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SOME TOPOLOGICAL AND GEOMETRIC PROPERTIES OF GENERALIZED EULER SEQUENCE SPACE

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ABSTRACT. In this paper, we introduce the Euler sequence space $e^r(p)$ of non-absolute type and prove that the spaces $e^r(p)$ and $l(p)$ are linearly isomorphic. Besides this, we compute the α-, β- and γ-duals of the space $e^r(p)$. The results proved herein are analogous to those in [ALTAY, B.—BAŞAR, F.: *On the paranormed Riesz sequence spaces of non-absolute type*, Southeast Asian Bull. Math. **26** (2002), 701–715] for the Riesz sequence space $r^q(p)$. Finally, we define a modular on the Euler sequence space $e^r(p)$ and consider it equipped with the Luxemburg norm. We give some relationships between the modular and Luxemburg norm on this space and show that the space $e^r(p)$ has property (H) but it is not rotund (R).

1. Introduction

By w, we denote the space of all real valued sequences. Any vector subspace of w is called as a sequence space. We write l_∞, c, c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively. Also by, bs, cs, l_1 and l_p we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively; where $1 < p < \infty$.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g\colon X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where

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θ is the zero vector in the linear space X. Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Then the linear spaces $l(p)$ and $l_\infty(p)$ were defined by Maddox [9] as follows:

$$l(p) = \Big\{x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty\Big\}$$

and

$$l_\infty(p) = \Big\{x = (x_k) \in w : \sup_{k\in\mathbb{N}} |x_k|^{p_k} < \infty\Big\}$$

which are the complete space paranormed by

$$g_1(x) = \left(\sum_k |x_k|^{p_k}\right)^{\frac{1}{M}} \qquad \text{and} \qquad g_2(x) = \sup_{k\in\mathbb{N}} |x_k|^{\frac{p_k}{M}}$$

iff $\inf p_k > 0$, respectively. We assume throughout $(p_k)^{-1} + (p'_k)^{-1} = 1$ provided $1 < \inf p_k \leq H < \infty$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathscr{F}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$.

For the sequence spaces λ and μ, define the set $S(\lambda, \mu)$ by

$$S(\lambda, \mu) = \big\{z = (z_k) \in w : \ xz = (x_k z_k) \in \mu \ \text{ for all } \ x \in \lambda\big\}. \tag{1.1}$$

With the notation of (1.1), the α-, β- and γ-duals of a sequence space λ, which are respectively denoted by $\lambda^\alpha, \lambda^\beta$ and λ^γ, are defined by

$$\lambda^\alpha = S(\lambda, l_1), \qquad \lambda^\beta = S(\lambda, cs), \qquad \lambda^\gamma = S(\lambda, bs).$$

If a sequence space λ paranormed by h contains a sequence (b_n) with the property that for every $x \in \lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} h\left(x - \sum_{k=0}^{n} \alpha_k b_k\right) = 0$$

then (b_n) is called a Schauder basis (or briefly basis) for λ. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum \alpha_k b_k$.

Let λ and μ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by writing $A\colon \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk} x_k \qquad (n \in \mathbb{N}). \tag{1.2}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A\colon \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1.2) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to α if Ax converges to α which is called as the A-limit of x.

The matrix domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \left\{x = (x_k) \in w : \ Ax \in \lambda\right\} \tag{1.3}$$

which is a sequence space. In the most cases, the new sequence space λ_A generated by the limitation matrix A from a sequence space λ is the expansion or the contraction of the original space λ.

Altay, Başar and Mursaleen [3], Altay and Başar [2], Altay and Polat [4] and Polat and Başar [11] introduced the Euler sequence spaces e_p^r and e_∞^r, e_0^r and e_c^r, $e_0^r(\Delta)$, $e_c^r(\Delta)$ and $e_\infty^r(\Delta)$, $e_0^r\left(\Delta^{(m)}\right)$ and $e_c^r\left(\Delta^{(m)}\right)$, respectively, where $1 \le p < \infty$. The main purpose of this paper is to introduce the Euler sequence space $e^r(p)$ and to determine its α-, β- and γ-duals. Furthermore, we show that the Euler sequence space $e^r(p)$ equipped with the Luxemburg norm has property (H) but it is not rotund, so it is not (LUR).

