

RELATIVELY UNIFORM CONVERGENCES IN ARCHIMEDEAN LATTICE ORDERED GROUPS

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ABSTRACT. For an archimedean lattice ordered group G let G^d and G^\wedge be the divisible hull or the Dedekind completion of G , respectively. Put $G^{d\wedge} = X$. Then X is a vector lattice. In the present paper we deal with the relations between the relatively uniform convergence on X and the relatively uniform convergence on G . We also consider the relations between the σ -convergence and the relatively uniform convergence on G . For any nonempty class τ of lattice ordered groups we introduce the notion of τ -radical class; we apply this notion by investigating relative uniform convergences.

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1. Introduction

For references concerning sequential convergences cf. the expository article [7].

The notion of relatively uniform convergence has been systematically used in the theory of vector lattices; cf. the monographs [2], [12] and [15].

The relatively uniform convergence in archimedean lattice ordered groups was dealt with in [1], [7], [8], [13] and [14]; for related results, cf. also [4], [5].

If G is an ℓ -subgroup of a lattice ordered group H and if α is a convergence on H , then α induces a convergence on G which will be denoted by $\alpha(H, G)$.

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Let G be an archimedean lattice ordered group. We denote by G^d and G^\wedge the divisible hull and the Dedekind completion of G , respectively. We put $G^{d\wedge} = X$. Then X is a vector lattice. Under the natural embedding, G is an ℓ -subgroup of X .

For $(x_n) \in G^{\mathbb{N}}$ and $x \in G$ we write $x_n \rightarrow_o x$ if (x_n) o -converges to the element x . Further, we denote by u the relatively uniform convergence on the vector lattice X . The relatively uniform convergence in G will be denoted by α_u .

For any nonempty class τ of lattice ordered groups we introduce the notion of τ -radical class; we apply this notion by investigating relatively uniform convergences. (For the corresponding definitions, cf. Section 5 below.)

Under the above notation we prove the following results:

- 1) $u(X, G) = \alpha_u$.
- 2) If G is either divisible or σ -complete, then the relation

$$x_n \rightarrow_{\alpha_u} x \implies x_n \rightarrow_o x$$

is valid in G .

- 3) The relatively uniform completion of G under the convergence α_u is equal to the intersection of all $G_i \subseteq X$ such that
 - (i) $G \subseteq G_i$, and
 - (ii) G_i is a relatively uniformly complete ℓ -subgroup of X under the convergence u .
- 4) Let \mathcal{C} be the class of all lattice ordered groups satisfying the condition (C) (given in Section 5) and let \mathcal{K} be the collection of all lattice ordered groups G such that G is archimedean and relatively uniformly complete. Then \mathcal{K} is a \mathcal{C} -radical class of lattice ordered groups.

2. Relatively uniform convergence

For lattice ordered groups and for vector lattices we apply the notation as in [2].

For the sake of completeness we recall some definitions concerning vector lattices and lattice ordered groups.

Let V be a vector lattice. We say that a sequence (x_n) in V *relatively uniformly converges* to an element $x \in V$ if there exists $a \in V$, $a > 0$, such that for each real $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|x_n - x| \leq \varepsilon a$ for all $n \in \mathbb{N}$, $n \geq n_0$.

It is easy to verify that the mentioned definition for V is equivalent with the following one.

Let $(x_n) \in V^{\mathbb{N}}$, $x \in V$. We say that the sequence (x_n) *relatively uniformly converges* to x in the vector lattice V if there exist an element $a > 0$ in V and a sequence of reals (λ_n) such that $\lambda_n \downarrow 0$ and $|x_n - x| \leq \lambda_n a$ for each $n \in \mathbb{N}$.

Under these assumptions we write $x_n \rightarrow_u x$ (or $x_n \xrightarrow{a}_u x$). If the role of the vector lattice V is to be emphasized then we write also $x_n \rightarrow_{u(V)} x$. (Cf. [2].)

Further, we say that the sequence (x_n) *o-converges* to the element x in the vector lattice V and we write $x_n \rightarrow_o x$ (or $x_n \rightarrow_{o(V)} x$, if the role of V is emphasized) if there exist sequences (u_n) and (v_n) in V such that $u_n \uparrow x$, $v_n \downarrow x$ and $u_n \leq x_n \leq v_n$ for each $n \in \mathbb{N}$.

