VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-010-0021-y
Math. Slovaca **60** (2010), No. 3, 411–418

REMARK TO AN UNAMBIGUITY OF ESTIMATORS IN THE UNIVERSAL LINEAR STATISTICAL MODELS WITH THE TYPE II CONSTRAINTS

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(*Communicated by Gejza Wimmer*)

ABSTRACT. In the universal linear statistical model with the type II constraints, estimates of the unbiasedly estimable linear functions of the parameters of the mean value vector are given by special types of generalized matrix inverse. Since there exists many versions of such matrix inverse it is of some interest to check the unambiguity of obtained estimators. The aim of the paper is to find the best unbiased linear estimators of unbiasedly estimable functions and to show that they do not depend on the choice of the used generalized inverse of the matrices.

1. Introduction

Universal linear statistical model with the type II constraints is given as follows

$$\mathbf{Y} \sim_n (\mathbf{X}\boldsymbol{\beta}_1, \boldsymbol{\Sigma}), \qquad \mathbf{b} + \mathbf{B}_1\boldsymbol{\beta}_1 + \mathbf{B}_2\boldsymbol{\beta}_2 = \mathbf{0}, \tag{1}$$

where

$\mathbf{Y}$ ... is a random n-dimensional vector (observation vector),

$\mathbf{X}$ ... $n \times k_1$ given matrix,

$\boldsymbol{\Sigma}$ ... given covariance matrix of the vector $\mathbf{Y}$,

$\boldsymbol{\beta}_1$ ... k_1 – dimensional unknown parameter,

$\boldsymbol{\beta}_2$ ... k_2 – dimensional unknown parameter,

2000 Mathematics Subject Classification: Primary 62J05.
Keywords: universal linear statistical model, type II constraints, unambiguity of estimators.
Supported by the Council of the Czech Government MSM 6 198 959 214.

$$
\begin{array}{rcl}
\mathbf{B}_1 & \dots & q \times k_1 \quad \text{given matrix,} \\
\mathbf{B}_2 & \dots & q \times k_2 \quad \text{given matrix,} \\
\mathbf{b} & \dots & \text{given vector with the property} \quad \mathbf{b} \in \mathscr{M}(\mathbf{B}_1, \mathbf{B}_2), \\
\mathscr{M}(\mathbf{B}_1, \mathbf{B}_2) & = & \{\mathbf{B}_1\mathbf{u} + \mathbf{B}_2\mathbf{v} : \ \mathbf{u} \in \mathbb{R}^{k_1}, \mathbf{v} \in \mathbb{R}^{k_2}\}, \\
\mathbb{R}^k & \dots & k - \text{dimensional real linear vector space.}
\end{array}
$$

The considered model can be expressed equivalently as follows.

$$
\begin{pmatrix} \mathbf{Y} \\ -\mathbf{b} \end{pmatrix} \sim_{n+q} \left[\begin{pmatrix} \mathbf{X}, & \mathbf{0} \\ \mathbf{B}_1, & \mathbf{B}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0} \end{pmatrix} \right]. \tag{2}
$$

Sometimes it is useful to write

$$
\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_{0,1} \sim_n (\mathbf{X}\mathbf{K}_1\boldsymbol{\gamma}, \boldsymbol{\Sigma}), \qquad \boldsymbol{\gamma} \in \mathbb{R}^{k_1+k_2-r(B_1,B_2)}, \tag{3}
$$

what is equivalent to the model (1) with respect to the parameter $\boldsymbol{\beta}_1$. Here $\mathbf{M}_{B_1'M_{B_2}} = \mathbf{I} - \mathbf{P}_{B_1'M_{B_2}}$ and $\mathbf{P}_{B_1'M_{B_2}}$ is a projection matrix (in the Euclidean norm) on the subspace $\mathscr{M}(\mathbf{B}_1'\mathbf{M}_{B_2})$,