2. The generalized Euler sequence space $e^r(p)$ of non-absolute type

We introduce the sequence space $e^r(p)$, as the set of all sequences such that E^r-transforms of them are in the space $l(p)$, that is

$$e^r(p) = \Big\{x = (x_k) \in w : \ \sum_k \Big|\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j x_j\Big|^{p_k} < \infty,$$
$$0 < p_k \le H < \infty\Big\}$$

where E^r denotes the method of Euler means of order r defined by the matrix $E^r = (e^r_{nk})$

$$e^r_{nk} = \begin{cases} \binom{n}{k} (1-r)^{n-k} r^k, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

for all $k \in \mathbb{N}$. It is known that the method E^r is regular for $0 < r < 1$ and we assume unless stated otherwise that $0 < r < 1$. With the notation (1.3) we can redefine the space $e^r(p)$ by

$$e^r(p) = (l(p))_{E^r} \tag{2.1}$$

If λ is any normed or paranormed sequence space then we call the matrix domain λ_{E^r} as the Euler sequence space.

If $(p_k) = p$ for every $k \in \mathbb{N}$, then we write e_p^r instead of $e^r(p)$ (see [3]).

Define the sequence $y = \{y_k(r)\}$, which will be frequently used, as the E^r-transform of a sequence $x = (x_k)$, i.e.,

$$y_k(r) = \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j x_j \qquad (k \in \mathbb{N}). \tag{2.2}$$

Now, we may begin with the following theorem which is essential in the text.

Theorem 1. *$e^r(p)$ is a complete linear topological space paranormed by g defined by*

$$g(x) = \left(\sum_k \left| \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j x_j \right|^{p_k} \right)^{\frac{1}{M}} \qquad (0 < p_k \le H < \infty).$$

P r o o f. The linearity of $e^r(p)$ with respect to the coordinatewise addition and scalar multiplication follows from the inequalities which are satisfied for $x, y \in e^r(p)$ (see [7, p. 30]).

$$\begin{aligned} g(x+y) &= \left(\sum_k \left| \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j (x_j + y_j) \right|^{p_k} \right)^{\frac{1}{M}} \\ &\le \left(\sum_k \left| \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j x_j \right|^{p_k} \right)^{\frac{1}{M}} \\ &\quad + \left(\sum_k \left| \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j y_j \right|^{p_k} \right)^{\frac{1}{M}} \end{aligned} \tag{2.3}$$

and for any $\alpha \in \mathbb{R}$ (see [8])

$$|\alpha|^{p_k} \le \max\left\{1, |\alpha|^M\right\}. \tag{2.4}$$

It is clear that $g(\theta) = 0$ and $g(x) = g(-x)$ for all $x \in e^r(p)$. Again the inequalities (2.3) and (2.4) yield the subaddivity of g and

$$g(\alpha x) = \max\{1, |\alpha|\} g(x).$$

Let $\{x^n\}$ be any sequence of the points of the space $e^r(p)$ such that $g(x^n - x) \to 0$ and (α_n) also be any sequence of scalars such that $\alpha_n \to \alpha$. Then, since the inequality

$$g(x^n) \le g(x) + g(x^n - x)$$

holds by subadditivity of g, $\{g(x^n)\}$ is bounded and thus we have

$$\begin{aligned} g(\alpha_n x^n - \alpha x) &= \left[\sum_k \left|\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j \left(\alpha_n x_j^n - \alpha x_j\right)\right|^{p_k}\right]^{\frac{1}{M}} \\ &\leq |\alpha_n - \alpha| \, g(x^n) + |\alpha| \, g(x^n - x) \end{aligned}$$

