VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-010-0028-4
Math. Slovaca **60** (2010), No. 4, 495–506

ON ABSOLUTE SUMMABILITY FOR DOUBLE TRIANGLE MATRICES

Ekrem Savaş — Hamdullah Şevli

(*Communicated by Michal Zajac*)

ABSTRACT. A lower triangular infinite matrix is called a triangle if there are no zeros on the principal diagonal. The main result of this paper gives a minimal set of sufficient conditions for a double triangle T to be a bounded operator on $\mathscr{A}_k^2$; i.e., $T \in B\left(\mathscr{A}_k^2\right)$ for the sequence space $\mathscr{A}_k^2$ defined below. As special summability methods T we consider weighted mean and double Cesàro, $(C,1,1)$, methods. As a corollary we obtain necessary and sufficient conditions for a double triangle T to be a bounded operator on the space $\mathscr{BV}$ of double sequences of bounded variation.

1. Introduction

Let $\sum a_v$ denote a series with partial sums s_n. For an infinite matrix T, t_n, the nth term of the T-transform of $\{s_n\}$ is denoted by

$$t_n = \sum_{v=0}^{\infty} t_{nv} s_v .$$

A series $\sum a_v$ is said to be absolutely T-summable if $\sum\limits_n |\Delta\, t_{n-1}| < \infty$, where Δ is the forward difference operator defined by $\Delta\, t_{n-1} = t_{n-1} - t_n$. Papers dealing with absolute summability date back at least as far as Fekete [3].

A sequence $\{s_n\}$ is said to be of bounded variation (bv) if $\sum\limits_n |\Delta\, s_n| < \infty$. Thus, to say that a series is absolutely summable by a matrix T is equivalent to saying that the T-transform of the sequence of its partial sums is in bv. Necessary and sufficient conditions for a matrix $T\colon bv \to bv$ are known. (See, e.g. Stieglitz and Tietz [13].)

2000 Mathematics Subject Classification: Primary 40C05, 40F05.
Keywords: bounded operator, double sequence space, triangular matrices, $\mathscr{A}_k$ spaces, weighted mean methods.

Let σ_n^α denote the nth terms of the transform of a Cesáro matrix (C, α) of a sequence $\{s_n\}$. In 1957 Flett [4] introduced the following definition. A series $\sum a_n$, with partial sums s_n, is said to be absolutely (C, α) summable of order $k \geq 1$, written $\sum a_n$ is summable $|C, \alpha|_k$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|\sigma_{n-1}^\alpha - \sigma_n^\alpha\right|^k < \infty. \tag{1}$$

He then proved the following inclusion theorem. If a series $\sum a_n$ is summable $|C, \alpha|_k$, then it is summable $|C, \beta|_r$ for each $r \geq k \geq 1$, $\alpha > -1$, $\beta > \alpha + 1/k - 1/r$. It then follows that, if one chooses $r = k$, then a series $\sum a_n$ which is $|C, \alpha|_k$ summable is also $|C, \beta|_k$ summable for $k \geq 1$, $\beta > \alpha > -1$.

There are many papers in the literature which establish sufficient conditions for a series $\sum a_n$ summable $|A|_k$ to imply that it is summable $|B|_k$, where A and B are particular lower triangular matrices. Two such papers are [7] and [9].

Let $\sum a_n$ be a series with partial sums s_n. Denote by $\mathscr{A}_k$ the sequence space defined by

$$\mathscr{A}_k = \left\{\{s_n\} : \sum_{n=1}^{\infty} n^{k-1} \left| a_n\right|^k < \infty, \ a_n = s_n - s_{n-1}\right\}.$$

If one sets $\alpha = 0$ in the inclusion statement involving (C, α) and (C, β), then one obtains the fact that $(C, \beta) \in B(\mathscr{A}_k)$ for each $\beta > 0$, where $B(\mathscr{A}_k)$ denotes the algebra of all matrices that map $\mathscr{A}_k$ to $\mathscr{A}_k$.

Let A be a sequence to sequence transformation mapping the sequence (s_n) into (t_n). If, whenever (s_n) converges absolutely, (t_n) converges absolutely, A is called absolutely conservative. If the absolute convergence of (s_n) implies absolute convergence of (t_n) to the same limit, A is called absolutely regular.