As usual, the notation $u_n \uparrow x$ means that $u_n \leq u_{n+1}$ for each $n \in \mathbb{N}$ and $\bigvee_{n \in \mathbb{N}} u_n = x$. The meaning of the notation $v_n \downarrow x$ is similar.

Now, let (x_n) be a sequence in a lattice ordered group G and $x \in G$. To avoid the trivial case, we always assume that G has more than one element. Suppose that there exists an element $b > 0$ in G such that for every $p \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ with

$$p|x_n - x| \leq b \quad \text{for each } n \in \mathbb{N}, \quad n \geq n_0.$$

Then we say that the sequence (x_n) *converges relatively uniformly* to the element x in the lattice ordered group G and we write $x_n \rightarrow_{\alpha_u} x$ (or $x_n \rightarrow_{\alpha_u(G)} x$). (Cf. [12].)

The *o*-convergence in the lattice ordered group G is defined analogously as in the case of vector lattices.

If V is a vector lattice, then we denote by V_0 the corresponding lattice ordered group (i.e., when dealing with V_0 , the multiplication of elements of V with reals is not taken into account).

The following assertion is easy to verify.

LEMMA 2.1. *Let (x_n) be a sequence in a vector lattice V and $x \in V$. Then the following conditions are equivalent:*

- (i) *The sequence (x_n) converges relatively uniformly to the element x in the vector lattice V .*
- (ii) *The sequence (x_n) converges relatively uniformly to the element x in the lattice ordered group V_0 .*

Suppose that G is an archimedean lattice ordered group. Consider the divisible hull G^d of G . Under the natural embedding, G is an ℓ -subgroup of G^d . If $y \in G^d$, $y > 0$, then there exist $n \in \mathbb{N}$ and $x \in G^+$ with $y = \frac{x}{n}$. (Cf. e.g., [10].) The lattice ordered group G^d is archimedean as well.

LEMMA 2.2. *Let (x_n) be a sequence in an archimedean lattice ordered group G and let $x \in G$. Then the following conditions are equivalent:*

- (i) *The sequence (x_n) converges relatively uniformly to the element x in the lattice ordered group G .*
- (ii) *The sequence (x_n) converges relatively uniformly to the element x in the lattice ordered group G^d .*

Proof. The implication (i) \implies (ii) is obvious, since G is embedded in G^d .

Assume that (ii) is valid. Thus there is an element b_1 in G^d such that for every $p \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ with

$$p|x_n - x| \leq b_1 \quad \text{for each } n \in \mathbb{N}, \quad n \geq n_0.$$

There are $n_1 \in \mathbb{N}$ and $b_2 \in G^+$ such that

$$b_1 = \frac{b_2}{n_1}.$$

Then $b_1 \leq b_2$. Hence

$$p|x_n - x| \leq b_2 \quad \text{for each } n \in \mathbb{N}, \quad n \geq n_0.$$

Therefore the condition (i) is satisfied. □

Let G be as above. We put $X = G^{d\wedge}$. In view of [10], X is a vector lattice. Under the natural embeddings, both G and G^d are ℓ -subgroups of the lattice ordered group X_0 . (The symbol X_0 has an analogous meaning with respect to X as the symbol V_0 with respect to V .)

LEMMA 2.3. *Let (x_n) be a sequence in G^d and $x \in G^d$. Then the following conditions are equivalent:*

- (i) *The sequence (x_n) converges relatively uniformly to the element x in the lattice ordered group G^d .*
- (ii) *The sequence (x_n) converges relatively uniformly to the element x in the vector lattice X .*

Proof. Assume that (i) is valid. Since G^d is embedded in X_0 , in view of Lemma 2.1 we conclude that (ii) holds.