which tends to zero as $n \to \infty$. That is to say that the scalar multiplication is continuous. Hence, g is a paranorm on the space $e^r(p)$. It remains to prove the completeness of the space $e^r(p)$. Let $\{x^i\}$ be any Cauchy sequence in the space $e^r(p)$, where $x^i = \left\{x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, \ldots\right\}$. Then, for a given $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$g\left(x^i - x^j\right) < \varepsilon \tag{2.5}$$

for all $i, j \geq n_0(\varepsilon)$. Using the definition of g we obtain for each fixed $k \in \mathbb{N}$ that

$$\left|\left(E^r x^i\right)_k - \left(E^r x^j\right)_k\right| \leq \left[\sum_k \left|\left(E^r x^i\right)_k - \left(E^r x^j\right)_k\right|^{p_k}\right]^{\frac{1}{M}} < \varepsilon$$

for $i, j \geq n_0(\varepsilon)$ which leads us to the fact that $\left\{\left(E^r x^0\right)_k, \left(E^r x^1\right)_k, \ldots\right\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $\left(E^r x^i\right)_k \to (E^r x)_k$ as $i \to \infty$. Using the infinitely many limits $(E^r x)_0, (E^r x)_1, \ldots$, we define the sequence $\{(E^r x)_0, (E^r x)_1, \ldots\}$. By the (2.5) for each $m \in \mathbb{N}$ and $i, j \geq n_0(\varepsilon)$

$$\sum_{k=0}^{m} \left|\left(E^r x^i\right)_k - \left(E^r x^j\right)_k\right|^{p_k} \leq \left(g\left(x^i - x^j\right)\right)^M < \varepsilon^M. \tag{2.6}$$

Take any $i \geq n_0(\varepsilon)$. First let $j \to \infty$ in (2.6) and then $m \to \infty$, to obtain $g\left(x^i - x\right) \leq \varepsilon$. Finally, taking $\varepsilon = 1$ in (2.6) and letting $i \geq n_0(1)$ we have by Minkowski's inequality for each $m \in \mathbb{N}$ that

$$\left[\sum_{k=0}^{m} |(E^r x)_k|^{p_k}\right]^{\frac{1}{M}} \leq g\left(x^i - x\right) + g\left(x^i\right) \leq 1 + g\left(x^i\right)$$

which implies that $x \in e^r(p)$. Since $g\left(x^i - x\right) < \varepsilon$ for all $i \geq n_0(\varepsilon)$ it follows that $x^i \to x$ as $i \to \infty$ whence we have shown that $e^r(p)$ is complete. □

Therefore, one can easily check that the absolute property does not hold on the space $e^r(p)$, that is $g(x) \neq g(|x|)$ for at least one sequence in the space $e^r(p)$, and this says that $e^r(p)$ is a sequence space of non-absolute type; where $|x| = (|x_k|)$.

Theorem 2. *The Euler sequence space $e^r(p)$ of non-absolute type is linearly isomorphic to the space $l(p)$, where $0 < p_k \leq H < \infty$.*

P r o o f. To prove the theorem, we should show the existence of a linear bijection between the spaces $e^r(p)$ and $l(p)$ for $0 < p_k \leq H < \infty$. With the transformation T from $e^r(p)$ to $l(p)$ by $x \mapsto y = Tx$. The linearity of T is trivial. Further, it is obvious that $x = \theta$ whenever $Tx = \theta$ and hence T is injective. Let $y \in l(p)$ and define the sequence $x = \{x_k(r)\}$ by

$$x_k(r) = \sum_{j=0}^{k} \binom{k}{j} (r-1)^{k-j} r^{-k} y_j \qquad (k \in \mathbb{N}).$$

Then,

$$\begin{aligned}\left(\sum_k \left|\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j x_j\right|^{p_k}\right)^{\frac{1}{M}} &= \left(\sum_k \left|\sum_{j=0}^{k} \delta_{kj} x_j\right|^{p_k}\right)^{\frac{1}{M}} \\ &= \left(\sum_k |y_k|^{p_k}\right)^{\frac{1}{M}} = g_1(y) < \infty\end{aligned}$$

where δ_{kj} are Kronecker delta's. Thus, we have that $x \in e^r(p)$. Consequently, T is surjective and is paranorm preserving. Hence, T is a linear bijection and this says us that the spaces $e^r(p)$ and $l(p)$ are linearly isomorphic, as was desired. □

Suppose that $1 < p_k \leq s_k$ for every $k \in \mathbb{N}$. Then, it was well-known that $l(p) \subset l(s)$ which leads us to the immediate consequence that $e^r(p) \subset e^r(s)$.

