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QUASICONTINUOUS FUNCTIONS, MINIMAL USCO MAPS AND TOPOLOGY OF POINTWISE CONVERGENCE

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ABSTRACT. In [HOLÁ, Ľ.—HOLÝ, D.: *Pointwise convergence of quasicontinuous mappings and Baire spaces*, Rocky Mountain J. Math.] a complete answer is given, for a Baire space X, to the question of when the pointwise limit of a sequence of real-valued quasicontinuous functions defined on X is quasicontinuous. In [HOLÁ, Ľ.—HOLÝ, D.: *Minimal USCO maps, densely continuous forms and upper semicontinuous functions*, Rocky Mountain J. Math. **39** (2009), 545–562], a characterization of minimal USCO maps by quasicontinuous and subcontinuous selections is proved. Continuing these results, we study closed and compact subsets of the space of quasicontinuous functions and minimal USCO maps equipped with the topology of pointwise convergence. We also study conditions under which the closure of the graph of a set-valued mapping which is the pointwise limit of a net of set-valued mappings, is a minimal USCO map.

1. Introduction

In the paper [24] Kempisty introduced a notion similar to continuity for real-valued functions defined in $\mathbb{R}$. For general topological spaces this notion can be given by the following equivalent formulation.

A function $f: X \to Y$ is called quasicontinuous at $x \in X$ if for every open set $V \subset Y$, $f(x) \in V$, and open set $U \subset X$, $x \in U$, there is a nonempty open set $W \subset U$ such that $f(W) \subset V$. If f is quasicontinuous at every point of X, we say that f is quasicontinuous.

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The notion of quasicontinuity was perhaps the first time used by R. Baire in [3] in the study of points of continuity of separately continuous functions. As Baire indicated in his paper [3] the condition of quasicontinuity has been suggested by Vito Volterra.

The notion of quasicontinuity recently turned out to be instrumental in the proof that some semitopological groups are actually topological ones (see Bouziad [8]), in the proof of some generalizations of Michael's selection theorem (see Giles, Bartlett [13]) and in characterizations of minimal usco maps via their selections (see Holá, Holý [18]). In the paper of Matejdes [28] a characterization is given for the minimality of maps via their quasicontinuous selections.

Quasicontinuity of real-valued separately continuous functions of two variables has been studied very frequently in connection with the existence of points of joint continuity for such functions (see Martin [29], Mibu [30], Piotrowski [36]).

Continuity points of quasicontinuous mappings were studied in many papers; see for example Bledsoe [5], Holá, Piotrowski [21], Kenderov, Kortezov, Moors [26], Levine [27], see also a survey paper of Neubrunn [34].

Of course it is very easy to verify that the pointwise limit of a sequence of functions that are even continuous need not be quasicontinuous.

However it is known that the pointwise limit of an equicontinuous sequence of functions is continuous. Of course equicontinuity is too strong; it is not necessary to guarantee continuity of the pointwise limit of a sequence of continuous functions.

In the paper of Beer and Levi [6] necessary and sufficient conditions for continuity of the pointwise limit of a net of continuous functions are given.

Minimal USCO maps are a very convenient tool in the theory of games (see Christensen [9], Saint-Raymond [38]) or in functional analysis (see Borwein, Moors [7]), where a differentiability property of single-valued functions is characterized by their Clarke subdifferentials being convex minimal USCO maps).

There are many books and papers concerning topologies and convergences on spaces of set-valued maps: Attouch [1], Aubin, Frankowska [2], Rockafellar, Wets [37], Hammer, McCoy [17], Holá [15], [16], Holý [14], Holý, Vadovič [23]. In particular, graph convergence has found many applications to variational and optimization problems, differential equations and approximation theory.

For topologies on the space of set-valued maps, there are mainly two approaches in the literature. For the first approach, hyperspace topologies on set-valued maps with closed graphs, which were studied by DiMaio, Meccariello, Naimpally [11], [33], Holá [15], McCoy [31], in which multifunctions are identified with their graphs and are considered as elements of the hyperspace. For the second approach, there are extensions of natural topologies on the space of continuous functions to the space of densely continuous forms, and to the spaces of USCO and minimal USCO maps, this approach was studied for example by Holá [16] and Holý [14]. This is also studied in our paper.

In [19], for Baire spaces, necessary and sufficient conditions for quasicontinuity of the pointwise limit of a net of quasicontinuous functions are given. Continuing these results, we study closed and compact subsets of the space of quasicontinuous functions equipped with the topology of pointwise convergence.

