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VERY TRUE ON CBA FUZZY LOGIC

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ABSTRACT. CBA logic was introduced as a non-associative generalization of the Łukasiewicz many-valued propositional logic. Its algebraic semantic is just the variety of commutative basic algebras. Petr Hájek introduced vt-operators as models for the "very true" connective on fuzzy logics. The aim of the paper is to show possibilities of using vt-operators on commutative basic algebras, especially we show that CBA logic endowed with very true connective is still fuzzy.

1. Introduction

Recently, many algebraic structures intensively studied in (quantum) logic theories induce lattices with antitone involutions on their principal filters (e.g. boolean algebras, MV-algebras, orthomodular lattices, lattice ordered effect algebras, etc.). It leads to generalization of those theories to the unique one. In 2005, I. Chajda introduced so-called *basic algebras* which form an MV-like variety containing all mentioned classes of algebras (see [9]).

Basic algebras have in general non-commutative and non-associative binary operation $\oplus$. By *a block* in a basic algebra we mean a maximal subset of its support formed by mutually commuting elements. An importance of blocks in the basic algebras' theory involves the research of commutative basic algebras (basic algebras with commutative $\oplus$).

Similarly as MV-algebras are algebraic semantic of the Łukasiewicz many-valued logic, the commutative basic algebras are algebraic counterparts for a non-associative fuzzy logic CBA ([4]). Recall that there are lot of cases when it is more then natural to use a non-associative logical deduction (see e.g. [11, 12, 14]). On the contrary, the theory of non-associative fuzzy logic has not been

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studied yet. Basic algebras can be therefore considered as an uncommon model of such a non-associative deduction. This fact enables us to study basic algebras in a new logical context.

The problem of stressing very true sentences at the expance of almost false ones was solved e.g. by P. Hájek [10]. The main instrument of his theory is a new unary connective vt ("very true") added to the language of the studied logic with the three extra axioms and extra deductive rule (truth confirmation). Now, it may be interesting to study properties of a logic created this way.

In the presented paper we solve problems related to the implementation of the very true connective into CBA logic theory. Especially, we show that the newly created logic is still fuzzy.

MV-algebras were introduced by C. C. Chang [8] as the semantics for the Łukasiewicz many-valued propositional calculus in the end of fifties of the last century. Recall that an *MV-algebra* is an algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ of the type $\langle 2, 1, 0\rangle$ satisfying identities

(MV1) $x \oplus y = y \oplus x$,

(MV2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,

(MV3) $x \oplus 0 = x$,

(MV4) $\neg\neg x = x$,

(MV5) $x \oplus \neg 0 = \neg 0$,

(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

It is well known that we can define a partial order relation $\leq$ on every MV-algebra, namely

$$x \leq y \iff \neg x \oplus y = 0.$$

The induced lattice operations are then introduced as

$$x \vee y \stackrel{\text{def}}{=} \neg(\neg x \oplus y) \oplus y,$$

$$x \wedge y \stackrel{\text{def}}{=} \neg(\neg x \vee \neg y).$$

Moreover, we standardly define also further binary operations $\odot$, $\to$ for the MV-algebras

$$x \odot y \stackrel{\text{def}}{=} \neg(\neg x \oplus \neg y),$$

$$x \to y \stackrel{\text{def}}{=} \neg x \oplus y.$$

For any MV-algebra A, we can find an *antitone involution* $f_y\colon [y,1] \to [y,1]$ (i.e. $f_y(f_y(x)) = x$ and $x_1 \leq x_2 \implies f_y(x_1) \geq f_y(x_2)$ for each $x, x_1, x_2 \in A$) on every pricipal filter (called section) $[y,1]$ of A. Namely, for a given section $[y,1] \subseteq A$, its antitone involution is a mapping $f_y\colon [y,1] \to [y,1]$ such that $f_y(x) = \neg x \oplus y$.

Definition 1.1. ([9]) By a *lattice with the sectionally antitone involutions* we understand a structure $(A; \vee, \wedge, ({}^{a})_{a\in A}, 0, 1)$, where

1. $(A; \vee, \wedge, 0, 1)$ is a bounded lattice,
2. $\forall a \in A,\ \ {}^{a}\colon [a, 1] \to [a, 1]$ is an antitone involution.

The following theorem shows that the class of lattices with sectionally antitone involutions forms a variety.

Theorem 1.1.

