr = 0$, $p+r = p \vee r \in P$ and $q + r = q \vee r \in P$, whence, for all $a \in A$,

$$J_{p+r}(J_{q+r}(a)) = (p+r)(q+r)a(q+r)(p+r) = rar = J_r(a),$$

and it follows that $(J_p)_{p \in P}$ is a compression base for A ([5, Definition 2]).

If $e \in E$, then by Theorem 2.10.(vii), e^{o} is the smallest projection p such that $e \leq p$; hence the compression base $(J_p)_{p \in P}$ has the projection cover property ([8, Definition 1.4]).

Let $a \in A$ and let $p := (a^{+})^{\mathrm{o}}$. Then by parts (i), (iii), and (vi) of Theorem 3.3, we have $C(a) \subseteq C(p)$, $J_p(a) = pap = pa = a^{+} \geq 0$, and $J_{1-p}(a) = (1-p)a(1-p) = (1-p)a = a - pa = a - a^{+} = -a^{-} \leq 0$. Thus, the compression base $(J_p)_{p \in P}$ has the comparability property ([8, Definition 1.6]), and therefore $(J_p)_{p \in P}$ is a spectral compression base for A. □

If $a \in A$, it is clear that, for all $p \in P$,

$$p \leq 1 - a^{\mathrm{o}} \iff a^{\mathrm{o}}p = 0 \iff ap = pa = 0 \iff \big(a \in C(p) \ \&\ J_p(a) = pap = 0\big).$$

Therefore, as per [8, Theorem 2.1 and ff.], the mapping $'\colon A \to P$ defined by $a' := 1 - a^{\mathrm{o}}$ for all $a \in A$ is effective as the *Rickart mapping* on A. We note that, for $p \in P$, we have $p' = 1 - p = p^{\perp}$.

Let $a, b \in A$. In [8] the notation $b \in CPC(a)$ means that, for all $p \in P$, $a \in C(p) \implies b \in C(p)$. Thus, $b \in CPC(a) \iff C(a) \cap P \subseteq C(b)$; hence, $CC(a) \subseteq CPC(a)$. For instance, by [8, Lemmas 2.1.(vi), 2.4.(iv)], $a^{\mathrm{o}}, |a|, a^{+} \in CPC(a)$, but for our synaptic algebra A, we have the (possibly) stronger conditions a^{o}, $|a|$, $a^{+} \in CC(a)$.

In view of the remarks above, we can translate the results in [8] into our present formalism by replacing a' by $1 - a^{\mathrm{o}}$, a'' by a^{o}, and p' by $p^{\perp} = 1 - p$ for all $a \in A$ and all $p \in P$. Moreover, if $a\,C\,p$, we can replace $J_p(a)$ by pa (or by ap).

Definition 8.2. Let $a \in A$ and $\lambda \in \mathbb{R}$. Then:

(i) The *spectral lower and upper bounds* L and U for a are defined by $L := \sup\{\lambda \in \mathbb{R} :\ \lambda \leq a\}$ and $U := \inf\{\lambda \in \mathbb{R} :\ a \leq \lambda\}$.

(ii) The family of projections $(p_\lambda)_{\lambda \in \mathbb{R}}$ defined by $p_\lambda := 1 - ((a - \lambda)^{+})^{\mathrm{o}}$ is called the *spectral resolution* of a.

(iii) The family of projections $(d_\lambda)_{\lambda \in \mathbb{R}}$ defined by $d_\lambda := 1 - (a - \lambda)^{\mathrm{o}}$ is called the family of *eigenprojections* of a. If $d_\lambda \neq 0$, then λ is called an *eigenvalue* of a.

Standing Assumptions 8.3. In what follows: $a \in A$; L and U are the spectral bounds for a; $(p_\lambda)_{\lambda \in \mathbb{R}}$ is the spectral resolution of a; and $(d_\lambda)_{\lambda \in \mathbb{R}}$ is the family of eigenprojections for a.