In the paper [18] a characterization of minimal USCO maps by quasicontinuous and subcontinuous selections is given. Using results of papers [18] and [19] we study conditions under which the closure of the graph of a set-valued mapping, which is the pointwise limit of a net of set-valued mappings, is a minimal USCO map. We also study closed and compact subsets of the space of minimal USCO maps with topology of pointwise convergence.

2. Preliminaries

In what follows let X, Y be Hausdorff topological spaces and $\mathbb{R}$ be the space of real numbers with the usual metric. Also, for $x \in X$, $\mathscr{U}(x)$ is always used to denote a base of open neighborhoods of x in X. The symbol $\overline{A}$ and $\operatorname{Int} A$ will stand for the closure and interior of the set A in a topological space, respectively.

We say that a function $f\colon X \to Y$ is subcontinuous ([12], [35]) at $x \in X$ if for every net $\{x_\sigma :\ \sigma \in \Sigma\}$ in X converging to x, there is a convergent subnet of $\{f(x_\sigma) :\ \sigma \in \Sigma\}$. A function f is subcontinuous if it is subcontinuous at every point of X.

Remark 2.1. Evidently if $Y = \mathbb{R}$, then notions subcontinuous function and locally bounded function coincide.

Following [10] the term map is reserved for set-valued mappings. If $F\colon X \to Y$ is a (set-valued) map, then

$$\operatorname{Gr} F = \{(x, y) \in X \times Y :\ y \in F(x)\}$$

is the graph of F. Notice that if $f\colon X \to Y$ is a single-valued function we will use the symbol $\operatorname{Gr} f$ also for the graph of f.

We say that a mapping $F\colon X \to \mathbb{R}$ (single-valued or set-valued) is locally bounded at $x \in X$ if there is $U \in \mathscr{U}(x)$ such that $F(U)$ is a bounded subset of $\mathbb{R}$. A map F is locally bounded if it is locally bounded at every point of X.

A map $F\colon X \to Y$ is upper semicontinuous at a point $x \in X$ ([4]), if, for every open set V containing $F(x)$, there exists $U \in \mathscr{U}(x)$ such that

$$F(U) = \bigcup \{F(u) :\ u \in U\} \subset V.$$

A map F is upper semicontinuous if it is upper semicontinuous at each point of X.

Following Christensen [9] we say, that a map F is USCO if it is upper semicontinuous and takes nonempty compact values.

Finally, a map F is said to be minimal USCO if it is a minimal element in the family of all USCO maps (with domain X and range Y); that is, if it is USCO and does not contain properly any other USCO map from X into Y. By an easy application of the Kuratowski-Zorn principle we can guarantee that every USCO map from X to Y contains a minimal USCO map from X to Y (see Drewnowski and Labuda [10]).

We denote by $2^{\mathbb{R}}$ the space of all closed subsets of $\mathbb{R}$ and by $CL(\mathbb{R})$ the space of all nonempty closed subsets of $\mathbb{R}$. Denote by d the usual metric on $\mathbb{R}$. The open d-ball with center $z_0 \in \mathbb{R}$ and radius $\varepsilon > 0$ will be denoted by $S_\varepsilon(z_0)$ and the ε-parallel body $\bigcup\limits_{a\in A} S_\varepsilon(a)$ for a subset A of $\mathbb{R}$ will be denoted by $S_\varepsilon(A)$.

If $A \in CL(\mathbb{R})$, the distance functional $d(\cdot, A):\ \mathbb{R} \to [0,\infty)$ is described by the familiar formula

$$d(z,A) = \inf\bigl\{d(z,a):\ a \in A\bigr\}.$$

The Hausdorff (extended-valued) metric H_d on $2^{\mathbb{R}}$ ([4]) is defined by

$$H_d(A,B) = \max\bigl\{\sup\{d(a,B):\ a \in A\},\ \sup\{d(b,A):\ b \in B\}\bigr\},$$

if A and B are nonempty. If $A \neq \emptyset$ take $H_d(A,\emptyset) = H_d(\emptyset, A) = \infty$. We will often use the following equality on $CL(\mathbb{R})$:

$$H_d(A,B) = \inf\bigl\{\varepsilon > 0:\ A \subset S_\varepsilon(B)\ \text{ and }\ B \subset S_\varepsilon(A)\bigr\}.$$

The topology generated by H_d is called the Hausdorff metric topology.