We shall quote some lemmas which are needed in proving our theorems.

Lemma 1.

(i) *Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (l(p) : l_1)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{K \in \mathscr{F}} \sum_k \left|\sum_{n \in K} a_{nk} B^{-1}\right|^{p'_k} < \infty.$$

(ii) *Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (l(p) : l_1)$ if and only if*

$$\sup_{K \in \mathscr{F}} \sup_{k \in \mathbb{N}} \left|\sum_{n \in K} a_{nk}\right|^{p_k} < \infty.$$

Lemma 2.

(i) *Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (l(p) : l_\infty)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{n \in \mathbb{N}} \sum_k \left|a_{nk} B^{-1}\right|^{p'_k} < \infty \tag{2.7}$$

(ii) *Let* $0 < p_k \leq 1$ *for every* $k \in \mathbb{N}$. *Then* $A \in (l(p) : l_\infty)$ *if and only if*

$$\sup_{n,k \in \mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{2.8}$$

Lemma 3. *Let* $1 < p_k \leq H < \infty$ *for every* $k \in \mathbb{N}$. *Then* $A \in (l(p) : c)$ *if and only if* (2.7), (2.8) *hold, and*

$$\lim_{n \to \infty} a_{nk} = \beta_k \qquad \text{for} \quad k \in \mathbb{N} \tag{2.9}$$

also holds.

Theorem 3. *Let* $1 < p_k \leq H < \infty$ *for every* $k \in \mathbb{N}$. *Define the set* $D_1(p)$ *as follows:*

$$D_1(p) = \bigcup_{B>1} \Big\{ a = (a_k) \in w : \sup_{K \in \mathscr{F}} \sum_k \Big| \sum_{n \in K} \tbinom{n}{k} (r-1)^{n-k} r^{-n} a_n B^{-1} \Big|^{p'_k} < \infty \Big\}.$$

Then, $[e^r(p)]^\alpha = D_1(p)$.

P r o o f. Let us define the matrix B^r whose rows are the product of the rows of the matrix $E^{\frac{1}{r}}$ with the sequence $a = (a_n)$. Therefore, we easily obtain by bearing in mind the relation (2.2) that ([3])

$$a_n x_n = \sum_{k=0}^{n} \binom{n}{k} (r-1)^{n-k} r^{-n} a_n y_k = (B^r y)_n \qquad (n \in \mathbb{N}). \tag{2.10}$$

Thus, we observe by (2.10) that $(a_n x_n) = ax \in l_1$ whenever $x \in e^r(p)$ if and only if $B^r y \in l_1$ whenever $y \in l(p)$. This means that $a = (a_n) \in [e^r(p)]^\alpha$ if and only if $B^r \in (l(p) : l_1)$. Then we derive by Lemma 1 with B^r instead of A that

$$\sup_{K \in \mathscr{F}} \sum_k \Big| \sum_{n \in K} \binom{n}{k} (r-1)^{n-k} r^{-n} a_n B^{-1} \Big|^{p'_k} < \infty.$$

This yields desired consequence that $[e^r(p)]^\alpha = D_1(p)$. □

Theorem 4. *Let* $1 < p_k \leq H < \infty$ *for every* $k \in \mathbb{N}$. *Define the sets* $D_2(p)$ *and* D *as follows:*

$$D = \Big\{ a = (a_k) \in w : \sum_{j=k}^{\infty} \tbinom{j}{k} (r-1)^{j-k} r^{-j} a_j \ \text{exists for each} \ k \in \mathbb{N} \Big\}$$

and

$$D_2(p) = \bigcup_{B>1} \Big\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_k \Big| \sum_{j=k}^{n} \tbinom{j}{k} (r-1)^{j-k} r^{-j} a_j B^{-1} \Big|^{p'_k} < \infty \Big\}.$$

Then $[e^r(p)]^\beta = D \cap D_2(p)$ *and* $[e^r(p)]^\gamma = D_2(p)$.