$$
\begin{array}{rcl}
\mathscr{K}er(\mathbf{B}_1, \mathbf{B}_2) & = & \left\{ \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} : \ \mathbf{B}_1\mathbf{u} + \mathbf{B}_2\mathbf{v} = \mathbf{0} \right\} = \mathscr{M} \begin{pmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{pmatrix}, \\
\boldsymbol{\beta}_1 & = & \boldsymbol{\beta}_{0,1} + \mathbf{K}_1\boldsymbol{\gamma}, \\
\mathscr{M}(\mathbf{K}_1) & = & \mathscr{M}(\mathbf{M}_{B_1'M_{B_2}}), \qquad \mathscr{M}(\mathbf{K}_2) = \mathscr{M}(\mathbf{M}_{B_2'M_{B_1}})
\end{array}
$$

and $\boldsymbol{\beta}_{0,1}$ is an arbitrary vector satisfying the equality $\mathbf{B}_1\boldsymbol{\beta}_{0,1} + \mathbf{B}_2\boldsymbol{\beta}_2 + \mathbf{b} = \mathbf{0}$ (in more detail cf. [2]).

If no assumptions on a regularity of the model are considered, then the parameters $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, respectively, need not be unbiasedly estimable. The estimates of linear functions of these parameters which are unbiasedly estimable are given by a help of special generalized inverse (g-inverse) of the model matrices (in more detail cf. [1] and [4]). Such inverse is not given unambiguously, nevertheless the estimators do not depend on the choice of them. The aim of the paper is to show this fact algebraically.

The Moore-Penrose g-inverse $\mathbf{A}^+$ of a matrix $\mathbf{A}$ is sometimes used instead of g-inverse $\mathbf{A}^-$ of the matrix $\mathbf{A}$ if it is commonly used practice or the matrix $\mathbf{A}$ is of a special statistical meaning, e.g. it is a covariance matrix.

2. Estimators of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$

Lemma 1.

(i) *The parameter space of $\boldsymbol{\beta}_1$ is*

$$
\underline{\beta}_1 = \left\{ \boldsymbol{\beta}_{0,1} + \mathbf{M}_{B_1'M_{B_2}}\mathbf{u} : \ \mathbf{u} \in \mathbb{R}^{k_1} \right\},
$$

where $\boldsymbol{\beta}_{0,1}$ *is any solution of the equation* $\mathbf{B}_1\boldsymbol{\beta}_{0,1} + \mathbf{B}_2\boldsymbol{\beta}_2 + \mathbf{b} = \mathbf{0}$.

(ii) *The parameter space of the parameter* $\boldsymbol{\beta}_2$ *is*

$$\underline{\beta}_2 = \left\{\boldsymbol{\beta}_{0,2} + \mathbf{M}_{B_2' M_{B_1}}\mathbf{u} : \ \mathbf{u} \in \mathbb{R}^{k_2}\right\},$$

where $\boldsymbol{\beta}_{0,2}$ *is any solution of the equation* $\mathbf{B}_1\boldsymbol{\beta}_1 + \mathbf{B}_2\boldsymbol{\beta}_{0,2} + \mathbf{b} = \mathbf{0}$.

P r o o f. Cf. in [1, p. 196]. □

Lemma 2.

(i) *A function* $\mathbf{h}_1'\boldsymbol{\beta}_1, \boldsymbol{\beta}_1 \in \underline{\beta}_1$, *is unbiasedly estimable iff*

$$\mathbf{h}_1 \in \mathscr{M}(\mathbf{X}', \mathbf{B}_1'\mathbf{M}_{B_2}).$$

(ii) *A function* $\mathbf{h}_2'\boldsymbol{\beta}_2, \boldsymbol{\beta}_2 \in \underline{\beta}_2$, *is unbiasedly estimable iff*

$$\mathbf{h}_2 \in \mathscr{M}(\mathbf{B}_2'\mathbf{M}_{B_1 M_{X'}}).$$

P r o o f. Cf. [1]. □

Lemma 3. *It is valid that*

$$P\left\{\mathbf{Y} + \mathbf{X}\mathbf{B}_1'(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^{+}\mathbf{b} \in \mathscr{M}(\mathbf{X}\mathbf{M}_{B_1' M_{B_2}}, \boldsymbol{\Sigma})\right\} = 1.$$