In 1970, using the same definition as Flett, Das [2] defined such a matrix to be absolutely kth power conservative for $k \geq 1$, if $T \in B(\mathscr{A}_k)$; i.e., if $\{s_n\}$ is a sequence satisfying

$$\sum_{n=1}^{\infty} n^{k-1} \left| s_n - s_{n-1}\right|^k < \infty, \tag{2}$$

then

$$\sum_{n=1}^{\infty} n^{k-1} \left| t_n - t_{n-1}\right|^k < \infty.$$

For $k = 1$ condition (2) guarantees the convergence of (s_n). Note that when $k > 1$, (2) does not necessarily imply the convergence of (s_n). For example, take

$$s_n = \sum_{v=1}^{n} \frac{1}{v \log(v+1)}.$$

Then (2) holds but (s_n) does not converge. Thus, since the limit of (s_n) need not exist, we cannot introduce the concept of absolute kth power regularity when $k > 1$.

In that same paper Das proved that every conservative Hausdorff matrix $H \in B(\mathscr{A}_k)$, which contains as a special case the fact that $(C, \beta) \in B(\mathscr{A}_k)$ for $\beta > 0$. In [10] it was shown that $(C, \alpha) \in B(\mathscr{A}_k)$ for each $\alpha > -1$. Also in [14] a further extension of absolute Cesaro summability was obtained.

Given a matrix T one can find a matrix B such that the statement $T \in B(\mathscr{A}_k)$ is equivalent to $B \in B\left(\ell^k\right)$. Since necessary and sufficient conditions are not known for an arbitrary $B \in B\left(\ell^k\right)$ for $k > 1$, it is not reasonable to expect to find necessary and sufficient conditions for $T \in B(\mathscr{A}_k)$.

A lower triangular matrix with nonzero principal diagonal entries is called a triangle. Recently, in [11], a minimal set of sufficient conditions was established for a triangle $T \in B(\mathscr{A}_k)$ as follows.

Theorem 1. *$T = (t_{nv})$ be a triangle satisfying*

(i) $|t_{nn}| = O(1)$,

(ii) $\sum\limits_{v=0}^{n-1} |t_{vv}| \left|\hat{t}_{nv}\right| = O(|t_{nn}|)$,

(iii) $\sum\limits_{n=v+1}^{\infty} (n|t_{nn}|)^{k-1} \left|\hat{t}_{nv}\right| = O(v|t_{vv}|)^{k-1}$.

Then, $T \in B(\mathscr{A}_k)$, $k \geq 1$.

As corollaries, other $A \in B(\mathscr{A}_k)$ results were obtained, including that of [8].

2. The main result

It is the purpose of this work to extend Theorem 1 to doubly infinite matrices. Let $\sum\limits_{m=0}^{\infty} \sum\limits_{n=0}^{\infty} a_{mn}$ be an infinite double series with real or complex numbers, and with partial sums

$$s_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij}.$$

For any double sequence $\{x_{mn}\}$ we shall define

$$\Delta_{11} x_{mn} = x_{mn} - x_{m+1,n} - x_{m,n+1} + x_{m+1,n+1}.$$

The series $\sum\sum a_{mn}$ is said to be summable $|C, \alpha, \beta|_k$, $k \geq 1$, $\alpha, \beta > -1$, if (see [6])

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \left|\Delta_{11} \sigma_{m-1,n-1}^{\alpha\beta}\right|^k < \infty,$$

where $\sigma_{mn}^{\alpha\beta}$ denotes the mn-term of the (C,α,β) transform of a sequence (s_{mn}); i.e.,

$$\sigma_{mn}^{\alpha\beta} = \frac{1}{E_m^{\alpha} E_n^{\beta}} \sum_{i=0}^{m} \sum_{j=0}^{n} E_{m-i}^{\alpha-1} E_{n-j}^{\beta-1} s_{ij}.$$

Define

$$\mathscr{A}_k^2 := \left\{ (s_{mn}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \left| a_{mn} \right|^k < \infty, \ a_{mn} = \Delta_{11} s_{m-1,n-1} \right\} \tag{3}$$

for $k \geq 1$.