(i) *If we put* $\neg x := x^0$ *and* $x \oplus y := (x^0 \vee y)^y$ *in a given lattice with sectionally antitone involutions* $\mathscr{A} = (A; \vee, \wedge, ({}^{a})_{a\in A}, 0, 1)$, *then we obtain an algebra* $\mathscr{Z}(\mathscr{A}) = (A; \oplus, \neg, 0)$ *satisfying*
  (B1) $x \oplus 0 = x$,
  (B2) $\neg\neg x = x$,
  (B3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,
  (B4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$.

(ii) *Consider an algebra* $\mathscr{Z} = (Z; \oplus, \neg, 0)$ *satisfying* (B1)–(B4) *and put* $x \vee y := \neg(\neg x \oplus y) \oplus y$, $x \wedge y := \neg(\neg x \vee \neg y)$, $1 := \neg 0$ *and for each* $x \in [a, 1]$ *also* $x^a := \neg x \oplus a$. *Then the system* $\mathscr{A}(\mathscr{Z}) = (Z; \vee, \wedge, ({}^{a})_{a\in A}, 0, 1)$ *is a lattice with sectionally antitone involutions.*

(iii) *The correspondence* $\mathscr{A} \longleftrightarrow \mathscr{Z}$ *from the foregoing is one-to-one.*

Proof. See [9]. □

An algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ satisfying identities (B1)–(B4) from the last theorem is called a *basic algebra*. We standardly define also other binary operations in basic algebras, namely:

$$x \odot y \overset{\text{def}}{=} \neg(\neg x \oplus \neg y);$$

$$x \to y \overset{\text{def}}{=} \neg x \oplus y;$$

$$x \ominus y \overset{\text{def}}{=} x \odot \neg y;$$

$$1 \overset{\text{def}}{=} \neg 0.$$

As it was mentioned the variety of all basic algebras contains some important subvarieties, e.g. orthomodular lattices, effect algebras, Boolean algebras, MV-algebras and also the variety of commutative basic algebras, briefly CB-algebras.

2. Commutative basic algebras and non-associative fuzzy logics

As it was said, the commutative basic algebras have been studied during the foregoing years, which has brought answers to many important problems. Mainly:

Remark 2.1. ([2]) Finite commutative basic algebras are just finite MV-algebras.

Remark 2.2. ([5]) There are non-associative basic algebras. So the class of MV-algebras is not identical to the class of basic algebras.

In [6], the question of subdirect representation was solved. Recall some, for us most important, results.

Definition 2.1. Let us consider a commutative basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ and $F \subseteq A$, $F \neq \emptyset$. Then F is called a $\odot$-*filter* if

(F1) $\forall x \in F,\ \forall y \in A: \ y \geq x \implies y \in F$,

(F2) $\forall x, y \in F: \ x \odot y \in F$.

If moreover

(F3) $\forall x, y \in A: \ x \odot (y \odot F) = (x \odot y) \odot F$,

F is said to be a *filter* of $\mathscr{A}$.

Consider following denotation:

$$\alpha_a^b(x) \stackrel{\text{def}}{=} (a \odot b) \to (a \odot (b \odot x)),$$

$$\beta_a^b(x) \stackrel{\text{def}}{=} b \to (a \to (a \odot b) \odot x).$$

Theorem 2.3. *Let F be a $\odot$-filter of a commutative basic algebra $\mathscr{A}$. Then F is a filter of $\mathscr{A}$ iff $\alpha_a^b(x), \beta_a^b(x) \in F$ for each $a, b \in A$ and for each $x \in F$.*

The mutual relationship of filters and congruences in commutative basic algebras is described in the following theorem:

Theorem 2.4.

(i) *Let F be a filter of a commutative basic algebra $\mathscr{A}$. Then the relation*

$$\Theta(F) = \big\{(x, y) : \ (\exists f_1, f_2 \in F)(x \odot f_1 = y \odot f_2)\big\}$$

is a congruence on $\mathscr{A}$.

(ii) *Let $\theta \in \operatorname{Con} \mathscr{A}$. Then $1/\theta$ is a filter of $\mathscr{A}$ such that $\theta = \Theta(1/\theta)$.*

For the next result we introduce for each $M \subseteq A$ the set

$$M^\perp \stackrel{\text{def}}{=} \big\{x : \ (\forall y \in M)(x \vee y = 1)\big\}.$$

Theorem 2.5.