P r o o f. Consider the equation

$$\begin{aligned}\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n}\left[\sum_{j=0}^{k}\binom{k}{j}(r-1)^{k-j} r^{-k} y_j\right] a_k \\ &= \sum_{k=0}^{n}\left[\sum_{j=k}^{n}\binom{j}{k}(r-1)^{j-k} r^{-j} a_j\right] y_k = (T^r y)_n \qquad (2.11)\end{aligned}$$

where $T^r = (t^r_{nk})$ is defined by

$$t^r_{nk} = \begin{cases}\sum\limits_{j=k}^{n}\binom{k}{j}(r-1)^{k-j} r^{-k} a_j, & 0 \le k \le n\\ 0, & k > n\end{cases}$$

for all $k, n \in \mathbb{N}$ ([3]). Thus, we deduce from Lemma 3 with (2.11) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in e^r(p)$ if and only if $T^r y \in c$ whenever $y = (y_k) \in l(p)$. That is to say that $a = (a_k) \in [e^r(p)]^\beta$ if and only if $T^r \in (l(p) : c)$. Therefore, we derive from Lemma 3 with (2.7) and (2.9) that

$$\sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=k}^{n}\binom{j}{k}(r-1)^{j-k} r^{-j} a_j B^{-1}\right|^{p'_k} < \infty$$

and

$$\sum_{j=k}^{\infty}\binom{j}{k}(r-1)^{j-k} r^{-j} a_j \qquad \text{exists for each} \quad k \in \mathbb{N}$$

which shows that $[e^r(p)]^\beta = D \cap D_2(p)$.

As this, we see from Lemma 2 with (2.11) that $a = (a_k) \in bs$ whenever $x = (x_k) \in e^r(p)$ if and only if $T^r y \in l_\infty$ whenever $y = (y_k) \in l(p)$. That is to say that $a = (a_k) \in [e^r(p)]^\gamma$ if and only if $T^r \in (l(p) : l_\infty)$. Therefore, we derive from Lemma 2 with (2.7) that

$$\sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=k}^{n}\binom{j}{k}(r-1)^{j-k} r^{-j} a_j B^{-1}\right|^{p'_k} < \infty$$

which shows that $[e^r(p)]^\gamma = D_2(p)$. □

Theorem 5. *Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$. Define the set $D_3(p)$ and $D_4(p)$ by*

$$D_3(p) = \left\{a = (a_k) \in w : \sup_{K\in\mathscr{F}}\sup_{k\in\mathbb{N}}\left|\sum_{n\in K}\binom{n}{k}(r-1)^{n-k} r^{-n} a_n\right|^{p_k} < \infty\right\}$$

$$D_4(p) = \left\{a = (a_k) \in w : \sup_{n,k\in\mathbb{N}}\left|\sum_{j=k}^{n}\binom{j}{k}(r-1)^{j-k} r^{-j} a_j\right|^{p_k} < \infty\right\}$$

Then, $[e^r(p)]^\alpha = D_3(p)$, $[e^r(p)]^\beta = D \cap D_4(p)$ *and* $[e^r(p)]^\gamma = D_4(p)$.