Further, suppose that (ii) is satisfied. Hence according to Lemma 2.1, (x_n) relatively uniformly converges to x in the lattice ordered group X_0 . In view of the construction of Dedekind completion of G^d , for each element $0 < b_1 \in G^{d\wedge}$ there exists an element $b_2 \in G^d$ such that $b_1 < b_2$. Now, applying the analogous argument as in the proof of Lemma 2.2, we conclude that the condition (i) is valid. \square

PROPOSITION 2.4. *Let (x_n) be a sequence in an archimedean lattice ordered group G and $x \in G$. Put $G^{d\wedge} = X$. Then the following conditions are equivalent:*

- (i) *The sequence (x_n) converges relatively uniformly to the element x in the lattice ordered group G .*
- (ii) *The sequence (x_n) converges relatively uniformly to the element x in the vector lattice X .*

Proof. This is a consequence of Lemmas 2.2 and 2.3. \square

In other words, we can say that the convergence α_u on the lattice ordered group G is induced by the convergence u on the vector lattice X .

The following example shows that if X is replaced by a vector lattice containing G and different from X , then Proposition 2.4 need not hold.

Example 1. Let \mathbb{R} be the additive group of all reals with the natural linear order. For each $i \in \mathbb{N}$ let $G_i = \mathbb{R}$. We put

$$H = \prod_{i \in \mathbb{N}} G_i.$$

Evidently, H is a vector lattice. For $h \in H$ and $i \in \mathbb{N}$ we denote by $h(i)$ the component of h in the direct factor G_i ; further we put $\text{supp}(h) = \{i \in \mathbb{N} : h(i) \neq 0\}$. Let G be the set of all $h \in H$ such that $\text{supp}(h)$ is finite. Then G is an ℓ -subgroup of H ; moreover, G is archimedean.

For $n \in \mathbb{N}$ we define $x_n \in G$ as follows: if $i \in \mathbb{N}$ and $i \neq n$, then the component $x_n(i)$ of x_n in G_i is equal to 0; if $i = n$, then $x_n(i) = \frac{1}{n}$.

There exists $a \in H$ such that $a(i) = 1$ for each $i \in \mathbb{N}$. Put $\lambda_n = \frac{1}{n}$ for each $n \in \mathbb{N}$. Thus

$$x_n = |x_n| \leq \lambda_n a \quad \text{for each } n \in \mathbb{N},$$

hence $x_n \rightarrow_u 0$ in the vector lattice H . On the other hand, the relation $x_n \rightarrow_{\alpha_u} 0$ fails to be valid in the lattice ordered group G .

3. o -convergence

Again, let G be an archimedean lattice ordered group and let G^d , X be as above.

In view of [2, Chap. X, §9], we have:

LEMMA 3.1. *Let L be a lattice and let L_1 be the Dedekind completion of L . Then the o -convergence in L is induced by the o -convergence in L_1 .*

Hence, in particular, we obtain:

LEMMA 3.2. *Let G be an archimedean lattice ordered group. Let (x_n) be a sequence in G and $x \in G$. Then the following conditions are equivalent:*

- (i) *The sequence (x_n) o -converges to x in G .*
- (ii) *The sequence (x_n) o -converges to x in G^\wedge .*

THEOREM 3.3. (Cf. [2, Chap. XV, Theorem 19].) *Let (x_n) be a sequence in archimedean vector lattice V and $x \in V$. Assume that (x_n) relatively uniformly converges to x in V . Then (x_n) o -converges to x in V .*

PROPOSITION 3.4. *Assume that G is a divisible lattice ordered group. Let (x_n) be a sequence in G and $x \in G$. Then we have*

$$x_n \rightarrow_{\alpha_u(G)} x \implies x_n \rightarrow_{o(G)} x.$$

Proof. Since G is divisible, $G^d = G$. Put $G^{d\wedge} = X$. Hence X is a vector lattice and $X = G^\wedge$.

Assume that the relation $x_n \rightarrow_{\alpha_u(G)} x$ is valid. Then according to Proposition 2.4 we have $x_n \rightarrow_{u(X)} x$. Thus Theorem 3.3 yields $x_n \rightarrow_{o(X)} x$. Applying Lemma 3.2 we obtain $x_n \rightarrow_{o(G)} x$. \square

Now, let us suppose that G is σ -complete lattice ordered group. Let (x_n) be a sequence in G , $x \in G$, and assume that the relation $x_n \rightarrow_{\alpha_u} x$ is valid in G .

We apply the notation as above. Further, we put $|x_n - x| = z_n$. Thus there is $b \in G$, $b > 0$, such that for each $k \in \mathbb{N}$ there exists $n_0(k) \in \mathbb{N}$ such that for each $n \geq n_0(k)$ we have $kz_n \leq b$.