P r o o f. Cf. in [2]. □

Since the relation $\mathscr{M}(\mathbf{B}_1') \subset \mathscr{M}(\mathbf{X}', \mathbf{B}_1'\mathbf{M}_{B_2})$ need not be valid, the vector $\mathbf{B}_1\boldsymbol{\beta}_1$ need not be unbiasedly estimable. The estimators of unbiasedly estimable linear functions $\mathbf{h}_1'\boldsymbol{\beta}_1, \boldsymbol{\beta}_1 \in \underline{\beta}_1$ and $\mathbf{h}_2'\boldsymbol{\beta}_2, \boldsymbol{\beta}_2 \in \underline{\beta}_2$ respectively, are given in [1]. Another form of estimation, mainly for the parameter $\boldsymbol{\beta}_2$, can be obtained as follows.

In general it holds that (in more detail cf. [3])

$$\widehat{\mathbf{h}_1'\boldsymbol{\beta}_1 + \mathbf{h}_2'\boldsymbol{\beta}_2} = (\mathbf{u}', \mathbf{v}')\begin{pmatrix} \mathbf{X}, & \mathbf{0} \\ \mathbf{B}_1, & \mathbf{B}_2 \end{pmatrix}\begin{pmatrix} \boxed{\mathbf{A}}, & \boxed{\mathbf{B}} \\ \boxed{\mathbf{C}}, & \boxed{\mathbf{D}} \end{pmatrix}\begin{pmatrix} \mathbf{Y} \\ -\mathbf{b} \end{pmatrix},$$

where

$$\begin{pmatrix} \boxed{\mathbf{A}}, & \boxed{\mathbf{B}} \\ \boxed{\mathbf{C}}, & \boxed{\mathbf{D}} \end{pmatrix} = \left[\begin{pmatrix} \mathbf{X}', & \mathbf{B}_1' \\ \mathbf{0}, & \mathbf{B}_2' \end{pmatrix}^{-}_{m\begin{pmatrix} \Sigma, & 0 \\ 0, & 0 \end{pmatrix}}\right]'$$

and

$$\mathbf{u}'\mathbf{X} + \mathbf{v}'\mathbf{B}_1 = \mathbf{h}_1', \qquad \mathbf{v}'\mathbf{B}_2 = \mathbf{h}_2'.$$

The expressions for $\boxed{\mathbf{A}}$ and $\boxed{\mathbf{B}}$ can be obtained easily from the model (3) Obviously

$$\begin{aligned}\boxed{\mathbf{A}}\,\mathbf{Y}+\boxed{\mathbf{B}}\,(-\mathbf{b}) &= \big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{Y}+\Big\{\mathbf{I}-\big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{X}\Big\}\\ &\quad\times\mathbf{B}_1'(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+(-\mathbf{b}).\end{aligned}$$

Thus

$$\begin{aligned}\boxed{\mathbf{A}} &= \big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]',\\ \boxed{\mathbf{B}} &= \Big\{\mathbf{I}-\big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{X}\Big\}\mathbf{B}_1'(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+.\end{aligned}$$