Denote by $F(X, 2^Y)$ the set of all maps from a topological space X to a topological space Y with closed values. Following the paper [17] we will define the topology τ_p of pointwise convergence on $F(X, 2^{\mathbb{R}})$. We use the symbols τ_p ($\mathfrak{U}_p$), also for the topology (uniformity) of pointwise convergence on the space $F(X, 2^{\mathbb{R}})$. The topology τ_p of pointwise convergence on $F(X, 2^{\mathbb{R}})$ is induced by the uniformity $\mathfrak{U}_p$ of pointwise convergence which has a base consisting of sets of the form

$$W(A,\varepsilon) = \bigl\{(\Phi,\Psi):\ \forall\, x \in A \quad H_d(\Phi(x),\Psi(x)) < \varepsilon\bigr\}$$

where A is a finite subset of X and $\varepsilon > 0$. The general τ_p-basic neighborhood of $\Phi \in F(X, 2^{\mathbb{R}})$ will be denoted by $W(\Phi, A, \varepsilon)$, i.e.

$$W(\Phi, A, \varepsilon) = W(A,\varepsilon)[\Phi] = \bigl\{\Psi:\ H_d(\Phi(x),\Psi(x)) < \varepsilon\ \text{ for every } x \in A\bigr\}.$$

In paper [18] the following characterization of minimal USCO maps is given:

Proposition 2.1. *Let X, Y be topological spaces and Y be a T_1 regular space. Let F be a map from X to Y. Then the following are equivalent:*

(1) *F is a minimal USCO map;*

(2) *There exists a quasicontinuous and subcontinuous function f from X to Y such that $\overline{\operatorname{Gr} f} = \operatorname{Gr} F$;*

(3) *Every selection f of F is quasicontinuous, subcontinuous and $\overline{\operatorname{Gr} f} = \operatorname{Gr} F$.*

3. Quasicontinuous functions and topology of pointwise convergence

The Definition 3.1 and Proposition 3.1 are generalizations for nets of [19, Definition 2.1, Proposition 2.1].

Definition 3.1. ([19]) Let $\{f_\lambda : \lambda \in \Lambda\}$ be a net of real-valued functions defined on a topological space X. We say that the net $\{f_\lambda : \lambda \in \Lambda\}$ is equi-quasicontinuous at $x \in X$ if for every $\varepsilon > 0$ and every $U \in \mathscr{U}(x)$ there is $\lambda_0 \in \Lambda$ and a nonempty open set $W \subset U$ such that $|f_\lambda(z) - f_\lambda(x)| < \varepsilon$ for every $z \in W$ and for every $\lambda \geq \lambda_0$. We say that $\{f_\lambda : \lambda \in \Lambda\}$ is equi-quasicontinuous if it is equi-quasicontinuous at every $x \in X$.

Proposition 3.1. *Let $\{f_\lambda : \lambda \in \Lambda\}$ be a net of real-valued functions defined on a topological space X convergent in the topology of pointwise convergence to a real-valued function f. Let $\{f_\lambda : \lambda \in \Lambda\}$ be equi-quasicontinuous at x. Then f is quasicontinuous at x.*

Proof. Let $\varepsilon > 0$ and let $U \in \mathscr{U}(x)$. Since $\{f_\lambda : \lambda \in \Lambda\}$ is equi-quasicontinuous at x, there exist $\lambda_0 \in \Lambda$ and a non empty open set $W \subset U$ such that $|f_\lambda(x) - f_\lambda(w)| < \frac{\varepsilon}{3}$ for every $\lambda \geq \lambda_0$ and every $w \in W$. Choice arbitrary point $w \in W$.

The net $\{f_\lambda : \lambda \in \Lambda\}$ converges in the topology of poinwise convergence to f, i.e. there exist $\lambda_1 \geq \lambda_0$ such that $|f(w) - f_{\lambda_1}(w)| < \frac{\varepsilon}{3}$ and $|f(x) - f_{\lambda_1}(x)| < \frac{\varepsilon}{3}$. Thus

$$\begin{aligned} |f(x) - f(w)| &\leq |f(x) - f_{\lambda_1}(x)| + |f_{\lambda_1}(x) - f_{\lambda_1}(w)| + |f_{\lambda_1}(w) - f(w)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

□

Theorem 3.1. ([19]) *Let X be a Baire space. Let $\{f_n : n \in \mathbb{Z}^+\}$ be a sequence of real-valued quasicontinuous functions defined on X pointwise convergent to a function $f\colon X \to \mathbb{R}$. Then the following are equivalent:*

(1) *f is quasicontinuous;*

(2) *$\{f_n : n \in \mathbb{Z}^+\}$ is equi-quasicontinuous.*

If $F\colon X \to Y$ is a mapping (single-valued or set-valued), then $\overline{\mathrm{Gr}\,F}$ can be considered as map from X to $\mathbb{R}$, where $\overline{\mathrm{Gr}\,F}(x) = \{y \in \mathbb{R} : (x, y) \in \overline{\mathrm{Gr}\,F}\}$.