A four dimensional matrix $T = \left(t_{mnij}\right)_{m,n,i,j=0,1,\dots}$ is said to be absolutely kth power conservative for $k \geq 1$, if $T \in B\left(\mathscr{A}_k^2\right)$; i.e.,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \left| \Delta_{11} s_{m-1,n-1} \right|^k < \infty$$

implies that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} \left| \Delta_{11} t_{m-1,n-1} \right|^k < \infty,$$

where

$$t_{mn} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} t_{mnij} s_{ij} \qquad (m,n = 0,1,\dots).$$

Quite recently in [12], it is shown that $(C,\alpha,\beta) \in B\left(\mathscr{A}_k^2\right)$ for each $\alpha, \beta > -1$.

Let $\sum\sum a_{mn}$ be a doubly infinite series with partial sums $\{s_{mn}\}$. Denote by T the doubly infinite matrix with entries t_{mnij}, $0 \leq i \leq m$, $0 \leq j \leq n$. For a four-fold sequence, like t_{mnij}, it will be understood that Δ_{11} operates only on the first two subscript; i.e.,

$$\Delta_{11} t_{mnij} = t_{mnij} - t_{m+1,n,i,j} - t_{m,n+1,i,j} + t_{m+1,n+1,i,j}.$$

We may associate with T two doubly infinite matrices $\bar{T}$ and $\hat{T}$ as follows:

$$\bar{t}_{mnij} = \sum_{\mu=i}^{m} \sum_{\nu=j}^{n} t_{mn\mu\nu} \qquad m,n,i,j = 0,1,2,\dots$$

and

$$\hat{t}_{m-1,n-1,i,j} = \Delta_{11} \bar{t}_{m-1,n-1,i,j} \qquad m,n = 1,2,3,\dots,$$

where $\hat{t}_{0000} = \bar{t}_{0000} = t_{0000}$.

If $T = (t_{mnij})$ is a lower triangular matrix, then $\bar{T} = (\bar{t}_{mnij})$ and $\hat{T} = \left(\hat{t}_{m-1,n-1,i,j}\right)$ are also lower triangular matrices. If T is a triangle, then for each $m,n \in \mathbb{N}^0$

$$\hat{t}_{m-1,n-1,m,n} = \bar{t}_{mnmn} = t_{mnmn} \neq 0,$$

and $\bar{T}$ and $\hat{T}$ are triangles.

We shall say that the series $\sum\sum a_{mn}$ is absolutely T-summable of order $k \geq 1$ if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left|\Delta_{11}T_{m-1,n-1}\right|^{k}<\infty,$$

where

$$T_{mn}=\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}t_{mn\mu\nu}s_{\mu\nu}.$$

Theorem 2. *Let $T=(t_{mnij})$ be a double triangle satisfying*

(i) $\sum\limits_{i=0}^{m}\sum\limits_{j=0}^{n}|t_{ijij}|\left|\hat{t}_{m-1,n-1,i,j}\right|=O\left(|t_{mnmn}|\right)$

and

(ii) $\sum\limits_{m=i}^{\infty}\sum\limits_{n=j}^{\infty}(mn|t_{mnmn}|)^{k-1}\left|\hat{t}_{m-1,n-1,i,j}\right|=O\left(ij\left|t_{ijij}\right|\right)^{k-1}.$

Then, $T\in B\left(\mathscr{A}_k^2\right)$, $k\geq 1$.

Proof. If y_{mn} denotes the mn-term of the T-transform of a double sequence $\{s_{mn}\}$, then

$$\begin{aligned} y_{mn} &= \sum_{\mu=0}^{m}\sum_{\nu=0}^{n}t_{mn\mu\nu}s_{\mu\nu}=\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}t_{mn\mu\nu}\sum_{i=0}^{\mu}\sum_{j=0}^{\nu}a_{ij}\\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}a_{ij}\sum_{\mu=i}^{m}\sum_{\nu=j}^{n}t_{mn\mu\nu}=\sum_{i=0}^{m}\sum_{j=0}^{n}a_{ij}\bar{t}_{mnij}. \end{aligned}$$

It then follows that

$$\begin{aligned} Y_{mn} &= \Delta_{11}y_{m-1,n-1}=y_{m-1,n-1}-y_{m-1,n}-y_{m,n-1}+y_{m,n}\\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\left(\bar{t}_{m-1,n-1,i,j}-\bar{t}_{m-1,n,i,j}-\bar{t}_{m,n-1,i,j}+\bar{t}_{m,n,i,j}\right)a_{ij}\\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\Delta_{11}\bar{t}_{m-1,n-1,i,j}a_{ij}=\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{t}_{m-1,n-1,i,j}a_{ij}. \end{aligned}$$