The problem is to find the expression $\boxed{\mathbf{C}}\,\mathbf{Y}+\boxed{\mathbf{D}}\,(-\mathbf{b})$. However (cf. properties of the minimum seminorm g-inverse in [4]),

$$\begin{pmatrix}\mathbf{X}, & \mathbf{0}\\ \mathbf{B}_1, & \mathbf{B}_2\end{pmatrix}\begin{pmatrix}\boxed{\mathbf{A}}, & \boxed{\mathbf{B}}\\ \boxed{\mathbf{C}}, & \boxed{\mathbf{D}}\end{pmatrix}\begin{pmatrix}\mathbf{X}, & \mathbf{0}\\ \mathbf{B}_1, & \mathbf{B}_2\end{pmatrix}=\begin{pmatrix}\mathbf{X}, & \mathbf{0}\\ \mathbf{B}_1, & \mathbf{B}_2\end{pmatrix},$$

$$\begin{aligned}&\begin{pmatrix}\mathbf{X}, & \mathbf{0}\\ \mathbf{B}_1, & \mathbf{B}_2\end{pmatrix}\begin{pmatrix}\boxed{\mathbf{A}}, & \boxed{\mathbf{B}}\\ \boxed{\mathbf{C}}, & \boxed{\mathbf{D}}\end{pmatrix}\begin{pmatrix}\mathbf{\Sigma}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{\Sigma}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{0}\end{pmatrix}\begin{pmatrix}\boxed{\mathbf{A}}', & \boxed{\mathbf{C}}'\\ \boxed{\mathbf{B}}', & \boxed{\mathbf{D}}'\end{pmatrix}\begin{pmatrix}\mathbf{X}', & \mathbf{B}_1'\\ \mathbf{0}, & \mathbf{B}_2'\end{pmatrix},\end{aligned}$$

i.e.,

$$\begin{aligned}&(ii) && \mathbf{X}\boxed{\mathbf{A}}\,\mathbf{X}+\mathbf{X}\boxed{\mathbf{B}}\,\mathbf{B}_1=\mathbf{X},\\ &(ij) && \mathbf{X}\boxed{\mathbf{B}}\,\mathbf{B}_2=\mathbf{0},\\ &(ji) && \mathbf{B}_1\boxed{\mathbf{A}}\,\mathbf{X}+\mathbf{B}_1\boxed{\mathbf{B}}\,\mathbf{B}_1+\mathbf{B}_2\boxed{\mathbf{C}}\,\mathbf{X}+\mathbf{B}_2\boxed{\mathbf{D}}\,\mathbf{B}_1=\mathbf{B}_1,\\ &(jj) && \mathbf{B}_1\boxed{\mathbf{B}}\,\mathbf{B}_2+\mathbf{B}_2\boxed{\mathbf{D}}\,\mathbf{B}_2=\mathbf{B}_2,\\ &(\beta\alpha) && \mathbf{B}_1\boxed{\mathbf{A}}\,\mathbf{\Sigma}+\mathbf{B}_2\boxed{\mathbf{C}}\,\mathbf{\Sigma}=\mathbf{0}.\end{aligned}$$

Let $\mathbf{v}\in\mathbb{R}^n$ and $\mathbf{u}\in\mathbb{R}^k$. Then

$$\begin{aligned}(\beta\alpha)\mathbf{v}+(ji)\mathbf{u} &= \mathbf{B}_1\boxed{\mathbf{A}}\,\mathbf{X}\mathbf{u}+\mathbf{B}_1\boxed{\mathbf{B}}\,\mathbf{B}_1\mathbf{u}+\mathbf{B}_2\boxed{\mathbf{C}}\,\mathbf{X}\mathbf{u}+\mathbf{B}_2\boxed{\mathbf{D}}\,\mathbf{B}_1\mathbf{u}\\ &\quad+\mathbf{B}_1\boxed{\mathbf{A}}\,\mathbf{\Sigma}\mathbf{v}+\mathbf{B}_2\boxed{\mathbf{C}}\,\mathbf{\Sigma}\mathbf{v}=\mathbf{B}_1\mathbf{u},\end{aligned}$$

$$\mathbf{B}_1\boxed{\mathbf{A}}\,(\mathbf{X}\mathbf{u}+\mathbf{\Sigma}\mathbf{v})+\mathbf{B}_1\boxed{\mathbf{B}}\,\mathbf{B}_1\mathbf{u}+\mathbf{B}_2\boxed{\mathbf{C}}\,(\mathbf{X}\mathbf{u}+\mathbf{\Sigma}\mathbf{v})+\mathbf{B}_2\boxed{\mathbf{D}}\,\mathbf{B}_1\mathbf{u}=\mathbf{B}_1\mathbf{u}.$$