Denote by $F(X,\mathbb{R})$ the set of all functions from a topological space X to $\mathbb{R}$, by $Q(X)$ the space of all quasicontinuous function from X to $\mathbb{R}$. We use the symbol τ_p ($\mathfrak{U}_p$) also for the topology (uniformity) of pointwise convergence on $F(X,\mathbb{R})$.

For the Tychonoff space X and for the space $C(X)$ of continuous functions from X to $\mathbb{R}$ the metrizability of $(C(X), \tau_p)$, the first countability of $(C(X), \tau_p)$ and countability of X are equivalent ([32]).

Theorem 3.2. *For a topological space X the following are equivalent:*

(1) *$(F(X,\mathbb{R}), \mathfrak{U}_p)$ is metrizable;*

(2) *$(F(X,\mathbb{R}), \tau_p)$ is first countable;*

(3) *$(Q(X), \mathfrak{U}_p)$ is metrizable;*

(4) *$(Q(X), \tau_p)$ is first countable;*

(5) *X is countable.*

Proof. The proof is similar to the proof of [22, Theorem 2.3]. Since we have $Q(X) \subseteq F(X,\mathbb{R})$, it is clear that (1) $\Longrightarrow$ (3) $\Longrightarrow$ (4) and (1) $\Longrightarrow$ (2) $\Longrightarrow$ (4).

We first show that (4) $\Longrightarrow$ (5). Let f be the constant function mapping each point to 0. Let $\{W(f, A_n, 1/m) : n, m \in \mathbb{Z}^+\}$ be a local base at f, where A_n is a finite subset of X. Suppose that X is not countable; hence $X \setminus \bigcup A_n \neq \emptyset$ and there is $x \notin \bigcup A_n$. For each $n \in \mathbb{Z}^+$ there is $U_n \subseteq X$ open such that $x \in U_n \subseteq \overline{U_n} \subseteq A_n^c$. Define the function $g_n\colon X \to Y$ by

$$g_n(z) = \begin{cases} 1, & \text{if } z \in \overline{U_n}, \\ 0, & \text{otherwise.} \end{cases}$$

Then g_n is quasicontinuous. Since $g_n \notin W(f, x, 1)$ and $g_n \in W(f, A_n, 1/m)$ for every $n \in \mathbb{Z}^+$ and $m \in \mathbb{Z}^+$, the family $\{W(f, A_n, 1/m) : n, m \in \mathbb{Z}^+\}$ can not be a local base at f, a contradiction.

(5) $\Longrightarrow$ (1): It is evident that $W\big(A, \frac{1}{n}\big) = \big\{(f, g) : \forall\, x{\in}A \ |f(x) - g(x)| < \frac{1}{n}\big\}$, where A is a finite subset of X and $n \in \mathbb{Z}^+$ is a countable base for the uniformity $\mathfrak{U}_p$ ([25]). This proves the theorem. $\square$

Proposition 3.2. *Let X be a topological space and let B be a closed subset of $(Q(X), \tau_p)$. If every net in B which is convergent in $(F(X,\mathbb{R}), \tau_p)$ is equi-quasicontinuous, then B is closed also in $(F(X,\mathbb{R}), \tau_p)$.*

Proof. The proof follows from the Proposition 3.1. □

Theorem 3.3. *Let X be a countable Baire space and let B be a subset of $Q(X)$. Then B is closed in $(F(X,\mathbb{R}), \tau_p)$ if and only if the following conditions are satisfied*

(a) *B is a closed subset of $(Q(X), \tau_p)$;*

(b) *every sequence in B which is convergent in $(F(X,\mathbb{R}), \tau_p)$ is equi-quasi-continuous.*

Proof. Let $\{f_n : n \in \mathbb{Z}^+\}$ be a sequence in B convergent to a function $f \in F(X,\mathbb{R})$. Since B is closed in $F(X,\mathbb{R})$ we have that f is quasicontinuous. By Theorem 3.1, $\{f_n : n \in \mathbb{Z}^+\}$ is equi-quasicontinuous. The converse follows from the Proposition 3.2 and the Theorem 3.2 □

If $B \subseteq F(X,\mathbb{R})$ and $x \in X$, then denote by $B[x]$ the set $\{f(x) : f \in B\}$.