By [8, Theorem 3.1], $-\infty < L \leq U < \infty$, $\|a\| = \max\{|L|, |U|\}$, and $L \leq a \leq U$. The following theorem is a consequence of [8, Theorems 3.3, 3.5, and 3.6].

Theorem 8.4. *For all $\lambda, \mu \in \mathbb{R}$:*

(i) $p_\lambda, d_\lambda \in CC(a)$*; hence* $p_\lambda C p_\mu$, $p_\lambda\, C\, d_\mu$, *and* $d_\lambda\, C\, d_\mu$.

(ii) $p_\lambda(a - \lambda) \leq 0 \leq (1 - p_\lambda)(a - \lambda)$.

(iii) $\lambda \leq \mu \implies p_\lambda \leq p_\mu$ *and* $p_\mu - p_\lambda = p_\mu \wedge (1 - p_\lambda)$.

(iv) $\lambda < \mu \implies d_\lambda \leq p_\lambda \leq 1 - d_\mu \implies d_\lambda \perp d_\mu$.

(v) $\mu \geq U \iff p_\mu = 1$.

(vi) $\lambda < L \implies p_\lambda = 0$, *and* $L < \lambda \implies 0 < p_\lambda$.

(vii) *If* $\alpha \in \mathbb{R}$, *then* $p_\alpha = \bigwedge\{p_\mu :\ \alpha < \mu \in \mathbb{R}\}$.

(viii) *If* $\alpha \in \mathbb{R}$, *then* $p_\alpha - d_\alpha = \bigvee\{p_\lambda :\ \alpha > \lambda \in \mathbb{R}\}$.

By [8, Theorem 3.4, Remark 3.1, and Corollary 3.1], we have the following theorem and corollary.

Theorem 8.5. *Suppose that $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\lambda_0 < L < \lambda_1 < \cdots < \lambda_n = U$, and let $\gamma_i \in \mathbb{R}$ with $\lambda_{i-1} \le \gamma_i \le \lambda_i$ for $i = 1, 2, \ldots, n$. Define $u_i := p_{\lambda_i} - p_{\lambda_{i-1}}$ for $i = 1, 2, \ldots, n$, and let $\epsilon := \max\{\lambda_i - \lambda_{i-1} : \ i = 1, 2, \ldots, n\}$. Then:*

$$u_1, u_2, \ldots, u_n \in P \cap CC(a), \quad \sum_{i=1}^{n} u_i = 1, \quad \text{and} \quad \left\| a - \sum_{i=1}^{n} \gamma_i u_i \right\| \le \epsilon.$$

According to Theorem 8.5, a can be written as a norm-convergent integral $a = \int_{L-0}^{U} \lambda \, \mathrm{d}p_\lambda$ of Riemann-Stieltjes type; hence a not only determines, but it is determined by its spectral resolution.

Corollary 8.6. *There exists an ascending sequence $a_1 \le a_2 \le \cdots$ in $CC(a)$ such that each a_n is a finite linear combination of projections in the family $(p_\lambda)_{\lambda \in \mathbb{R}}$ and $\lim_{n \to \infty} a_n = a$.*

Definition 8.7. A real number ρ belongs to the *resolvent set* of a iff there is an open interval I in $\mathbb{R}$ with $\rho \in I$ such that $p_\lambda = p_\rho$ for all $\lambda \in I$. The *spectrum* of a, in symbols $\mathrm{spec}(a)$, is defined to be the complement in $\mathbb{R}$ of the resolvent set of a.