Any realization of $\mathbf{Y}$ can be expressed as $\mathbf{X}\mathbf{u}+\mathbf{\Sigma}\mathbf{v}$,

$$\mathbf{u}=\mathbf{B}_1'(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+(-\mathbf{b})+\mathbf{M}_{B_1'M_{B_2}}\boldsymbol{\gamma}$$

(cf. also Lemma 3). Thus

$$\mathbf{B}_2\boxed{\mathbf{C}}\,\mathbf{Y}+\mathbf{B}_2\boxed{\mathbf{D}}\,(-\mathbf{B}_2\boldsymbol{\beta}_2-\mathbf{b})+\mathbf{B}_1\boxed{\mathbf{A}}\,\mathbf{Y}+\mathbf{B}_1\boxed{\mathbf{B}}\,(-\mathbf{B}_2\boldsymbol{\beta}_2-\mathbf{b})=-\mathbf{B}_2\boldsymbol{\beta}_2-\mathbf{b}.$$

Since

$$\boxed{\mathbf{B}}\,\mathbf{B}_2 = \Big\{\mathbf{I} - \big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{X}\Big\}\mathbf{B}_1'(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+\mathbf{B}_2 = \mathbf{0}$$

and $\mathbf{B}_2\boxed{\mathbf{D}}\,\mathbf{B}_2 = \mathbf{B}_2$, we have

$$\mathbf{B}_2\boxed{\mathbf{C}}\,\mathbf{Y} + \mathbf{B}_2\boxed{\mathbf{D}}\,(-\mathbf{b}) = -\Big\{\mathbf{B}_1\Big[\,\boxed{\mathbf{A}}\,\mathbf{Y} + \boxed{\mathbf{B}}\,(-\mathbf{b})\Big] + \mathbf{b}\Big\}.$$

If $(\mathbf{h}_1', \mathbf{h}_2') = (\mathbf{u}', \mathbf{v}')\begin{pmatrix} \mathbf{X}, & \mathbf{0} \\ \mathbf{B}_1, & \mathbf{B}_2 \end{pmatrix}$, then

$$\mathbf{h}_1'\widehat{\boldsymbol{\beta}_1 + \mathbf{h}}_2'\boldsymbol{\beta}_2 = \mathbf{h}_1'\Big[\,\boxed{\mathbf{A}}\,\mathbf{Y} + \boxed{\mathbf{B}}\,(-\mathbf{b})\Big] - \mathbf{v}'\Big\{\mathbf{B}_1\Big[\,\boxed{\mathbf{A}}\,\mathbf{Y} + \boxed{\mathbf{B}}\,(-\mathbf{b})\Big] + \mathbf{b}\Big\}.$$

Let $\mathbf{h}_1 = \mathbf{0}$, i.e.,

$$\begin{aligned} &\mathbf{X}'\mathbf{u} + \mathbf{B}_1'\mathbf{v} = \mathbf{0} \quad \& \quad \mathbf{B}_2'\mathbf{v} = \mathbf{h}_2 \\ &\Longrightarrow\ \mathbf{v} = \mathbf{M}_{B_1M_{X'}}\mathbf{s} \quad \& \quad \mathbf{h}_2' = \mathbf{s}'\mathbf{M}_{B_1M_{X'}}\mathbf{B}_2, \qquad \mathbf{s} \in \mathbb{R}^q. \end{aligned}$$