Theorem 3.4. *Let X be a topological space. Let B be a subset of $Q(X)$. Then B is compact relative to the pointwise topology if the following conditions are satisfied*

(a) *B is a closed subset of $(Q(X), \tau_p)$;*

(b) *$\overline{B[x]}$ is a compact for every $x \in X$;*

(c) *every net in B has a equi-quasicontinuous subnet.*

Proof. Let $\{f_\lambda : \lambda \in \Lambda\}$ be a net in B. By the assumption, $\{f_\lambda : \lambda \in \Lambda\}$ has a equi-quasicontinuous subnet $\{h_\omega : \omega \in \Omega\}$. The product $\prod_{x \in X} \overline{B[x]}$ is a compact subset of $\mathbb{R}^X = \prod_{x \in X} \mathbb{R}_x$, where $\mathbb{R}_x = \mathbb{R}$ for every $x \in X$, relative to the product topology. Since the topology of pointwise convergence for every subset of $F(X,\mathbb{R})$ is the relativized product topology, net $\{h_\omega : \omega \in \Omega\}$ has a convergent subnet $\{f_\delta : \delta \in \Delta\}$. The net $\{h_\omega : \omega \in \Omega\}$ is equi-quasicontinuous and hence $\{f_\delta : \delta \in \Delta\}$ is equi-quasicontinuous, too. The net $\{f_\delta : \delta \in \Delta\}$ converges to a function f. By Proposition 3.1, f is quasicontinuous. Since B is closed in $(Q(X), \tau_p)$ we have that $f \in B$ and thus B is compact. □

Theorem 3.5. *Let X be a countable Baire space. Let B be a subset of $Q(X)$. Then B is compact relative to the pointwise topology if and only if the following conditions are satisfied*

(a) *B is a closed subset of $(Q(X), \tau_p)$;*

(b) *$\overline{B[x]}$ is a compact for every $x \in X$;*

(c) *every sequence in B has a equi-quasicontinuous subsequence.*

Proof. Suppose B is compact. The space $(Q(X), \tau_p)$ is Hausdorff so B is a closed subset of $(Q(X), \tau_p)$. Projection $p_x\colon \prod\limits_{x \in X} \mathbb{R} \to \mathbb{R}$ is continuous and hence the image $B[x]$ of B is compact. Let $\{f_n : n \in \mathbb{Z}^+\}$ be a sequence in B. Since B is compact and, by Theorem 3.2, is metrizable, the sequence $\{f_n : n \in \mathbb{Z}^+\}$ has a convergent subsequence. Thus by Theorem 3.3 this subsequence is equi-quasicontinuous. The converse follows from the Theorems 3.4 and 3.2. □

4. Minimal usco maps and topology of pointwise convergence

Let F be a USCO map from a topological space X to $\mathbb{R}$. Define the function f^F as follows:

$$f^F(x) = \sup\{y : y \in F(x)\}.$$

Then of course f^F is a selection of F and f^F is upper semicontinuous. If F is a minimal USCO map from a topological space X to $\mathbb{R}$, then by [18, Theorem 2.6], f^F is also quasicontinuos and locally bounded.

We denote by $UC(X)$ the space of all real valued upper semicontinuous functions, by $Q^\star(X)$ the space of all locally bounded functions from $Q(X)$ and by $M(X)$ the space of all minimal USCO maps.

Define $\Omega\colon M(X) \to Q^\star(X) \cap UC(X)$ as follows: $\Omega(F) = f^F$ ([18]).
By Proposition 2.1, $\overline{\operatorname{Gr} f^F} = \operatorname{Gr} F$.

Proposition 4.1. ([18]) *The mapping $\Omega\colon M(X) \to Q^\star(X) \cap UC(X)$ is a bijection.*

Proposition 4.2. ([18]) *Let X be a topological space. Then the mapping Ω from $(M(X), \mathfrak{U}_p)$ onto $(Q^\star(X) \cap UC(X), \mathfrak{U}_p)$ is uniformly continuous.*

The mapping Ω^{-1} from $(Q^\star(X) \cap UC(X), \tau_p)$ onto $(M(X), \tau_p)$ need not be continuous, see [18, Example 4.3].

Theorem 4.1. *Let X be a Fréchet topological space. Then the following are equivalent:*

(1) *The spaces $(M(X),\mathfrak{U}_p)$ and $(Q^\star(X)\cap UC(X),\mathfrak{U}_p)$ are uniformly isomorphic.*

(2) *X is a discrete topological space.*

P r o o f.

(1) $\Longrightarrow$ (2): Suppose that X is not a discrete topological space. Since X is a Hausdorff non discrete topological space, then there is a point $x_0 \in X$ such that every $U \in \mathscr{U}(x_0)$ is infinite set. Moreover X is a Fréchet space, so there exist a sequence $\{x_n :\ n \in \mathbb{Z}^+\}$ converging to x_0 and open sets $W_n \subset X$, where $n \in \mathbb{Z}^+$, with the following properties: $x_n \in W_n$ for every $n \in \mathbb{Z}^+$, $\{W_n :\ n \in \mathbb{Z}^+\}$ is a pairwise disjoint family and $x_0 \notin \bigcup\{W_n :\ n \in \mathbb{Z}^+\}$.