Hence for any $n \geq n_0(1)$ we get $z_n \leq b$. Since G is σ -complete, for each $n \in \mathbb{N}$ with $n \geq n_0(1)$ there exists the join

$$t_n = \bigvee_{m \geq n} z_m$$

in G . We put

$$\bar{t} = t_{n_0(1)}$$

and for $n = 1, 2, \dots, n_0(1) - 1$ we set

$$t_n = z_n \vee z_{n+1} \vee \dots \vee z_{n_0(1)-1} \vee \bar{t}.$$

Therefore $t_n \geq t_{n+1}$ and $t_n \geq z_n \geq 0$ for each $n \in \mathbb{N}$.

Further, there exists $q \in G$ with

$$q = \bigwedge_{n \in \mathbb{N}} t_n;$$

clearly $q \geq 0$.

Let $k \in \mathbb{N}$. Take any $n \in \mathbb{N}$ with $n \geq n_0(k)$; we get

$$kq \leq kt_n = k\left(\bigvee_{m \geq n} z_m\right) = \bigvee_{m \geq n} kz_m \leq b.$$

Since G is archimedean, we conclude that $q = 0$. Therefore we get that the relation

$$z_n \rightarrow_o 0$$

is valid in G . Thus we have also $-z_n \rightarrow_o 0$. Clearly

$$-z_n \leq x - x_n \leq z_n$$

for each $n \in \mathbb{N}$. Thus $x - x_n \rightarrow_o 0$ in G . This yields that $x_n \rightarrow_o x$ is valid in G .

It is well-known that σ -complete lattice ordered groups are archimedean. Hence, summarizing the above results, we have:

PROPOSITION 3.5. *Let G be a σ -complete lattice ordered group. Let (x_n) be a sequence in G and $x \in G$ such that (x_n) converges relatively uniformly to the element x . Then (x_n) o -converges to the element x .*

4. Relative uniform completion of G

We recall the definitions of a relatively uniformly Cauchy sequence in vector lattices and in lattice ordered groups (cf., e.g., [12] and [6]).

Let V be a vector lattice. A sequence (x_n) in V is called *relatively uniformly Cauchy* if there exists $b \in V$, $b > 0$, such that for every real $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|x_n - x_m| \leq \varepsilon b$ for each $m, n \in \mathbb{N}$, $m, n \geq n_0$.

Let G be a lattice ordered group. It is said that a sequence (x_n) in G is *relatively uniformly Cauchy* if there exists $b \in G$, $b > 0$, such that for every $p \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ with the property $p|x_n - x_m| \leq b$ for each $m, n \in \mathbb{N}$, $m, n \geq n_0$.

If every relatively uniformly Cauchy sequence in a vector lattice V converges relatively uniformly in V then V is called *relatively uniformly complete*. The concept of a relatively uniformly complete lattice ordered group is defined analogously.

A. I. Veksler [14] discussed the notion of a relative uniform completion of vector lattices. This notion can be applied to lattice ordered groups by a slight modification (cf. [1], [6], [12]):

Let H be an archimedean lattice ordered group with the properties

- (a) G is an ℓ -subgroup of H .
- (b) H is relatively uniformly complete.
- (c) If G is an ℓ -subgroup of K , K is an ℓ -subgroup of H and K is relatively uniformly complete, then $K = H$.

Then H is said to be a *relative uniform completion* of G .

A relative uniform completion exists and it is uniquely determined up to isomorphisms over G (cf. [6]).

Let V_0 and X_0 be as in Section 2. The following result is easy to verify.

LEMMA 4.1. *Let (x_n) be a sequence in a vector lattice V . Then the following conditions are equivalent:*

- (i) (x_n) is a relatively uniformly Cauchy sequence in the vector lattice V .
- (ii) (x_n) is a relatively uniformly Cauchy sequence in the lattice ordered group V_0 .

As a consequence of Lemma 2.1 and Lemma 4.1 we get:

LEMMA 4.2. *The following conditions are equivalent:*

- (i) V is a relatively uniformly complete vector lattice.
- (ii) V_0 is a relatively uniformly complete lattice ordered group.