Since $\operatorname{Var}\bigg(\mathbf{v}'\Big\{\mathbf{B}_1\Big[\,\boxed{\mathbf{A}}\,\mathbf{Y} + \boxed{\mathbf{B}}\,(-\mathbf{b})\Big] + \mathbf{b}\Big\}\bigg)$ must be minimum and at the same time $\mathbf{B}_2'\mathbf{v} = \mathbf{h}_2$, the vector $\mathbf{v}$ must be chosen as

$$\mathbf{v} = (\mathbf{B}_2')^-_{m[\mathrm{Var}(B_1\boxed{A}\,Y)]}\mathbf{h}_2.$$

Thus

$$\widehat{\mathbf{h}_2'\boldsymbol{\beta}_2} = -\mathbf{s}'\mathbf{M}_{B_1M_{X'}}\mathbf{B}_2\big[(\mathbf{B}_2')^-_{m[\mathrm{Var}(B_1\boxed{A}\,Y)]}\big]'\Big\{\mathbf{B}_1\Big[\,\boxed{\mathbf{A}}\,\mathbf{Y} + \boxed{\mathbf{B}}\,(-\mathbf{b})\Big] + \mathbf{b}\Big\}.$$

In the following text the notation

$$\widehat{\mathbf{B}_1\boldsymbol{\beta}_1} = \mathbf{B}_1\Big[\,\boxed{\mathbf{A}}\,\mathbf{Y} + \boxed{\mathbf{B}}\,(-\mathbf{b})\Big]$$

will be used, even the estimator $\widehat{\mathbf{B}_1\boldsymbol{\beta}_1}$ is not given unambiguosly, if the subspace $\mathscr{M}(\mathbf{B}_1')$ is not embedded into $\mathscr{M}(\mathbf{X}', \mathbf{B}_1'\mathbf{M}_{B_2})$.

Theorem 4. *Let $\mathbf{h}_1 \in \mathscr{M}(\mathbf{X}', \mathbf{B}_1'\mathbf{M}_{B_2})$; then the estimator*

$$\begin{aligned} \widehat{\mathbf{h}_1'\boldsymbol{\beta}_1} \quad = \quad & \mathbf{h}_1'\bigg(\big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{Y} + \Big\{\mathbf{I} - \big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{X}\Big\} \\ & \times\mathbf{B}_1'(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+(-\mathbf{b})\bigg) \end{aligned}$$

is invariant with respect to the g-inverse of the matrix $\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}'$.

P r o o f. With respect to assumptions the vector $\mathbf{h}_1$ can be expressed as

$$\mathbf{h}_1 = \mathbf{X}'\mathbf{c} + \mathbf{B}_1'\mathbf{M}_{B_2}\mathbf{d},$$

where $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{d} \in \mathbb{R}^q$, and thus

$$\begin{aligned}
\widehat{\mathbf{h}_1'\boldsymbol{\beta}_1} &= (\mathbf{c}'\mathbf{X} + \mathbf{d}'\mathbf{M}_{B_2}\mathbf{B}_1)\Bigg(\big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{Y} \\
&\quad + \Big\{\mathbf{I} - \big[(\mathbf{M}_{B_1'M_{B_2}})^-_{m(\Sigma)}\big]'\mathbf{X}\Big\}\mathbf{B}_1'(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+(-\mathbf{b})\Bigg) \\
&= \mathbf{c}'\mathbf{X}(\mathbf{M}_{B_1'M_{B_2}} + \mathbf{P}_{B_1'M_{B_2}})\big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]' \\
&\quad \times\big[\mathbf{X}\mathbf{B}_1'(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+(-\mathbf{b}) + \mathbf{X}\mathbf{M}_{B_1'M_{B_2}}\boldsymbol{\gamma} + \boldsymbol{\Sigma}\mathbf{v}\big] \\
&\quad -\mathbf{c}'\mathbf{X}\mathbf{M}_{B_1'M_{B_2}}\big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{X}\mathbf{B}_1'(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+(-\mathbf{b}) \\
&\quad +\mathbf{c}'\mathbf{X}\mathbf{B}_1'(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+(-\mathbf{b}) \\
&\quad +\mathbf{d}'\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2}(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+(-\mathbf{b}) \\
&= \mathbf{c}'\mathbf{X}\mathbf{M}_{B_1'M_{B_2}}\boldsymbol{\gamma} + \mathbf{c}'\mathbf{X}\mathbf{M}_{B_1'M_{B_2}}\big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\boldsymbol{\Sigma}\mathbf{v} \\
&\quad +\mathbf{c}'\mathbf{X}\mathbf{B}_1'(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+(-\mathbf{b}) + \mathbf{d}'\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2} \\
&\quad \times(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+(-\mathbf{b}).
\end{aligned}$$