Using Hölder's inequality, (i) and (ii), we get

$$\begin{aligned} &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left|Y_{mn}\right|^{k}\\ &=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\left|\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{t}_{m-1,n-1,i,j}a_{ij}\right|^{k} \end{aligned}$$

$$\leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}\sum_{i=0}^{m}\sum_{j=0}^{n}\left|\hat{t}_{m-1,n-1,i,j}\right|\left|t_{ijij}\right|^{1-k}\left|a_{ij}\right|^{k}\times$$

$$\times\left(\sum_{i=0}^{m}\sum_{j=0}^{n}\left|t_{ijij}\right|\left|\hat{t}_{m-1,n-1,i,j}\right|\right)^{k-1}$$

$$=O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn|t_{mnmn}|)^{k-1}\sum_{i=0}^{m}\sum_{j=0}^{n}\left|\hat{t}_{m-1,n-1,i,j}\right|\left|t_{ijij}\right|^{1-k}\left|a_{ij}\right|^{k}$$

$$=O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|t_{ijij}\right|^{1-k}\left|a_{ij}\right|^{k}\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}(mn|t_{mnmn}|)^{k-1}\left|\hat{t}_{m-1,n-1,i,j}\right|$$

$$=O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left|t_{ijij}\right|^{1-k}\left|a_{ij}\right|^{k}\left(ij\left|t_{ijij}\right|\right)^{k-1}$$

$$=O(1)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij)^{k-1}\left|a_{ij}\right|^{k}=O(1),$$

since $(s_{mn})\in\mathscr{A}_k^2$. This completes the proof. $\square$

Using an argument similar to that of [8] we now establish another set of sufficient conditions for a nonnegative double triangle T to be in $B\left(\mathscr{A}_k^2\right)$.

Theorem 3. *Let $T=(t_{mnij})$ be a double triangle with nonnegative entries satisfying*

(i) $mnt_{mnmn}=O(1)$,

(ii) $\sum\limits_{\mu=0}^{m}t_{mn\mu j}=\sum\limits_{\mu=0}^{m-1}t_{m-1,n,\mu,j}=a(n,j)$ *and*
$\sum\limits_{\nu=0}^{n}t_{mni\nu}=\sum\limits_{\nu=0}^{n-1}t_{m,n-1,i,\nu}=b(m,i)$,
$0\leq i\leq m,\ 0\leq j\leq n,\ m,n=0,1,\dots,$

(iii) $\sum\limits_{i=0}^{m}\sum\limits_{j=0}^{n}t_{mnij}=O(1),\ m,n=0,1,\dots,$

(iv) $\Delta_{11}t_{m-1,n-1,i,j}\geq 0$ *for* $0\leq i\leq m,\ 0\leq j\leq n,\ m,n=1,2,\dots$ *and*

(v) $\sum\limits_{i=0}^{m}\sum\limits_{j=0}^{n}t_{ijij}\hat{t}_{m-1,n-1,i,j}=O(t_{mnmn})$.

Then, $T\in B\left(\mathscr{A}_k^2\right)$, $k\geq 1$.

Proof. Since $T=(t_{mnij})$ is a positive matrix, from (ii) and (iv),

$$\hat{t}_{m-1,n-1,i,j}=\bar{t}_{m-1,n-1,i,j}-\bar{t}_{m,n-1,i,j}-\bar{t}_{m-1,n,i,j}+\bar{t}_{m,n,i,j}$$