Equip $\mathscr{U}(x_0)$ with the natural direction: if $U, V \in \mathscr{U}(x_0)$, then $U \geq V$ if and only if $U \subseteq V$. Let f be the function identically equal to 1. For each $U \in \mathscr{U}(x_0)$ define a function $f_U : X \to \mathbb{R}$ as follows:

$$f_U(x) = \begin{cases} 0, & x \in \operatorname{Int} \overline{U} \cap \operatorname{Int} \overline{\bigcup\{W_{2i} :\ i \in \mathbb{Z}^+\}}, \\ 1, & \text{otherwise.} \end{cases}$$

The net $\{f_U :\ U \in \mathscr{U}(x_0)\}$ converges to f in $(Q^\star(X)\cap UC(X),\tau_p)$. Since $0 \in \overline{\operatorname{Gr} f_U}(x_0)$ for every $U \in \mathscr{U}(x_0)$, the net $\{\overline{\operatorname{Gr} f_U} :\ U \in \mathscr{U}(x_0)\}$ does not converge to $\overline{\operatorname{Gr} f}$ in $(M(X),\tau_p)$. Hence mapping Ω^{-1} from $(Q^\star(X)\cap UC(X),\tau_p)$ onto $(M(X),\tau_p)$ is not continuous and so the spaces $(M(X),\mathfrak{U}_p)$ and $(Q^\star(X)\cap UC(X),\mathfrak{U}_p)$ are not uniformly isomorphic.

(2) $\Longrightarrow$ (1): If X is a discrete topological space then the space $Q^\star(X)\cap UC(X)$ and the space $M(X)$ are equal to the space of all continuous functions from X to $\mathbb{R}$. So we are done. □

Theorems 4.2, 4.3 and 4.4 give conditions under which a limit map of a net of maps with nonempty compact values in $(F(X,2^{\mathbb{R}}),\tau_p)$ is minimal USCO.

Denote by $F_{LB}(X,2^{\mathbb{R}})$ the set of all locally bounded maps from a topological space X to $\mathbb{R}$ with closed values.

Theorem 4.2. *Let $\{F_\lambda :\ \lambda \in \Lambda\}$ be a net of maps from a topological space X to $\mathbb{R}$ with non empty compact values pointwise convergent to a map $F \in F_{LB}(X,2^{\mathbb{R}})$. Let the net $\{f^{F_\lambda} :\ \lambda \in \Lambda\}$ be equi-quasicontinuous. Then $F \in M(X)$ if and only if $\overline{\operatorname{Gr} f^F} = \operatorname{Gr} F$.*

P r o o f. Suppose that F is a minimal USCO map. Since f^F is a selection of F, by [18, Proposition 2.1], $\overline{\operatorname{Gr} f^F} = \operatorname{Gr} F$.

We prove the converse. Since the space of all nonempty compact sets of $\mathbb{R}$ is a closed set in $(2^{\mathbb{R}},H_d)$, $F(x)$ must be a nonempty compact set for every $x \in X$.

Since F is locally bounded, f^F is locally bounded, too, and by Remark 2.1, f^F is subcontinuous. It is easy to see that $\{f^{F_\lambda}: \lambda \in \Lambda\}$ pointwise converges to f^F. So by Proposition 3.1, f^F is quasicontinuous. Since f^F is quasicontinuous and subcontinuous selection of F and $\overline{\operatorname{Gr} f^F} = \operatorname{Gr} F$, by Proposition 2.1, $F \in M(X)$. □

Let $\mathscr{F} = \{F_\lambda: \lambda \in \Lambda\}$ be a net of maps from X to $\mathbb{R}$. Let $x \in X$, denote by $\mathscr{F}[x] = \{y \in \mathbb{R}: y \in F_\lambda(x), \lambda \in \Lambda\}$.

Theorem 4.3. *Let $\mathscr{F} = \{F_\lambda: \lambda \in \Lambda\}$ be a net of maps from a topological space X to $\mathbb{R}$ with non empty compact values pointwise convergent to a map $F \in F_{LB}(X, 2^{\mathbb{R}})$ and let $\mathscr{F}[x]$ be a bounded subset of $\mathbb{R}$. Let f_λ be a selection of F_λ for every $\lambda \in \Lambda$ and let $\{f_\lambda: \lambda \in \Lambda\}$ be an equi-quasicontinuous net of functions. Then $F \in M(X)$ if and only if $\overline{\operatorname{Gr} f} = \operatorname{Gr} F$, where f is a cluster point of the net $\{f_\lambda: \lambda \in \Lambda\}$ in the topology of pointwise convergence.*

Proof. The proof is similar to the proof of Theorem 4.2, we only note the following observation.

