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UNIQUE SOLVABILITY OF A NON-LINEAR NON-LOCAL BOUNDARY-VALUE PROBLEM FOR SYSTEMS OF NON-LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

N. Dilna* — A. Ronto**

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ABSTRACT. General conditions for the unique solvability of a non-linear non-local boundary-value problem for systems of non-linear functional differential equations are obtained.

The main goal of this paper is to establish new general condition sufficient for the unique solvability of the non-local non-linear boundary-value problem (2) for the non-linear functional differential equations (1). For non-linear functional differential systems determined by operators that may be defined on the space of the absolutely continuous functions only, we prove several new theorems close to some results of [2, 5, 7, 14, 15, 13, 12].

The main theorems established here generalize some recent results from [5, 14, 3, 6, 15] and are proved by using an abstract theorem from [9].

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1. Introduction

We consider the non-local boundary-value problem

$$u_k'(t) = (f_k u)(t), \qquad t \in [a,b], \;\; k = 1,2,\dots,n, \tag{1}$$

$$u_k(a) = \varphi_k(u), \qquad k = 1,2,\dots,n, \tag{2}$$

where $f_k \colon D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R})$, $k = 1,2,\dots,n$, are, generally speaking, non-linear operators and $\varphi_k \colon D([a,b],\mathbb{R}^n) \to \mathbb{R}$, $k = 1,2,\dots,n$, are non-linear functionals defined on the space $D([a,b],\mathbb{R}^n)$ of vector functions with absolutely continuous components.

2. Notation

(1) $\|u\| := \max_{k=1,2,\dots,n} |u_k|$ for any $u = (u_k)_{k=1}^n$ from $\mathbb{R}^n$.

(2) The set $D^+([a,b],\mathbb{R}^n)$ is defined by the formula

$$D^+([a,b],\mathbb{R}^n) := \Big\{ u = (u_k)_{k=1}^n \in D([a,b],\mathbb{R}^n) : \min_{\xi \in [a,b]} u_k(\xi) \geq 0 \text{ for all } k = 1,2,\dots,n \Big\}. \tag{3}$$

(3) The set $D^{++}([a,b],\mathbb{R}^n)$ is defined by the formula

$$D^{++}([a,b],\mathbb{R}^n) := \Big\{ u = (u_k)_{k=1}^n \in D^+([a,b],\mathbb{R}^n) : \operatorname*{vrai\,min}_{\xi \in [a,b]} u_k'(\xi) \geq 0 \text{ for all } k = 1,2,\dots,n \Big\}. \tag{4}$$

Definition 1. (see, e.g., [1])] A vector function $u = (u_k)_{k=1}^n : [a,b] \to \mathbb{R}^n$ is said to be a solution of the problem (1), (2) if it satisfies system (1) almost everywhere on the interval $[a,b]$ and possesses property (2).

A special class of linear operators is essentially used in what follows.

Definition 2. Let $h = (h_k)_{k=1}^n \colon D([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ be a linear mapping. We say that a linear operator $p = (p_k)_{k=1}^n : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$ belongs to the set $\mathscr{S}_{a,h}([a,b],\mathbb{R}^n)$ if the boundary-value problem

$$u_k'(t) = (p_k u)(t) + q_k(t), \qquad t \in [a,b], \;\; k = 1,2,\dots,n, \tag{5}$$

$$u_k(a) = h_k(u) + c_k, \qquad k = 1,2,\dots,n, \tag{6}$$

has a unique solution $u = (u_k)_{k=1}^n$ for any $\{q_k : k = 1, 2, \dots, n\} \subset L_1([a, b], \mathbb{R})$ and $\{c_k : k = 1, 2, \dots, n\} \subset \mathbb{R}$ and, moreover, the solution of (5), (6) possesses the property

$$\min_{t\in[a,b]} u_k(t) \geq 0, \qquad k = 1, 2, \dots, n, \tag{7}$$

whenever the functions q_k, $k = 1, 2, \dots, n$, and the constants c_k, $k = 1, 2, \dots, n$, appearing in (5) and (6) are non-negative.

We shall use in the sequel a natural notion of positivity of a linear operator.

Definition 3. A linear operator $p = (p_k)_{k=1}^n : D([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R}^n)$ is said to be positive if

$$\operatorname*{vrai\,min}_{t\in[a,b]} (p_k u)(t) \geq 0, \qquad k = 1, 2, \dots, n,$$

for any $u = (u_k)_{k=1}^n$ from $D([a, b], \mathbb{R}^n)$ satisfying (7).