$$
\begin{aligned}
&= \sum_{r=i}^{m-1}\sum_{s=j}^{n-1} t_{m-1,n-1,r,s} - \sum_{r=i}^{m}\sum_{s=j}^{n-1} t_{m,n-1,r,s} - \sum_{r=i}^{m-1}\sum_{s=j}^{n} t_{m-1,n,r,s} + \sum_{r=i}^{m}\sum_{s=j}^{n} t_{mnrs} \\
&= \sum_{r=i}^{m}\sum_{s=j}^{n} (t_{m-1,n-1,r,s} - t_{m,n-1,r,s} - t_{m-1,n,r,s} + t_{mnrs}) \\
&= \sum_{r=i}^{m} \Bigg(\sum_{s=0}^{n-1} t_{m-1,n-1,r,s} - \sum_{s=0}^{j-1} t_{m-1,n-1,r,s} - \sum_{s=0}^{n-1} t_{m,n-1,r,s} + \sum_{s=0}^{j-1} t_{m,n-1,r,s} \\
&\qquad - \sum_{s=0}^{n} t_{m-1,n,r,s} + \sum_{s=0}^{j-1} t_{m-1,n,r,s} + \sum_{s=0}^{n} t_{mnrs} - \sum_{s=0}^{j-1} t_{mnrs} \Bigg) \\
&= \sum_{r=i}^{m} \Bigg(b(m-1,r) - \sum_{s=0}^{j-1} t_{m-1,n-1,r,s} - b(m,r) + \sum_{s=0}^{j-1} t_{m,n-1,r,s} \\
&\qquad - b(m-1,r) + \sum_{s=0}^{j-1} t_{m-1,n,r,s} + b(m,r) - \sum_{s=0}^{j-1} t_{mnrs} \Bigg) \\
&= \sum_{s=0}^{j-1} \Bigg(- \sum_{r=0}^{m-1} t_{m-1,n-1,r,s} + \sum_{r=0}^{i-1} t_{m-1,n-1,r,s} + \sum_{r=0}^{m} t_{m,n-1,r,s} - \sum_{r=0}^{i-1} t_{m,n-1,r,s} \\
&\qquad + \sum_{r=0}^{m-1} t_{m-1,n,r,s} - \sum_{r=0}^{i-1} t_{m-1,n,r,s} - \sum_{r=0}^{m} t_{mnrs} + \sum_{r=0}^{i-1} t_{mnrs} \Bigg) \\
&= \sum_{s=0}^{j-1} \Bigg(-a(n-1,s) + \sum_{r=0}^{i-1} t_{m-1,n-1,r,s} + a(n-1,s) - \sum_{r=0}^{i-1} t_{m,n-1,r,s} \\
&\qquad + a(n,s) - \sum_{r=0}^{i-1} t_{m-1,n,r,s} - a(n,s) + \sum_{r=0}^{i-1} t_{mnrs} \Bigg) \\
&= \sum_{r=0}^{i-1}\sum_{s=0}^{j-1} (t_{m-1,n-1,r,s} - t_{m,n-1,r,s} - t_{m-1,n,r,s} + t_{mnrs}) \\
&= \sum_{r=0}^{i-1}\sum_{s=0}^{j-1} \Delta_{11} t_{m-1,n-1,r,s} \geq 0,
\end{aligned}
$$

and $\hat{T} = \left(\hat{t}_{m-1,n-1,i,j}\right)$ is a positive matrix. As in the proof of Theorem 2,

$$
Y_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} \hat{t}_{m-1,n-1,i,j} a_{ij}.
$$

Using Hölder's inequality, (i) and (v), we get

$$\begin{aligned}
&\sum_{m=1}^{M}\sum_{n=1}^{N}(mn)^{k-1}\left|Y_{mn}\right|^{k}\\
&=\sum_{m=1}^{M}\sum_{n=1}^{N}(mn)^{k-1}\left|\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{t}_{m-1,n-1,i,j}a_{ij}\right|\\
&=O\left(1\right)\sum_{m=1}^{M}\sum_{n=1}^{N}(mn)^{k-1}\sum_{i=0}^{m}\sum_{j=0}^{n}\left(t_{ijij}\right)^{1-k}\left(\hat{t}_{m-1,n-1,i,j}\right)\left|a_{ij}\right|^{k}\times\\
&\qquad\times\left(\sum_{i=0}^{m}\sum_{j=0}^{n}t_{ijij}\hat{t}_{m-1,n-1,i,j}\right)^{k-1}\\
&=O\left(1\right)\sum_{m=1}^{M}\sum_{n=1}^{N}\left(mnt_{mnmn}\right)^{k-1}\sum_{i=0}^{m}\sum_{j=0}^{n}\left(t_{ijij}\right)^{1-k}\left(\hat{t}_{m-1,n-1,i,j}\right)\left|a_{ij}\right|^{k}\\
&=O\left(1\right)\sum_{i=0}^{M}\sum_{j=0}^{N}\left(t_{ijij}\right)^{1-k}\left|a_{ij}\right|^{k}\sum_{m=i}^{M}\sum_{n=j}^{N}\hat{t}_{m-1,n-1,i,j}.
\end{aligned}$$