Since the matrix $\mathbf{X}\mathbf{M}_{B_1'M_{B_2}}\big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\boldsymbol{\Sigma}$ is invariant with respect to the choice of the g-inverse of the matrix $\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}'$ (in more detail cf. [4]), the estimator $\widehat{\mathbf{h}_1'\boldsymbol{\beta}_1}$ is given unambiguously. □

THEOREM 5. *Let* $\mathbf{h}_2 \in \mathscr{M}(\mathbf{B}_2'\mathbf{M}_{B_1M_{X'}})$*; then*

$$\widehat{\mathbf{h}_2'\boldsymbol{\beta}_2} = -\mathbf{h}_2'\Big[(\mathbf{B}_2')^-_{m[\mathrm{Var}(\widehat{B_1\beta_1})]}\Big]'(\widehat{\mathbf{B}_1\boldsymbol{\beta}_1} + \mathbf{b})$$

is given unambiguously even if the estimator $\widehat{\mathbf{B}_1\boldsymbol{\beta}_1}$ *need not be given unambiguously and thus also the matrix* $(\mathbf{B}_2')^-_{m[\mathrm{Var}(\widehat{B_1\beta_1})]}$ *need not be given unambiguously.*

P r o o f. It is valid that

$$\begin{aligned}
&\mathbf{M}_{B_2}(\widehat{\mathbf{B}_1\boldsymbol{\beta}_1} + \mathbf{b}) = \\
&\quad = \mathbf{M}_{B_2}\Bigg[\mathbf{B}_1\Bigg(\big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{Y} + \Big\{\mathbf{I} - \big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{X}\Big\}\mathbf{B}_1' \\
&\qquad\qquad \times(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+(-\mathbf{b})\Bigg) + \mathbf{b}\Bigg]
\end{aligned}$$

$$= \quad \mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2}(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+(-\mathbf{b}) + \mathbf{M}_{B_2}\mathbf{b} = \mathbf{0}$$

and thus $\widehat{\widehat{\mathbf{B}_1\boldsymbol{\beta}_1}} + \mathbf{b} \in \mathscr{M}(\mathbf{B}_2)$. Here the equalities

$$\begin{aligned}
\big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]' &= \mathbf{M}_{B_1'M_{B_2}}\big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]',\\
(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+ &= \mathbf{M}_{B_2}(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+ = (\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+\mathbf{M}_{B_2},
\end{aligned}$$

were utilized. Since

$$\widehat{\widehat{\mathbf{B}_1\boldsymbol{\beta}_1}} + \mathbf{b} \in \mathscr{M}(\mathbf{B}_2) \implies \mathbf{B}_2\Big[(\mathbf{B}_2')^-_{m[\mathrm{Var}(\widehat{\widehat{B_1\beta_1}})]}\Big]'(\widehat{\widehat{\mathbf{B}_1\boldsymbol{\beta}_1}} + \mathbf{b}) = \widehat{\widehat{\mathbf{B}_1\boldsymbol{\beta}_1}} + \mathbf{b}$$