(i) *$M^{\perp}$ is a filter of $\mathscr{A}$ for any $M \subseteq A$.*

(ii) *Let $\mathscr{A}$ be subdirectly irreducible commutative basic algebra. Then it is linearly ordered.*

The foregoing results show the possibilities of using commutative basic algebras in the fuzzy logics' theory. Till now, most of logics that have been introduced are associative. A non-associative fuzzy logic, whose algebraic model are just commutative basic algebras, was introduced in [4]. Let us denote it CBA.

The language of CBA is built from a set of propositional variables, connectives $\neg$ (a negation), $\to$ (an implication). Axioms are

(A1) $\varphi \to (\psi \to \varphi)$,

(A2) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$,

(A3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$,

(A4) $\neg\neg\varphi$,

(A5) $(\varphi \to \psi) \to (\varphi \to \neg\neg\psi)$

and the deductive rules are

(MP) $\varphi, \varphi \to \psi \vdash \psi$,

(SF) $\varphi \to \psi \vdash (\psi \to \chi) \to (\varphi \to \chi)$.

By Block and Pigozzi's sense [1], commutative basic algebras can be considered as the semantics of the CBA logic. We introduce a binary relation $\models_{CBA}$ such that for an arbitrary set Σ of CBA formulas and an arbitrary CBA formula φ we have:

If the identity $\varphi = 1$ holds for any commutative basic algebras satisfying identities $\phi = 1$ for each $\phi \in \Sigma$ then we put

$$\Sigma \models_{CBA} \varphi.$$

Then we have

$$\Sigma \vdash \varphi \iff \Sigma \models_{CBA} \varphi.$$

Further, we introduce an analogy of vt connective (introduced by Hájek in [10] for fuzzy logics) for the non-associative logic CBA.

Definition 2.2. By a CBA_{vt} we mean a logic of a language $\{\to, \neg, vt\}$ with the deduction rules (MP), (SF) and truth confirmation which from φ infer $vt(\varphi)$. The axioms are (A1)–(A5) and also

(VT1) $vt(\varphi) \to \varphi$,

(VT2) $vt(\varphi \to \psi) \to (vt(\varphi) \to vt(\psi))$,

(VT3) $vt(\varphi \vee \psi) \to (vt(\varphi) \vee vt(\psi))$,

where $\psi \vee \varphi$ is $(\psi \to \varphi) \to \varphi$.

For the algebraization of CBA_{vt} we need to describe the very true connective in an algebraic sense. It leads to the following definition of a vt-operator on a commutative basic algebra.

Definition 2.3. Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a commutative basic algebra. Then a mapping $t\colon A \to A$ is said to be a *vt-operator* on $\mathscr{A}$ if it satisfies for each $x, y \in A$ the following:

(T1) $t(1) = 1$;

(T2) $t(x) \le x$;

(T3) $t(x \to y) \le t(x) \to t(y)$;

(T4) $t(x \vee y) \le t(x) \vee t(y)$.

We say that $\mathscr{A} = (A; \oplus, \neg, t, 0)$ is a CB_{vt}-algebra if

(i) $\mathscr{A} = (A; \oplus, \neg, 0)$ is a CB-algebra,

(ii) t is a vt-operator on $\mathscr{A}$.

We can easily see that the class of all CB_{vt}-algebras forms a variety.

Theorem 2.6. *CB_{vt}-algebras are algebraic semantics for the logic CBA_{vt}.*

Proof. First, consider the fact that for a vt-operator t on a commutative basic algebra A we have

$$t(x) = 1 \iff x = 1,$$

which is equivalent with the deductive rule "truth confirmation". Therefore we get:

(i) $t(x) \le x \iff 1 = t(x) \to x$;

(ii) $t(x \to y) \le t(x) \to t(y) \iff 1 = t(x \to y) \to (t(x) \to t(y))$;

(iii) $t(x \vee y) \le t(x) \vee t(y) \iff 1 = t(x \vee y) \to (t(x) \vee t(y))$.

The theorem follows from the axiomatization of CBA logic ([1], [4]). □

3. vt-operators on commutative basic algebras

Fuzzy truth values as very true, quite true, etc., were introduced and studied by Zadeh as fuzzy sets on $[0, 1]$. The aim of the following is to show that CBA_{vt} logic is also fuzzy, that means that the only subdirectly irreducible members in the variety of commutative basic algebra (the algebraic semantic for CBA_{vt}) are the linearly ordered ones (see [1]).

Let us begin with some properties of commutative basic algebras that are useful for the next results.