Using (ii) and (iii),

$$\begin{aligned}
&\sum_{m=i}^{M}\sum_{n=j}^{N}\hat{t}_{m-1,n-1,i,j}\\
&=\sum_{m=i}^{M}\sum_{n=j}^{N}\sum_{r=0}^{i-1}\sum_{s=0}^{j-1}\Delta_{11}t_{m-1,n-1,r,s}\\
&=\sum_{r=0}^{i-1}\sum_{s=0}^{j-1}\sum_{m=i}^{M}\left(t_{m-1,j-1,r,s}-t_{m,j-1,r,s}-t_{m-1,N,r,s}+t_{MNrs}\right)\\
&=\sum_{r=0}^{i-1}\sum_{s=0}^{j-1}\left(t_{i-1,j-1,r,s}-t_{M,j-1,r,s}-t_{i-1,N,r,s}+t_{MNrs}\right)\\
&=\sum_{r=0}^{i-1}\left(b(i-1,r)-b(M,r)-b(i-1,r)+\sum_{s=j}^{N}t_{i-1,Nrs}+b(M,r)-\sum_{s=j}^{N}t_{MNrs}\right)\\
&=\sum_{r=0}^{i-1}\sum_{s=j}^{N}\left(t_{i-1,N,r,s}-t_{MNrs}\right)\\
&=\sum_{s=j}^{N}\left(a(N,s)-\sum_{r=0}^{M}t_{MNrs}+\sum_{r=i}^{M}t_{MNrs}\right)
\end{aligned}$$

$$= \sum_{s=j}^{N} \left(a(N,s) - a(N,s) + \sum_{r=i}^{M} t_{MNrs} \right)$$
$$= \sum_{r=i}^{M} \sum_{s=j}^{N} t_{MNrs} = O\left(1\right) \sum_{r=0}^{M} \sum_{s=0}^{N} t_{MNrs} = O\left(1\right).$$

Using (i) and since $(s_{mn}) \in \mathscr{A}_k^2$,

$$\sum_{m=1}^{M} \sum_{n=1}^{N} (mn)^{k-1} \left|Y_{mn}\right|^k = O\left(1\right) \sum_{i=1}^{M} \sum_{j=1}^{N} (t_{ijij})^{1-k} \left|a_{ij}\right|^k$$
$$= O\left(1\right) \sum_{i=1}^{M} \sum_{j=1}^{N} (ij)^{k-1} \left|a_{ij}\right|^k = O\left(1\right), \qquad M, N \to \infty.$$

Hence the proof is complete. □

Corollary 1. *The conditions of* Theorem 3 *imply these of* Theorem 2.

P r o o f. Condition (v) of Theorem 3 with t_{mnij} nonnegative implies condition (i) of Theorem 2. Since $T = (t_{mnij})$ is a positive matrix, from (ii) and (iv) of Theorem 3, we have $\hat{T} = \left(\hat{t}_{m-1,n-1,i,j}\right)$ is a positive matrix. Using (i), (ii) and (iii) of Theorem 3,

$$\frac{1}{\left(ij\left|t_{ijij}\right|\right)^{k-1}} \sum_{m=i}^{\infty} \sum_{n=j}^{\infty} \left(mn\left|t_{mnmn}\right|\right)^{k-1} \left|\hat{t}_{m-1,n-1,i,j}\right|$$
$$= O\left(1\right) \sum_{m=i}^{\infty} \sum_{n=j}^{\infty} \hat{t}_{m-1,n-1,i,j} = O\left(1\right),$$

and condition (ii) of Theorem 2 is satisfied. □

A factorable double weighted mean matrix, written $\left(\bar{N}, p_{mn}\right) = \left(\bar{N}, p_m, q_n\right)$ is a double triangle with entries

$$t_{mnij} = \frac{p_{ij}}{P_{mn}} = \frac{p_i q_j}{P_m Q_n},$$

where $\{p_m\}$, $\{q_n\}$ are nonnegative sequences with $p_0, q_0 > 0$ and $P_m = \sum_{i=0}^{m} p_i \to \infty$, $Q_n = \sum_{j=0}^{n} q_j \to \infty$.

Corollary 2. *If $\{p_{mn}\}$ is a positive factorable double sequence satisfying*

$$mnp_{mn} \asymp P_{mn}, \tag{4}$$

then $\left(\bar{N}, p_{mn}\right) \in B\left(\mathscr{A}_k^2\right)$.