P r o o f. This is easily obtained by proceeding as in the proofs of Theorem 1 and Theorem 2 above by using the second parts of Lemmas 1, 2 and 3 instead of the first parts. So, we omit the detail. □

Theorem 6. *Define the sequence* $b^{(k)}(r) = \left\{b_n^{(k)}(r)\right\}_{n\in\mathbb{N}}$ *of elements of the space* $e^r(p)$ *for every fixed* $k \in \mathbb{N}$ *by*

$$b_n^{(k)}(r) = \begin{cases} 0, & n < k \\ \binom{n}{k}(r-1)^{n-k}\, r^{-n}, & n \geq k. \end{cases}$$

Then, the sequence $\left\{b^{(k)}(r)\right\}_{n\in\mathbb{N}}$ *is a basis for the space* $e^r(p)$ *and any* $x \in e^r(p)$ *has unique representation of the form*

$$x = \sum_k \alpha_k(r) b^{(k)}(r)$$

where $\alpha_k(r) = (E^r x)_k$ *for all* $k \in \mathbb{N}$. ([3])

3. Some geometric properties of the space $e^r(p)$

In this section, we study some geometric properties of the space $e^r(p)$. Firstly, we remind them.

Let $X = (X, \|\cdot\|)$ be a real Banach space and $B(X)$ and $S(X)$ be the closed unit ball and the unit sphere of X, respectively. A point $x \in S(X)$ is an *H-point* if for any sequence (x_n) in X such that $\|x_n\| \to 1$ as $n \to \infty$, the weak convergence of (x_n) to x implies $\|x_n - x\| \to 0$ as $n \to \infty$. If every point in $S(X)$ is an H-point of $B(X)$, then X is said to have the (H) *property.* A point $x \in S(X)$ is an *extreme point* if for every $x, y \in S(X)$ the equality $2x = y + z$ implies $z = y$. A point $x \in S(X)$ is a *locally uniformly rotund point* if for any sequence (x_n) in $B(X)$ such that $\|x_n + x\| \to 2$ as $n \to \infty$, there holds $\|x_n - x\| \to 0$ as $n \to \infty$. A Banach space X is said to be *rotund* if every point of the unit sphere is an extreme point of $B(X)$. If every of $S(X)$ is a (LUR)-point of $B(X)$, then X is said to be *locally uniformly rotund* (LUR). It is known that if X is LUR, then it is (R) and possesses property (H). ([12])

For $1 \leq p < \infty$, the Euler sequence space e_p^r was defined by Altay, Başar and Mursaleen [3] as follows:

$$e_p^r = \left\{x = (x_k) \in w : \sum_k \left|\sum_{j=0}^{k} \binom{k}{j}(1-r)^{k-j}\, r^j x_j\right|^p < \infty\right\}$$

and this space equipped with the norm

$$\|x\|_{e_p^r} = \left(\sum_k \left| \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j x_j \right|^p \right)^{\frac{1}{p}}.$$

In [10], Altay, Başar and Mursaleen proved some geometric properties, Banach-Saks and weak Banach-Saks properties of the space e_p^r. Now we assume that $p_k \geq 1$ for all $k \in \mathbb{N}$. For $x \in e^r(p)$, let

$$\rho(x) = \sum_k \left| \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j x_j \right|^{p_k}$$

and define the Luxemburg norm on $e^r(p)$ by

$$\|x\| = \inf\left\{\varepsilon > 0 : \ \rho\left(\tfrac{x}{\varepsilon}\right) \leq 1\right\}, \qquad x \in e^r(p).$$

Now, we give some propositions which are related with $\rho(x)$ and Luxemburg norm.

Proposition 1. *The functional ρ is convex modular on $e^r(p)$.*

P r o o f. It is obvious that $\rho(x) = 0 \iff x = 0$ and $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$. Let $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. By the convexity of the function $t \mapsto |t|^{p_k}$ for every $k \in \mathbb{N}$, we have

$$\begin{aligned}
&\rho(\alpha x + \beta y) = \\
&= \sum_k \left| \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j (\alpha x_j + \beta y_j) \right|^{p_k} \\
&= \sum_k \left| \alpha \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j x_j + \beta \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j y_j \right|^{p_k} \\
&\leq \alpha \sum_k \left| \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j x_j \right|^{p_k} + \beta \sum_k \left| \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j y_j \right|^{p_k} \\
&= \alpha\rho(x) + \beta\rho(y).
\end{aligned}$$