Lemma 3.1. *Given a commutative basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$, the following hold for each $x, y, z \in A$:*

(i) $x \odot y \leq z$ *iff* $x \leq y \to z$;

(ii) $x \wedge y = x \odot (x \to y)$;

(iii) $(x \to y) \vee (y \to x) = 1$;

(iv) $x \leq y \implies x \to z \geq y \to z, \;\; z \to x \leq z \to y$.

P r o o f. See [2, 3]. □

Further, we will study the correspondence between congruences and its kernels (so-called t-filters) on commutative basic algebra.

Definition 3.1. A filter F of a CB_{vt}-algebra $\mathscr{A} = (A; \oplus, \neg, t, 0)$ is said to be a *t-filter* if $t(x) \in F$ for each $x \in F$.

To prove the main result we need to describe the congruence $\Theta(F)$ from Th.2.4 in a different way.

Lemma 3.2. *Given a filter F in a commutative basic algebra $\mathscr{A}$, we have $\Theta^*(F) = \Theta(F)$ for the relation*

$$\Theta^*(F) = \{(x, y) \in A^2 : \; x \to y, \, y \to x \in F\}.$$

P r o o f. Let us consider arbitrary elements $x, y \in A$ such that $(x, y) \in \Theta^*(F)$. Then $x \to y$, $y \to x \in F$, and since $x \odot (x \to y) = x \wedge y = y \odot (y \to x)$, so $(x, y) \in \Theta(F)$.

Conversely, let $(x, y) \in \Theta(F)$. Then there are $f, g \in F$ such that $x \odot f = y \odot g$, hence by Lemma 3.1.(i),(iv), $f \leq x \to y \odot g \leq x \to y$. By (F1) from that, $x \to y \in F$. Analogously we get also $y \to x \in F$, which means that $(x, y) \in \Theta^*(F)$. □

Theorem 3.3.

(i) *Let $\mathscr{A} = (A; \oplus, \neg, t, 0)$ be a CB_{vt}-algebra and F its t-filter. Then $\Theta(F)$ is a congruence on $\mathscr{A}$.*

(ii) *Let θ be a congruence on a CB_{vt}-algebra $\mathscr{A}$. Then $1/\theta$ is a t-filter of $\mathscr{A}$ such that $\theta = \Theta(1/\theta)$.*

(iii) *There is a one-to-one correspondence between t-filters and congruences on the CB_{vt}-algebras.*

P r o o f.

(i) Let F be a t-filter of $\mathscr{A}$. By Th.2.4.(i), it is enough to show the compatibility of $\Theta(F)$ to t. Let $(x, y) \in \Theta(F)$, then by Lemma 3.1 $x \to y$, $y \to x \in F$, and also $t(x \to y)$, $t(y \to x) \in F$. By (T3) further, $t(x \to y) \leq t(x) \to t(y)$,

$t(y \to x) \leq t(y) \to t(x)$, hence $t(x) \to t(y)$, $t(y) \to t(x) \in F$, which is equivalent to $(t(x), t(y)) \in \Theta(F)$.

(ii) Let θ be a congruence on CB_{vt}-algebra $\mathscr{A}$ and let $x \in 1/\theta$. Then by Th.2.4.(ii), $1/\theta$ is a filter of $\mathscr{A}$ and we can consider the factor commutative basic algebra $\mathscr{A}/\theta$ endowed with the induced vt-operator $t_{\mathscr{A}/\theta}$ in the analogous way as in [13, Proposition 11]. Then

$$t(x)/\theta = t_{\mathscr{A}/\theta}(x/\theta) = t_{\mathscr{A}/\theta}(1/\theta) = t(1)/\theta = 1/\theta.$$

Hence, $t(x) \in 1/\theta$, so $1/\theta$ is a t-filter of CB_{vt}-algebra $\mathscr{A}$. The rest follows from Th.2.4.(ii).

(iii) It is a direct consequence of (i) and (ii). □

In general, it is not easy to generate a t-filter on a CB_{vt}-algebra, but the following theorem shows that there is an important set of t-filters that are easy to describe.

Theorem 3.4.

(i) *Let $\mathscr{A} = (A; \oplus, \neg, t, 0)$ be a CB_{vt}-algebra and $M \subseteq A$. Then $M^{\perp}$ is a t-filter.*

(ii) *Let $\mathscr{A} = (A; \oplus, \neg, t, 0)$ be a subdirectly irreducible CB_{vt}-algebra. Then $\mathscr{A}$ is linearly ordered.*

Proof.