THEOREM 4.3. *Let G be an archimedean lattice ordered group and $X = G^{d\wedge}$. Assume that $\{G_i : i \in I\}$ is the system of all ℓ -subgroups K of X_0 satisfying the following conditions:*

- (i) G is an ℓ -subgroup of K .
- (ii) K is a relatively uniformly complete ℓ -subgroup of X_0 .

Then $\bigcap_{i \in I} G_i$ is a relative uniform completion of G .

P r o o f. With respect to [14], X is a relatively uniformly complete vector lattice. By 4.2, X_0 is a relatively uniformly complete lattice ordered group. We have to prove that $H = \bigcap_{i \in I} G_i$ has the properties (a), (b) and (c). We consider the relative uniform convergence in X_0 .

(a) Evidently, H is an ℓ -subgroup of X_0 and G is an ℓ -subgroup of H .

(b) Let (x_n) be a sequence in H and let (x_n) be relatively uniformly Cauchy in H . Hence for each $i \in I$, (x_n) is a sequence in G_i and (x_n) is relatively uniformly Cauchy in G_i . According to the assumption (x_n) is relatively uniformly convergent in all G_i . Whence for each $i \in I$ there exist $x_i \in G_i$ and $0 < v_i \in G_i$ with $x_n \xrightarrow{v_i}_{\alpha_u} x_i$. Archimedeanity of X_0 implies that limits in X_0 are uniquely determined (cf. [7]). Hence $x_i = x$ for each $i \in I$, so $x \in H$. Let $i \in I$ be fixed. Then $x_n \xrightarrow{v_i}_{\alpha_u} x$. According to the proofs of 2.2 and 2.3 there exists $v \in G$, $v \geq v_i$. Therefore $x_n \xrightarrow{v}_{\alpha_u} x$. By (a), $G \subseteq H$. Hence (x_n) is relative uniformly convergent in H . This implies that H is relatively uniformly complete.

(c) This is clear. □

5. The \mathcal{C} -radical class \mathcal{K}

For a lattice ordered group G let $c(G)$ be the system of all convex ℓ -subgroups of G . This system is partially ordered by the set-theoretical inclusion. Then $c(G)$ is a complete lattice.

Let $(G_i)_{i \in I}$ be an indexed system of elements of $c(G)$. In the lattice $c(G)$ we have

$$\bigcap_{i \in I} G_i = \bigwedge_{i \in I} G_i.$$

We put $\bigcup_{i \in I} G_i = H_o$. Let H be the set of all elements $h \in G$ which can be written in the form $h = g_1 + g_2 + \cdots + g_n$ with $g_1, \dots, g_n \in H_o$. Then the relation

$$H = \bigvee_{i \in I} G_i$$

is valid in the lattice $c(G)$.

The notion of radical class of lattice ordered groups has been introduced in [11] and it was investigated in several papers.

We introduce a generalization of this notion as follows.

Let τ be a nonempty class of lattice ordered groups. A class \mathcal{A} of lattice ordered groups is said to be a τ -radical class if the following conditions are satisfied:

- (i) \mathcal{A} is closed with respect to isomorphisms.
- (ii) If $G \in \mathcal{A}$ and $G_1 \in c(G)$ then $G_1 \in \mathcal{A}$.
- (iii) If $G \in \tau$ and $\{G_i\}_{i \in I} \subseteq \mathcal{A} \cap c(G)$, $I \neq \emptyset$, then $\bigvee_{i \in I} G_i \in \mathcal{A}$.

Obviously, a class \mathcal{A} of lattice ordered groups is a radical class if and only if it is a \mathcal{G} -radical class, where \mathcal{G} is the class of all lattice ordered groups.

We denote by \mathcal{C} the class of all lattice ordered groups H fulfilling the condition

- (C) If $K \in c(H)$ and (x_n) is a sequence in K which is relatively uniformly Cauchy in H , then it is relatively uniformly Cauchy in K .

The condition (C) was applied in [8]. Some lattice ordered groups fulfil the condition (C) (e.g., the lattice ordered group R) and some do not (cf. [8, Example 4.9]).

Let \mathcal{K} be the class of all archimedean lattice ordered groups G such that G is relatively uniformly complete. In the present section we prove that \mathcal{K} is a \mathcal{C} -radical class.