Then we have $e^r(p)$ is a modular space. □

Proposition 2. *For $x \in e^r(p)$, the modular ρ on $e^r(p)$ satisfies the following properties:*

(i) *If $0 < \alpha < 1$, then $\alpha^M \rho\left(\frac{x}{\alpha}\right) \leq \rho(x)$ and $\rho(\alpha x) \leq \alpha\rho(x)$.*

(ii) *If $\alpha > 1$, then $\rho(x) \leq \alpha^M \rho\left(\frac{x}{\alpha}\right)$.*

(iii) *If $\alpha \geq 1$, then $\rho(x) \leq \alpha\rho(x) \leq \rho(\alpha x)$.*

Proof.

(i) Let $0 < \alpha < 1$. Then we have

$$\begin{aligned}
\rho(x) &= \sum_k \left|\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j x_j\right|^{p_k} \\
&= \sum_k \left|\alpha \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j \frac{x_j}{\alpha}\right|^{p_k} \\
&= \sum_k \alpha^{p_k} \left|\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j \frac{x_j}{\alpha}\right|^{p_k} \\
&\geq \sum_k \alpha^{M} \left|\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j \frac{x_j}{\alpha}\right|^{p_k} \\
&= \alpha^M \sum_k \left|\alpha \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j \frac{x_j}{\alpha}\right|^{p_k} \\
&= \alpha^M \rho\left(\frac{x}{\alpha}\right).
\end{aligned}$$

By the convexity of ρ, we have $\rho(\alpha x) \leq \alpha\rho(x)$, so (i) is obtained.

(ii) is an easy consequence of (i) when α is replaced by $\frac{1}{\alpha}$.

(iii) follows from the convexity of ρ. □

Proposition 3. *For any $x \in e^r(p)$, we have*

(i) *If $\|x\| < 1$, then $\rho(x) \leq \|x\|$.*
(ii) *If $\|x\| > 1$, then $\rho(x) \geq \|x\|$.*
(iii) *$\|x\| = 1$ if and only if $\rho(x) = 1$.*
(iv) *$\|x\| < 1$ if and only if $\rho(x) < 1$.*
(v) *$\|x\| > 1$ if and only if $\rho(x) > 1$.*
(vi) *If $0 < \alpha < 1$, $\|x\| > \alpha$, then $\rho(x) > \alpha^M$.*
(vii) *If $\alpha \geq 1$, $\|x\| < \alpha$, then $\rho(x) < \alpha^M$.*

Proof. See [12, Propositions 2.4, 2.5]. □

Proposition 4. *Let (x_n) be a sequence in $e^r(p)$.*

(i) *If $\lim_{n\to\infty} \|x_n\| = 1$, then $\lim_{n\to\infty} \rho(x_n) = 1$,*
(ii) *If $\lim_{n\to\infty} \rho(x_n) = 0$, then $\lim_{n\to\infty} \|x_n\| = 0$.*

Proof. See [12, Proposition 2.5]. □

Lemma 4. *Let (x_n) be a sequence in $e^r(p)$. If $\rho(x_n) \to \rho(x)$ and $x_n(k) \to x(k)$ for all k, then $x_n \to x$ as $n \to \infty$.* ([12])

Theorem 7. *The space $e^r(p)$ is a Banach space with respect to Luxemburg norm.*

Proof. Let $(x^n) = \left(x_j^n\right)$ be a Cauchy sequence in $e^r(p)$. Given $\varepsilon \in (0,1)$. Thus, there exists $N \in \mathbb{N}$ such that $\|x^n - x^m\| < \varepsilon^M$, for all $n, m \geq N$. By Proposition 3(i), we obtain

$$\rho(x^n - x^m) \leq \|x^n - x^m\| < \varepsilon^M \qquad \text{for all} \quad n, m \geq N. \tag{3.1}$$