(i) By Th.2.4.(ii), $M^{\perp}$ is a filter of $\mathscr{A}$. Let $x \in M^{\perp}$. Then we have $x \vee m = 1$ for any $m \in M$, so $t(x \vee m) = 1$. By (T4) and (T2), further $1 = t(x \vee m) \leq t(x) \vee t(m) \leq t(x) \vee m$, hence $t(x) \vee m = 1$, and consequently $t(x) \in M^{\perp}$.

(ii) Let us consider two uncomparable elements $x, y \in A$. Then we get from prelinearity that $\{x \to y\}^{\perp}$, $\{x \to y\}^{\perp\perp}$ are non-trivial t-filters of $\mathscr{A}$ such that $y \to x \in \{x \to y\}^{\perp}$, $x \to y \in \{x \to y\}^{\perp\perp}$. Since $\{x \to y\}^{\perp} \cap \{x \to y\}^{\perp\perp} = \{1\}$, $\mathscr{A}$ is subdirectly reducible. □

As a corollary we get the result that CBA_{vt} logic is fuzzy (by [1]).

4. vf-operators on commutative basic algebras

In the last chapter we will deal with the duals to the vt-operators on commutative basic algebras, which stays as algebraic counterparts for the "very false" connective of CBA_{vt}, where

$$vf(\varphi) \text{ is } \neg vt(\neg\varphi).$$

In general the correspondence between very true and very false operators need not be one-to-one (see e.g. [13]) and this is the reason why the very false operators are not always so easy to describe (opposite to very true operators). We will show that in our case the correspondence is one-to-one.

Lemma 4.1. *Given a CB_{vt}-algebra $\mathscr{A} = (A; \oplus, \neg, t, 0)$, the following hold for each $x, y \in A$:*

(i) $t(0) = 0$;

(ii) $x \leq y \implies t(x) \leq t(y)$;

(iii) $t(x) \odot t(y) \leq t(x \odot y)$;

(iv) $t(\neg x) \leq \neg t(x)$.

P r o o f. Let x, y, z be elements of the CB_{vt}-algebra $\mathscr{A}$.

(i) By (T2), $t(0) \leq 0$, then it is obvious.

(ii) Let $x \leq y$, so $x \to y = 1$. Then by (T1) and (T3), $1 = t(x \to y) \leq t(x) \vee t(y)$. Thus, $t(x) \to t(y) = 1$, and therefore $t(x) \leq t(y)$.

(iii) Let $x \odot y \leq z$. Then $x \leq y \to z$, and further by (ii) and (T3), $t(x) \leq t(y \to z) \leq t(y) \to t(z)$. Hence we get $t(x) \odot t(y) \leq t(z)$. If we put $z := x \odot y$ in the foregoing result, we obtain just the inequality.

(iv) It follows from Def.2.1 (of a basic algebra) that $\neg$ is order reversing on $\mathscr{A}$, therefore $\neg x \leq \neg t(x)$. From that using (T2) we get the result. $\square$

Now, for any set $A \neq \emptyset$ and any self-mapping g of A we define a new mapping $g \colon A \to A$ such that for each $x \in A$

$$g^{-}(x) \overset{\text{def}}{=} \neg g(\neg x). \qquad (\diamond)$$

Lemma 4.2. *Let $\mathscr{A} = (A; \oplus, \neg, t, 0)$ be a CB_{vt}-algebra and let t^{-} be a self-mapping of A defined via $(\diamond)$. Then the following hold for each $x, y, z \in A$:*

(i) $t^{-}(0) = 0,\ t^{-}(1) = 1$;

(ii) $x \leq t^{-}(x)$;

(iii) $x \leq y \implies t^{-}(x) \leq t^{-}(y)$;

(iv) $t^{-}(x \to y) \leq t(x) \to t^{-}(y)$;

(v) $x \odot y \leq z \implies t^{-}(x) \odot t(y) \leq t^{-}(z)$;

(vi) $t^{-}(x) \odot t(y) \leq t^{-}(x \odot y)$.

P r o o f. Choose arbitrary $x, y, z \in A$.

(i) $t^{-}(0) = \neg t(\neg 0) = \neg t(1) = \neg 1 = 0$,
$t^{-}(1) = \neg t(\neg 1) = \neg t(0) = \neg 0 = 1$.

(ii) Using (T2) and (B2), $t^-(x) = \neg t(\neg x) \geq \neg\neg x = x$.