P r o o f. In Theorem 2 set $T = \left(\bar{N}, p_{mn}\right)$. We may write condition (i) of Theorem 2 as

$$\begin{aligned}
&\frac{1}{|t_{mnmn}|}\sum_{i=0}^{m}\sum_{j=0}^{n}|t_{ijij}|\left|\hat{t}_{m-1,n-1,i,j}\right| \\
&=\frac{1}{|t_{mnmn}|}\Big\{|t_{mnmn}|\left|\hat{t}_{m-1,n-1,m,n}\right|+\sum_{i=0}^{m-1}|t_{inin}|\left|\hat{t}_{m-1,n-1,i,n}\right| \\
&\qquad +\sum_{j=0}^{n-1}|t_{mjmj}|\left|\hat{t}_{m-1,n-1,m,j}\right|+\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|t_{ijij}|\left|\hat{t}_{m-1,n-1,i,j}\right|\Big\} \\
&=I_1+I_2+I_3+I_4, \qquad \text{say.}
\end{aligned}$$

Since

$$\hat{t}_{m-1,n-1,m,n} = \bar{t}_{mnmn} = t_{mnmn},$$

we have

$$I_1 = |t_{mnmn}| = \frac{p_{mn}}{P_{mn}} = O\left(1\right).$$

$$\begin{aligned}
\hat{t}_{m-1,n-1,i,n} &= \sum_{r=0}^{i-1}\left(\frac{p_{rn}}{P_{m-1,n}}-\frac{p_{rn}}{P_{mn}}\right) \\
&= \frac{p_{mn}}{P_{mn}}\frac{P_{i-1}}{P_{m-1}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
I_2 = \frac{1}{|t_{mnmn}|}\sum_{i=0}^{m-1}|t_{inin}|\left|\hat{t}_{m-1,n-1,i,n}\right| &= \frac{P_{mn}}{p_{mn}}\sum_{i=0}^{m-1}\frac{p_{in}p_{mn}P_{i-1}}{P_{in}P_{mn}P_{m-1}} \\
&= O\left(1\right)\frac{q_n}{Q_nP_{m-1}}\sum_{i=0}^{m-1}p_i = O\left(1\right).
\end{aligned}$$

Similarly $I_3 = O\left(1\right)$.

$$\hat{t}_{m-1,n-1,i,j} = \frac{p_{mn}}{P_{mn}}\frac{P_{i-1}}{P_{m-1}}\frac{Q_{j-1}}{Q_{n-1}}, \tag{5}$$

$$\begin{aligned}
I_4 &= \frac{P_{mn}}{p_{mn}}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\frac{p_{ij}p_{mn}P_{i-1}Q_{j-1}}{P_{ij}P_{mn}P_{m-1}Q_{n-1}} \\
&= O\left(1\right)\frac{1}{P_{m-1}Q_{n-1}}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}p_iq_j = O\left(1\right),
\end{aligned}$$

and condition (i) of Theorem 2 is satisfied.

Using (4) and (5),

$$\left(\frac{P_{ij}}{ijp_{ij}}\right)^{k-1}\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\left(\frac{mnp_{mn}}{P_{mn}}\right)^{k-1}\frac{p_{mn}P_{i-1}Q_{j-1}}{P_{mn}P_{m-1}Q_{n-1}}$$
$$=O\left(1\right)\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{p_{mn}P_{i-1}Q_{j-1}}{P_{mn}P_{m-1}Q_{n-1}}$$
$$=O\left(1\right)P_{i-1}Q_{j-1}\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\frac{p_m}{P_mP_{m-1}}\frac{q_n}{Q_nQ_{n-1}}=O\left(1\right),$$

and condition (ii) of Theorem 2 is satisfied. □

From [12, Corollary 3], $(C,1,1)\in B\left(\mathscr{A}_k^2\right)$. For completeness, we show that Corollary 3 follows from Corollary 2.

Corollary 3. $(C,1,1)\in B\left(\mathscr{A}_k^2\right)$.

Proof. Set each $p_n=q_n=1$ for each m and n in Corollary 2. Then the condition (4) is satisfied. □

Since $\left(\bar{N},p_{mn}\right)$ and $(C,1,1)$ are nonnegative, we also obtain Corollary 2 and Corollary 3 from Theorem 3.