This implies that

$$\sum_k \left|\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j \left(x_j^n - x_j^m\right)\right|^{p_k} < \varepsilon^M \qquad \text{for} \quad n, m \geq N.$$

For fixed j, we get that

$$\left|x_j^n - x_j^m\right| < \varepsilon \qquad \text{for all} \quad n, m \geq N.$$

Thus, $\left(x_j^n\right)$ be a Cauchy sequence in $\mathbb{R}$ for all $j \in \mathbb{N}$. Since $\mathbb{R}$ is complete for each $j \in \mathbb{N}$, $x_j^m \to x_j$ as $m \to \infty$. So, for fixed j,

$$\left|x_j^n - x_j\right| < \varepsilon \qquad \text{for} \quad n \geq N.$$

By (7.1)

$$\sum_k \left|\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j \left(x_j^n - x_j^m\right)\right|^{p_k} < \varepsilon \qquad \text{for all} \quad n, m \geq N.$$

For every $j \in \mathbb{N}$, we have $x_j^m \to x_j$, so we obtain that $\rho(x^n - x^m) \to \rho(x^n - x)$ as $m \to \infty$. Thus,

$$\sum_k \left|\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j \left(x_j^n - x_j^m\right)\right|^{p_k} \to \sum_k \left|\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j \left(x_j^n - x_j\right)\right|^{p_k}$$

as $m \to \infty$. Hence, we have

$$\rho(x^n - x) < \varepsilon, \qquad \text{for all} \quad n \geq N.$$

So, $\|x^n - x\| < \varepsilon$. And now, by the linearity of the sequence space $e^r(p)$, we can write

$$x = (x - x^n) + x^n$$

$$\begin{aligned}
&\sum_k \Big|\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j x_j\Big|^{p_k} \\
&= \sum_k \Big|\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j \left(x_j - x_j^n\right) + \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j x_j^n\Big|^{p_k} \\
&\leq \sum_k \Big|\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j \left(x_j - x_j^n\right)\Big|^{p_k} + \sum_k \Big|\sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j x_j^n\Big|^{p_k} \\
&\leq \varepsilon.
\end{aligned}$$

So, x is in $e^r(p)$. Hence, the sequence space $e^r(p)$ is a Banach space with respect to Luxemburg norm. □

Theorem 8. *The space $e^r(p)$ has property* (H).

Proof. Let $x \in S(e^r(p))$, $x_n \in B(e^r(p))$ for all $n \in \mathbb{N}$ such that $x_n \to x$, weakly, and $\|x_n\| \to 1$ as $n \to \infty$. By Proposition 3(iii) , we have $\rho(x) = 1$. By Proposition 4(i), we obtain that $\rho(x_n) \to 1$ as $n \to \infty$. So $\rho(x_n) \to \rho(x)$ as $n \to \infty$. Since $x_n \to x$, weakly, ith coordinate mapping $\pi_i\colon e^r(p) \to \mathbb{R}$, defined by $\pi_i(x) = x_i$, is continuous, it implies that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$. It follows from Lemma 4 that $x_n \to x$ as $n \to \infty$. □

Corollary 1. *For $1 \leq p < \infty$, $\left(e_p^r, \|x\|_{e_p^r}\right)$ has property* (H).

Remark 1. For a bounded sequence of positive real numbers $p = (p_k)$ with $p_k = 1$ for all $k \in \mathbb{N}$, the space $e^r(p)$ equipped the Luxemburg norm is not retund, so it is not (LUR). To see this we put

$$x = x_k(r) = \left\{\left(1 + \frac{1}{r}\right)^k\right\}, \qquad y = y_k(r) = \left\{k\frac{(r-1)^{k-1}}{r^k}\right\}$$

Then $x, y \in S(e^r(p))$ because $\rho(x) = \rho(y) = 1$. Since $\rho\left(\frac{x+y}{2}\right) = 1$, we have by Proposition 3(iii) that $\left\|\frac{x+y}{2}\right\| = 1$. This shows that $e^r(p)$ is not rotund , so it is not (LUR).

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