The product $\prod_{x \in X} \overline{\mathscr{F}[x]}$ is a compact subset of $\mathbb{R}^X = \prod_{x \in X} \mathbb{R}_x$, where $\mathbb{R}_x = \mathbb{R}$ for every $x \in X$, relative to the product topology. Thus there exists a subnet of $\{f_\lambda: \lambda \in \Lambda\}$ which converges in the topology of pointwise convergence to a function f. Evidently f is a selection of F. □

Theorem 4.4. *Let $\{F_n: n \in \mathbb{Z}^+\}$ be a sequence of maps from a topological space X to $\mathbb{R}$ with non empty compact values pointwise convergent to a map $F \in F_{LB}(X, 2^{\mathbb{R}})$. Let f_n be a selection of F_n for every $n \in \mathbb{Z}^+$ and let $\{f_n: n \in \mathbb{Z}^+\}$ be an equi-quasicontinuous sequence of functions. Then $F \in M(X)$ if and only if $\overline{\operatorname{Gr} f} = \operatorname{Gr} F$, where f is a cluster point of the sequence $\{f_n: n \in \mathbb{Z}^+\}$ in the topology of pointwise convergence.*

Theorem 4.5. *For a topological space X the following are equivalent:*

(1) *$(F(X, 2^{\mathbb{R}}), \mathfrak{U}_p)$ is metrizable;*
(2) *$(F(X, 2^{\mathbb{R}}), \tau_p)$ is first countable;*
(3) *$(M(X), \mathfrak{U}_p)$ is metrizable;*
(4) *$(M(X), \tau_p)$ is first countable;*
(5) *X is countable.*

Proof. The proof is similar to the proof of Theorem 3.2, where in part (4) $\implies$ (5) we take $\overline{\operatorname{Gr} f}$ and $\overline{\operatorname{Gr} g_n}$ instead of f and g_n. □

Proposition 4.3. *Let X be a topological space and let B be a closed subset of $(M(X),\tau_p)$. Let for every net $\{F_\lambda :\ \lambda\in\Lambda\}$ in B which is pointwise convergent to a map $F\in F_{LB}(X,2^{\mathbb{R}})$, the net $\{f^{F_\lambda} :\ \lambda\in\Lambda\}$ be equi-quasicontinuous and let $\overline{\mathrm{Gr}\, f^F}=\mathrm{Gr}\, F$. Then B is closed also in $(F_{LB}(X,2^{\mathbb{R}}),\tau_p)$.*

Proof. The proof follows from Theorem 4.2. □

Theorem 4.6. *Let X be a countable Baire space and let B be a subset of $M(X)$. Then B is closed in $(F_{LB}(X,\mathbb{R}),\tau_p)$ if and only if the following conditions are satisfied*

(a) *B is a closed subset of $(M(X),\tau_p)$.*

(b) *For every sequence $\{F_n :\ n\in\mathbb{Z}^+\}$ in B, which is pointwise convergent to a map $F\in F_{LB}(X,2^{\mathbb{R}})$, the sequence $\{f^{F_n} :\ n\in\mathbb{Z}^+\}$ is equi-quasicontinuous and $\overline{\mathrm{Gr}\, f^F}=\mathrm{Gr}\, F$.*

Proof. Let $\{F_n :\ n\in\mathbb{Z}^+\}$ be a sequence in B, which is pointwise convergent to a map $F\in F_{LB}(X,2^{\mathbb{R}})$. Since B is closed in $(F_{LB}(X,2^{\mathbb{R}}),\tau_p)$, the map F is minimal USCO. So by Proposition 2.1 the function f^F is quasicontinuous and $\overline{\mathrm{Gr}\, f^F}=\mathrm{Gr}\, F$. Since the sequence $\{f^{F_n} :\ n\in\mathbb{Z}^+\}$ pointwise converges to f^F, by Theorem 3.1 it is equi-quasicontinuous.

The converse follows from Proposition 4.3 and Theorem 4.5. □

If $B\subseteq F(X,2^{\mathbb{R}})$ and $x\in X$, then denote by $B[x]=\{y\in\mathbb{R} :\ y\in F(x), F\in B\}$.