and $\mathbf{h}_2' = \mathbf{s}'\mathbf{M}_{B_1M_{X'}}\mathbf{B}_2$, we have

$$\begin{aligned}
\widehat{\widehat{\mathbf{h}_2'\boldsymbol{\beta}_2}} = {} & -\mathbf{s}'\mathbf{M}_{B_1M_{X'}}\Bigg[\mathbf{B}_1\Bigg(\big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{Y} + \Big\{\mathbf{I} - \big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{X}\Big\}\\
& \times \mathbf{B}_1'(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+(-\mathbf{b})\Bigg) + \mathbf{b}\Bigg]\\
= {} & -\mathbf{s}'\mathbf{M}_{B_1M_{X'}}\Bigg[\mathbf{B}_1(\mathbf{M}_{X'} + \mathbf{P}_{X'})\Bigg(\big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{Y}\\
& + \Big\{\mathbf{I} - \big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\mathbf{X}\Big\}\mathbf{B}_1'(\mathbf{M}_{B_2}\mathbf{B}_1\mathbf{B}_1'\mathbf{M}_{B_2})^+(-\mathbf{b})\Bigg) + \mathbf{b}\Bigg]\\
= {} & -\mathbf{s}'\mathbf{M}_{B_1M_{X'}}(\mathbf{B}_1\widehat{\widehat{\mathbf{P}_{X'}\boldsymbol{\beta}_1}} + \mathbf{b}),
\end{aligned}$$

where the expression $\widehat{\widehat{\mathbf{P}_{X'}\boldsymbol{\beta}_1}}$ does not depend on the choice of the g-inverse $(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}$. It is implied by the following consideration.

$$\widehat{\widehat{\mathbf{P}_{X'}\boldsymbol{\beta}_1}} \quad = \quad \mathbf{X}'(\mathbf{X}\mathbf{X}')^+\mathbf{X}\big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'(\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_{0,1}) + \mathbf{X}'(\mathbf{X}\mathbf{X}')^+\mathbf{X}\boldsymbol{\beta}_{0,1}$$

and $\mathbf{Y} - \mathbf{X}\boldsymbol{\beta}_{0,1} = \mathbf{X}\mathbf{M}_{B_1'M_{B_2}}\mathbf{u} + \boldsymbol{\Sigma}\mathbf{v}$. Thus

$$\begin{aligned}
\widehat{\widehat{\mathbf{P}_{X'}\boldsymbol{\beta}_1}} &= \mathbf{X}'(\mathbf{X}\mathbf{X}')^+\mathbf{X}\big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'(\mathbf{X}\mathbf{M}_{B_1'M_{B_2}}\mathbf{u} + \boldsymbol{\Sigma}\mathbf{v})\\
&\quad + \mathbf{X}'(\mathbf{X}\mathbf{X}')^+\mathbf{X}\boldsymbol{\beta}_{0,1}\\
&= \mathbf{X}'(\mathbf{X}\mathbf{X}')^+\mathbf{X}\mathbf{M}_{B_1'M_{B_2}}\mathbf{u} + \mathbf{X}'(\mathbf{X}\mathbf{X}')^+\mathbf{X}\mathbf{M}_{B_1'M_{B_2}}\\
&\quad \times\big[(\mathbf{M}_{B_1'M_{B_2}}\mathbf{X}')^-_{m(\Sigma)}\big]'\boldsymbol{\Sigma}\mathbf{v} + \mathbf{X}'(\mathbf{X}\mathbf{X}')^+\mathbf{X}\boldsymbol{\beta}_{0,1}.
\end{aligned}$$

Since $\widehat{\widehat{\mathbf{P}_{X'}\boldsymbol{\beta}_1}}$ is invariant on the choice of the minimum $\boldsymbol{\Sigma}$-seminorm g-inverse (cf. [4]), the proof is finished. □

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Received 4. 6. 2008
Accepted 3. 11. 2008

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