If we take $k=1$, then $\mathscr{A}_1=bv$ and $\mathscr{A}_1^2=\mathscr{BV}$. Some properties of the space $\mathscr{BV}$ of double sequences of bounded variation was examined in [1].

Corollary 4. *Let $T=(t_{mnij})$ be a double triangle. $T\in B\left(\mathscr{BV}\right)$ if and only if*

$$\sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\left|\hat{t}_{m-1,n-1,i,j}\right|=O\left(1\right).$$

Proof. Necessity. If y_{mn} denotes the mn-term of the T-transform of a double sequence $\{s_{mn}\}$. Then $Y_{mn}=\Delta_{11}y_{m-1,n-1}=\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{t}_{m-1,n-1,i,j}a_{ij}$. We are given that $T\in B\left(\mathscr{BV}\right)$. Hence $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|Y_{mn}|<\infty$ whenever $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{mn}|<\infty$. The space $\mathscr{BV}$ is a BK-space; i.e., a Banach space with continuous coordinates, with norm given by

$$\|x\|=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\Delta_{11}x_{m-1,n-1}\right|.$$

Applying the Banach-Steinhaus theorem, there exists a constant $K>0$ such that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|Y_{mn}|\le K\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{mn}|.$$

Let $e^{(ij)}$ denote the ijth coordinate sequence; i.e., the sequence with a 1 in position ij and zeros elsewhere. Then

$$K \geq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|Y_{mn}| = \sum_{m=i}^{\infty}\sum_{n=j}^{\infty}\hat{t}_{m-1,n-1,i,j}.$$

The sufficiency follows from Theorem 2. □

Patterson [5] has obtained a different necessary and sufficient condition for an arbitrary double matrix $T \in B\,(\mathscr{BV})$.

REFERENCES

[1] ALTAY, B.—BAŞAR, F.: *Some new spaces of double sequences*, J. Math. Anal. Appl. **309** (2005), 70–90.

[2] DAS, G.: *A Tauberian theorem for absolute summability*, Math. Proc. Cambridge Philos. Soc. **67** (1970), 321–326.

[3] FEKETE, M.: *Zur Theorie der divergenten Reihen*, Math. És Termész Ért. **29** (1911), 719–726 (Hungarian).

[4] FLETT, T. M.: *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. London Math. Soc. **7** (1957), 113–141.

[5] PATTERSON, R. F.: *Four dimensional matrix characterization of absolute summability*, Soochow J. Math. **30** (2004), 21–26.

[6] RHOADES, B. E.: *Absolute comparison theorems for double weighted mean and double Cesàro means*, Math. Slovaca **48** (1998), 285–301.

[7] RHOADES, B. E.: *Inclusion theorems for absolute matrix summability methods*, J. Math. Anal. Appl. **238** (1999), 82–90.

[8] RHOADES, B. E.: *On absolute normal double matrix summability methods*, Glas. Mat. Ser. III **38** (2003), 57–73.

[9] RHOADES, B. E.—SAVAŞ, E.: *General inclusion theorems for absolute summability of order $k \geq 1$*, Math. Inequal. Appl. **8** (2005), 505–520.

[10] SAVAŞ, E.—ŞEVLI, H.: *On extension of a result of Flett for Cesáro matrices*, Appl. Math. Lett. **20** (2007), 476–478.

[11] SAVAŞ, E.—ŞEVLI, H.—RHOADES, B. E.: *Triangles which are bounded operators on $\mathscr{A}_k$*, Bull. Malays. Math. Sci. Soc. (2) **32** (2009), 223–231.

[12] SAVAŞ, E.—ŞEVLI, H.—RHOADES, B. E.: *On the Cesáro summability of double series*, J. Inequal. Appl. (2008), Art. ID 257318, 4 pp.

[13] STIEGLITZ, M.—TIETZ, H.: *Matrixtransformationen von Folgenräumen. Eine Ergebnisübersicht*, Math. Z. **154** (1977), 1–16 (German).

[14] ŞEVLI, H.—SAVAŞ, E.: *On absolute Cesaro summability*, J. Inequal. Appl. (2009), Art. ID 279421, 7pp.

Received 8. 9. 2008
Accepted 13. 10. 2008

Department of Mathematics
İstanbul Ticaret University
Üsküdar-İstanbul
TURKEY
E-mail: ekremsavas@yahoo.com
hsevli@yahoo.com