Theorem 4.7. *Let X be a topological space. Let B be a subset of $M(X)$. Then B is compact relative to the topology of pointwise convergence if the following conditions are satisfied*

(a) *B is a closed subset of $(M(X),\tau_p)$.*

(b) *$\overline{B[x]}$ is a compact for every $x\in X$.*

(c) *If a net $\{F_\lambda :\ \lambda\in\Lambda\}$ in B, pointwise converges to a map $F\in F(X,2^{\mathbb{R}})$, then the net $\{f^{F_\lambda} :\ \lambda\in\Lambda\}$ is equi-quasicontinuous, f^F is locally bounded and $\overline{\mathrm{Gr}\, f^F}=\mathrm{Gr}\, F$.*

Proof. Let $\{F_\lambda :\ \lambda\in\Lambda\}$ be a net in B. By [4], $(\overline{B[x]},d)$ is compact if and only if $(CL(\overline{B[x]}),H_d)$ is compact. The product $\prod_{x\in X} CL(\overline{B[x]})$ is a compact subset of $CL(\mathbb{R}_x)^X=\prod_{x\in X} CL(\mathbb{R}_x)$, where $\mathbb{R}_x=\mathbb{R}$ for every $x\in X$, relative to the product topology, where the space $CL(\mathbb{R})$ is equipped with the topology induced by the metric H_d. Thus there exists a subnet $\{F_\sigma :\ \sigma\in\Sigma\}$ of $\{F_\lambda :\ \lambda\in\Lambda\}$ which converges in the topology of pointwise convergence to a map F. By above mentioned the map F has nonempty compact values. It is clear that $\{f^{F_\sigma} : \sigma \in \Sigma\}$ pointwise converges to f^F. So by Proposition 3.1, f^F is quasicontinuous and since by (c), $\overline{\operatorname{Gr} f^F} = \operatorname{Gr} F$, by Proposition 2.1, $F \in M(X)$. □

Theorem 4.8. *Let X be a countable Baire space. Let B be a subset of $M(X)$. Then B is compact relative to the pointwise topology if and only if the following conditions are satisfied*

(a) *B is a closed subset of $(M(X), \tau_p)$.*

(b) *$\overline{B[x]}$ is a compact for every $x \in X$.*

(c) *If a sequence $\{F_n : n \in \mathbb{Z}^+\}$, in B, pointwise converges to a map $F \in F(X, 2^{\mathbb{R}})$, then the sequence $\{f^{F_n} : n \in \mathbb{Z}^+\}$ is equi-quasicontinuous, f^F is locally bounded and $\overline{\operatorname{Gr} f^F} = \operatorname{Gr} F$.*

Proof. Suppose B is compact. The space $(M(X), \tau_p)$ is Hausdorff so B is a closed subset of $(M(X), \tau_p)$. For the proof of condition (b) see first part of the proof of Theorem 4.7. The projections $p_x\colon \prod_{x \in X} CL(\mathbb{R}_x) \to CL(\mathbb{R}_x)$, where $\mathbb{R}_x = \mathbb{R}$ for every $x \in X$, are continuous and hence the image $CL(\overline{B[x]})$ of B is compact and thus $\overline{B[x]}$ is compact. The condition (c) follows from Theorem 4.6. The converse follows from Theorems 4.7 and 4.5. □

At last we will mention the so-called densely continuous forms introduced by McCoy and Hammer in [17] and then studied by Holá, McCoy, Holý, Vadovič in their papers [16, 20, 22, 23].

Let X, Y be Hausdorff topological spaces. Densely continuous forms from X to Y can be considered as maps (set-valued mappings) from X to Y which have a kind of minimality property found in the theory of minimal USCO maps. In particular, every minimal USCO map from a Baire space X into a metric space Y is a densely continuous form.

For each function f from X to Y denote by

$$C(f) = \{x \in X : f \text{ is continuous at } x\}.$$

For every $f \in DC(X, Y)$, $\operatorname{Gr}(f\lceil C(f))$ is a subset of $X \times Y$.

Denote by $\overline{\operatorname{Gr}(f\lceil C(f))}$ the closure of $\operatorname{Gr}(f\lceil C(f))$ in $X \times Y$.

We define the set $D(X, Y)$ of densely continuous forms by

$$D(X, Y) = \{\overline{\operatorname{Gr}(f\lceil C(f))} : f \in DC(X, Y)\}.$$

The densely continuous forms from X to Y may be considered as maps: for each $x \in X$ and $\Phi \in D(X, Y)$ define $\Phi(x) = \{y \in Y : (x, y) \in \Phi\}$.

Now define $D^\star$ to by the set of all members of $D(X, \mathbb{R})$ that are locally bounded.

Remark 4.1. If X is Baire space, than $M(X) = D^{\star}(X)$ ([18]) and so in Theorems 4.6 and 4.8 we can replace $M(X)$ by $D^{\star}(X)$.

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