For the classical Riesz Decomposition Theorem concerning lattice ordered groups, different formulations (and different proofs) have been applied in the literature. Let us quote a rather simple formulation (together with the proof) as given in [3, Introduction, Section 10].

THEOREM 5.1. (Cf. [3].) *Let G be a lattice ordered group. If $0 < x \leq d_1 + \dots + d_n$ where $d_i \in G^+$ then $x = c_1 + \dots + c_n$ where $0 \leq c_i \leq d_i$.*

PROOF. Let $c_1 = x \wedge d_1$ and $c = -c_1 + x$. Then $0 \leq c_1 \leq d_1$ and

$$0 \leq c = -c_1 + x = -(x \wedge d_1) + x = (-x \vee -d_1) + x = 0 \vee (-d_1 + x) \leq d_2 + \dots + d_n.$$

Thus by induction $c = c_2 + \dots + c_n$ where $0 \leq c_i \leq d_i$, whence $x = c_1 + c_2 + \dots + c_n$ where $0 \leq c_i \leq d_i$. \square

We will apply the idea of this proof below.

In what follows we assume that G is an archimedean lattice ordered group.

LEMMA 5.2. *Assume that $x, y, b_1 \in G^+$ and $x \leq y$. Put*

$$x \wedge b_1 = x_1, \quad y \wedge b_1 = y_1, \quad x - x_1 = x'_1, \quad y - y_1 = y'_1.$$

Then $y_1 - x_1 \leq y - x$ and $x'_1 \leq y'_1$.

PROOF. (Cf. Fig. 1.) We put $x \vee y_1 = z$. Then we have $x \wedge y_1 = x_1$, hence $z - x = y_1 - x_1$ and $z - y_1 = x - x_1$. Since $x \leq z \leq y$, we obtain $z - x \leq y - x$, thus $y_1 - x_1 \leq y - x$.

Further, $y'_1 \geq z - y_1 = x - x_1 = x'_1$. \square

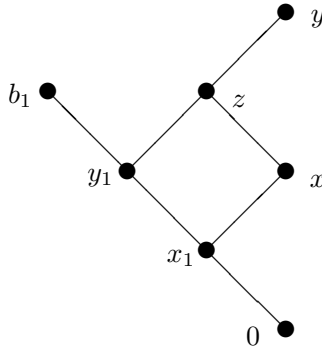


FIGURE 1

Applying Theorem 5.1 (together with the method of its proof) and using Lemma 5.2 we obtain:

LEMMA 5.3. *Assume that $p, q, d_1, \dots, d_m \in G^+$, $d = d_1 + \dots + d_m$, $p \leq q \leq d$. Then there are $c_i, c'_i \in G$ ($i = 1, 2, \dots, m$) such that*

$$\begin{aligned} p &= c_1 + \dots + c_m, \\ q &= c'_1 + \dots + c'_m, \end{aligned}$$

$$0 \leq c_i \leq d_i, \quad 0 \leq c'_i \leq d_i, \quad 0 \leq c'_i - c_i \leq q - p \text{ for } i = 1, 2, \dots, m.$$

LEMMA 5.4. *Assume that $x, y, d_1, \dots, d_m \in G^+$, $d = d_1 + \dots + d_m$, $x \leq d$, $y \leq d$. Then there are $x_i, y_i \in G^+$ ($i = 1, 2, \dots, m$) such that*

$$\begin{aligned} x &= x_1 + \dots + x_m, & y &= y_1 + \dots + y_m, \\ 0 &\leq x_i \leq d_i, & 0 &\leq y_i \leq d_i. \end{aligned}$$

Further we have

$$|x_i - y_i| \leq |x - y| \quad \text{for } i = 1, 2, \dots, m.$$

Proof. The first part of the assertion of the lemma is a consequence of Theorem 5.1. For proving the assertion concerning $|x_i - y_i|$ and $|x - y|$ it suffices to consider the elements $p = x \wedge y$, $q = x \vee y$ and to apply Lemma 5.3. We have clearly $|x - y| = q - p$ and analogously for $|x_i - y_i|$. \square

LEMMA 5.5. (Cf. [8].) *Suppose that G is relatively uniformly complete and that G_1 is a convex ℓ -subgroup of G . Then G_1 is relatively uniformly complete.*