(iii) Let $x \leq y$. Then $\neg x \geq \neg y$, and consequently $t(\neg x) \geq t(\neg y)$. From that further $\neg t(\neg x) \leq \neg t(\neg y)$, which is equivalent to $t^-(x) \leq t^-(y)$.

(iv) Using Lemma 4.1.(iii),(iv),

$$\begin{aligned} t^-(x \to y) = \neg t(\neg(x \to y)) &= \neg t(x \odot \neg y) \\ &\leq \neg(t(x) \odot t(\neg y)) \\ &= \neg(t(x) \odot \neg\neg t(\neg y)) = t(x) \to t^-(y). \end{aligned}$$

(v) Let $x \odot y \leq z$. Then by (ii) and (iv), $t^-(x) \leq t^-(y \to z) \leq t(y) \to t^-(z)$. Hence $t^-(x) \odot t(y) \leq t^-(z)$.

(vi) Using (v), where we put $z := x \odot y$. □

Definition 4.1. By a vf-operator of a commutative basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ we understand a self-mapping f of A satisfying

(VF1) $f(0) = 0$,

(VF2) $x \leq f(x)$,

(VF3) $f(x) \ominus f(y) \leq f(x \ominus y)$,

(VF4) $f(x \wedge y) = f(x) \wedge f(y)$.

Remark 4.3. Notice that the axioms (VF1)–(VF4) of the last definition are just the duals to the axioms (T1)–(T4) from the definition of the vt-operator.

Theorem 4.4. *Given a CB_{vt}-algebra $\mathscr{A} = (A; \oplus, \neg, t, 0)$, then t^- is a vf-operator on $\mathscr{A}$.*

Proof. By Lemma 4.2, t^- fulfills (VF1) and (VF2). Further, choose arbitrary $x, y \in A$. Then we have using Lemma 4.2.(vi),

$$t^-(x) \ominus t^-(y) = t^-(x) \odot t(\neg y) \leq t^-(x \odot \neg y) = t^-(x \ominus y),$$

and using (T4) also

$$\begin{aligned} t^-(x \wedge y) &= \neg t(\neg(x \wedge y)) = \neg t(\neg x \vee \neg y) \\ &= \neg(t(\neg x) \vee t(\neg y)) = \neg t(\neg x) \wedge \neg t(\neg y) = t^-(x) \wedge t^-(y). \end{aligned}$$

□

Consider further the converse situation. We can show that from a vf-operator on a CB_{vt}-algebra we can derive a vt-operator.

Theorem 4.5. *Given a commutative basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ and a vf-operator f on $\mathscr{A}$, then f^- is a vt-operator on $\mathscr{A}$.*

P r o o f. Choose arbitrary $x, y \in A$. We must check that f^- fulfills (T1)–(T4).

(T1): Using (VF1), $f^-(1) = \neg f(\neg 1) = \neg f(0) = \neg 0 = 1$.

(T2): Using (VF2), $f^-(x) = \neg f(\neg x) \leq \neg\neg x = x$.

(T3): According to Def.1.2, we have $x \to y = \neg y \to \neg x$ in $\mathscr{A}$. Then using this identity and also (VF3) we get

$$\begin{aligned} f^-(x \to y) &= f^-(\neg y \to \neg x) = \neg f(\neg(\neg y \to \neg x)) \\ &= \neg f(\neg y \ominus \neg x) \leq \neg(f(\neg y) \ominus f(\neg x)) \\ &= f(\neg y) \to f(\neg x) = \neg f(\neg x) \to \neg f(\neg y) = f^-(x) \to f^-(y). \end{aligned}$$

(T4): Using (VF4),

$$\begin{aligned} f^-(x \vee y) &= \neg f(\neg(x \vee y)) = \neg f(\neg x \wedge \neg y) \\ &= \neg(f(\neg x) \wedge f(\neg y)) = \neg f(\neg x) \vee \neg f(\neg y) = f^-(x) \vee f^-(y). \end{aligned}$$

□

COROLLARY 4.6. *There is a one-to-one correspondence between vt- and vf-operators on commutative basic algebras.*

P r o o f. It follows directly from Th.4.4 and Th.4.5 and from the fact that for any self-mapping g of A we have

$$g^{--}(x) = \neg\neg g(\neg\neg x) = g(x)$$

for each $x \in A$, hence g^{--} is the same mapping (operator) as the starting one g. □

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