As is proved in [8], $\mathrm{spec}(a)$ has all of the expected basic properties. For instance, by [8, Theorem 4.3], $\mathrm{spec}(a)$ is a closed nonempty subset of the closed interval $[L, U] \subseteq \mathbb{R}$, $L = \inf(\mathrm{spec}(a)) \in \mathrm{spec}(a)$, $U = \sup(\mathrm{spec}(a)) \in \mathrm{spec}(a)$, and $\|a\| = \sup\{|\alpha| : \ \alpha \in \mathrm{spec}(a)\}$. By [8, Theorem 4.4], $a \in A^+ \iff \mathrm{spec}(a) \subseteq \mathbb{R}^+$, and by [8, Corollary 5.1], $a \in P \iff \mathrm{spec}(a) \subseteq \{0, 1\}$. As a consequence of [8, Theorem 4.2], every isolated point of $\mathrm{spec}(a)$ is an eigenvalue of a, and every eigenvalue of a belongs to $\mathrm{spec}(a)$.

Definition 8.8. An element in A is *simple* iff it is a finite linear combination of pairwise commuting projections.

The following result is a consequence of [8, Theorems 5.2 and 5.3].

Theorem 8.9. *The simple elements of A are precisely those with finite spectrum. Let a be a simple element of A. Then a can be written uniquely as $a = \sum_{i=1}^{n} \alpha_i u_i$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, $0 \ne u_i \in P$, and $\sum_{i=1}^{n} u_i = 1$. Moreover, a is regular, $|a| = \sum_{i=1}^{n} |\alpha_i| u_i$, $\|a\| = \max\{|\alpha_i| : \ i = 1, 2, \ldots, n\}$, $a^{\mathrm{o}} = \sum_{\alpha_i \ne 0} u_i$, and $u_i = d_{\alpha_i}$ for $i = 1, 2, \ldots, n$.*

As a consequence of Corollary 8.6, each element $a \in A$ is the norm limit (hence by Theorem 4.7.(ii) also the supremum) of an ascending sequence of pairwise commuting simple elements, and it follows that the simple elements in A (hence by Theorem 8.9, also the regular elements in A) are norm-dense in A.

Theorem 8.10. *If $b \in A$, then $b\,C\,a$ iff $b\,C\,p_\lambda$ for all $\lambda \in \mathbb{R}$.*

Proof. For $\lambda \in \mathbb{R}$, we have $p_\lambda \in CC(a)$; hence $b\,C\,a$ implies that $b\,C\,p_\lambda$. Conversely, suppose that $b\,C\,p_\lambda$ for all $\lambda \in \mathbb{R}$ and let $(a_n)_{n\in\mathbb{N}}$ be the ascending sequence in Corollary 8.6. As $a_n \in CC(a)$ for all $n \in \mathbb{N}$, the elements of the sequence $(a_n)_{n\in\mathbb{N}}$ commute with each other. As each a_n is a finite linear combination of projections p_λ, we have $a_n \in C(b)$ for all $n \in \mathbb{N}$, and it follows from [SA8] that $a \in C(b)$. □

Theorem 8.11. *$C(a)$ is norm-closed in A and, with the partial order inherited from A, $C(a)$ is a synaptic algebra with unit 1 and enveloping algebra R. Let $b, c \in C(a)$. Then: $b \circ c$, b^{o}, $|b|$, b^+, b^-, $J_b(c)$, $\phi_b(c) \in C(a)$; $0 \le b \implies b^{1/2} \in C(A)$; and the spectral resolution and family of eigenprojections of b are the same whether calculated in A or in $C(a)$.*

Proof. Suppose that $(b_n)_{n\in\mathbb{N}}$ is a sequence in C(a) and $b_n \to b \in A$. Then by Theorem 8.10, $b_n \in C(p_\lambda)$ for all $n \in \mathbb{N}$ and all $\lambda \in \mathbb{R}$, and it follows from Lemma 4.6.(ii) that $b \in C(p_\lambda)$ for all $\lambda \in \mathbb{R}$. Therefore, $b \in C(a)$ by Theorem 8.10, whence $C(a)$ is norm-closed in A. The remainder of the proof is omitted as it is completely straightforward □

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Emeritus Professor of
Mathematics and Statistics
University of Massachusetts
USA

Postal Address:
1 Sutton Court
Amherst, MA 01002
USA

E-mail: foulis@math.umass.edu