LEMMA 5.6. *The following conditions are equivalent:*

- (i) *G is relatively uniformly complete.*
- (ii) *If (x_n) is a relatively uniformly Cauchy sequence in G such that $x_n \geq 0$ for each $n \in \mathbb{N}$, then (x_n) is relatively uniformly convergent in G .*

Proof. The implication (i) \implies (ii) is obvious. Assume that the condition (ii) is valid. Suppose that (y_n) is a relatively uniformly Cauchy sequence in G . Then (y_n) is bounded in G , hence there is $b \in G$ such that $b \leq y_n$ for each $n \in \mathbb{N}$. Put $y_n - b = x_n$; hence $0 \leq x_n$ for each $n \in \mathbb{N}$. Also, (x_n) is a relatively uniformly Cauchy sequence in G . In view of (ii), there exists $x \in G$ such that (x_n) relatively uniformly converges to x in G . Thus (y_n) relatively uniformly converges to $x + b$ in G . Hence the condition (i) is satisfied. \square

The validity of the following assertion is obvious.

LEMMA 5.7. *Let (x_n) and (y_n) be sequences in a lattice ordered group H . Suppose that (y_n) is a relatively uniformly Cauchy sequence in H and that $|x_n - x_m| \leq |y_n - y_m|$ for each $n, m \in \mathbb{N}$. Then (x_n) is a relatively uniformly Cauchy sequence in H .*

LEMMA 5.8. *Let H be a lattice ordered group satisfying the condition (C) and let $\{G_i\}_{i \in I} \subseteq c(H)$, $\bigvee_{i \in I} G_i = H$. Assume that all G_i are archimedean and relatively uniformly complete. Then H is archimedean and relatively uniformly complete.*

Proof. It is well-known that the collection of all archimedean lattice ordered groups is a radical class. Hence H is an archimedean lattice ordered group.

Let (x_n) be a relatively uniformly Cauchy sequence in H such that $0 \leq x_n$ for each $n \in \mathbb{N}$. Then (x_n) is upper-bounded in H . Hence there is $b \in H$ such that $b \geq x_n$ for each $n \in \mathbb{N}$.

In view of the relation $H = \bigvee_{i \in I} G_i$ there exist $b_1, \dots, b_m \in \bigcup_{i \in I} G_i$ such that $b = b_1 + \dots + b_m$. Without loss of generality we can suppose that $b_1 \geq 0, \dots, b_m \geq 0$.

There exist $i(1), \dots, i(m) \in I$ such that $b_1 \in G_{i(1)}, \dots, b_m \in G_{i(m)}$. Let $n \in \mathbb{N}$. Then $0 \leq x_n \leq b_1 + \dots + b_m$. Hence according to Theorem 5.1 there are elements x_{n1}, \dots, x_{nm} in H such that $0 \leq x_{nk} \leq b_k$ for each $k \in \{1, 2, \dots, m\}$ and

$$x_n = x_{n1} + \dots + x_{nm}.$$

If $s, n \in \mathbb{N}$, then in view of Lemma 5.4 we have

$$|x_{nj} - x_{sj}| \leq |x_n - x_s|$$

for each $j \in \{1, 2, \dots, m\}$. In view of Lemma 5.7 we obtain that (x_{nj}) is a relatively uniformly Cauchy sequence in the lattice ordered group H .

According to the condition (C), (x_{nj}) is a relatively uniformly Cauchy sequence in $G_{i(j)}$ ($j = 1, 2, \dots, m$). Since $G_{i(j)}$ is relatively uniformly complete, the sequence (x_{nj}) relatively uniformly converges in $G_{i(j)}$ to some element x^j . From the fact that $G_{i(j)} \subseteq H$ we conclude that (x_{nj}) relatively uniformly converges to x^j also in the lattice ordered group H . Therefore the sequence (x_n) relatively uniformly converges in H to the element $x^1 + \dots + x^m$.

For finishing the proof it suffices to apply Lemma 5.6. \square

In view of Lemma 5.5, Lemma 5.8 and the fact that if G satisfies the condition (C), so does each convex ℓ -subgroup of G , we get:

THEOREM 5.9. *The collection \mathcal{K} is a \mathcal{C} -radical class of lattice ordered groups.*

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