3. Main result

Theorem 1. *Assume that there exist certain linear operators* $p_i = (p_{ik})_{k=1}^n : D([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R}^n)$, $i = 1, 2$, *and linear functionals* $h_{ik} : D([a, b], \mathbb{R}^n) \to \mathbb{R}$, $i = 1, 2$, $k = 1, 2, \dots, n$, *such that for arbitrary functions* $u = (u_k)_{k=1}^n : [a, b] \to \mathbb{R}^n$, $v = (v_k)_{k=1}^n : [a, b] \to \mathbb{R}^n$ *with the properties*

$$u_k(t) \geq v_k(t) \qquad t \in [a, b], \quad k = 1, 2, \dots, n, \tag{8}$$

the estimates

$$\begin{aligned} p_{2k}(u - v)(t) &\leq (f_k u)(t) - (f_k v)(t) \\ &\leq p_{1k}(u - v)(t), \qquad t \in [a, b], \quad k = 1, 2, \dots, n, \end{aligned} \tag{9}$$

and

$$h_{2k}(u - v) \leq \varphi_k(u) - \varphi_k(v) \leq h_{1k}(u - v), \qquad k = 1, 2, \dots, n, \tag{10}$$

are true. Furthermore, assume that the following inclusions are fulfilled:

$$p_1 \in \mathscr{S}_{a,h_1}([a, b], \mathbb{R}^n), \qquad \frac{1}{2}(p_1 + p_2) \in \mathscr{S}_{a,\frac{1}{2}(h_1+h_2)}([a, b], \mathbb{R}^n). \tag{11}$$

Then the boundary-value problem (1), (2) *has a unique solution.*

Remark 1. Theorem 1 implies, in particular, [5, Theorem 1].

The proofs of Theorem 1 and the results stated in the next section are given in Section 6.

4. Corollaries

The following statements are true.

Corollary 1. *Assume that there exist certain linear operator* $l = (l_k)_{k=1}^n : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$ *and linear functional* $h = (h_k)_{k=1}^n : D([a,b],\mathbb{R}^n) \to \mathbb{R}$ *such that the inclusion*

$$l \in \mathscr{S}_{a,h}([a,b],\mathbb{R}^n) \tag{12}$$

holds and the estimates

$$0 \le \varphi_k(u) - \varphi_k(v) \le h_k(u-v), \qquad k = 1,2,\dots,n, \tag{13}$$

and

$$0 \le (f_k u)(t) - (f_k v)(t) \le l_k(u-v)(t), \qquad t \in [a,b], \quad k = 1,2,\dots,n, \tag{14}$$

are true for any absolutely continuous functions u *and* v *with property* (8)*.*

Then the boundary-value problem (1), (2) *has a unique solution.*

Theorem 2. *Let there exist linear operators* $l_i : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $l_i = (l_{ik})_{k=1}^n$, $i = 1,2$, *satisfying the inclusions*

$$l_1 + l_2 \in \mathscr{S}_{a,h_1}([a,b],\mathbb{R}^n), \qquad l_1 \in \mathscr{S}_{a,\frac{1}{2}(h_1+h_2)}([a,b],\mathbb{R}^n) \tag{15}$$

and such that the inequalities

$$|(f_k u)(t) - (f_k v)(t) - l_{1k}(u-v)(t)| \le l_{2k}(u-v)(t), \qquad k = 1,2,\dots,n, \tag{16}$$

are true for arbitrary absolutely continuous functions $u = (u_k)_{k=1}^n$ *and* $v = (v_k)_{k=1}^n$ *possessing the properties* (8) *and there exist linear functionals* $h_{ik} : D([a,b],\mathbb{R}^n) \to \mathbb{R}$, $i = 1,2$, $k = 1,2,\dots,n$, *which satisfy* (10)*.*

Then the boundary-value problem (1), (2) *is uniquely solvable.*

Remark 2. Theorem 2 implies, in particular, the result established in [3, Theorem 1] and [15, Theorem 3.3].

Theorem 3. *Assume that there exist some positive linear operators* $g_i = (g_{ik})_{i=1}^n : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i = 1,2$, *such that the inequalities*

$$|(f_k u)(t) - (f_k v)(t) + g_{2k}(u-v)(t)| \le g_{1k}(u-v)(t), \qquad k = 1,2,\dots,n, \tag{17}$$

hold on $[a,b]$ *for any vector functions* $u = (u_k)_{k=1}^n$ *and* $v = (v_k)_{k=1}^n$ *from* $D([a,b],\mathbb{R}^n)$ *with the properties* (8) *and there exist linear functionals* $h_{ik} : D([a,b],\mathbb{R}^n) \to \mathbb{R}$, $i = 1,2$, $k = 1,2,\dots,n$, *which satisfy* (10)*. Moreover, let there exist some* $\theta \in (0,1)$ *such that the inclusions*

$$g_1 + (1-2\theta)\, g_2 \in \mathscr{S}_{a,h_1}([a,b],\mathbb{R}^n), \qquad -\theta g_2 \in \mathscr{S}_{a,\frac{1}{2}(h_1+h_2)}([a,b],\mathbb{R}^n) \tag{18}$$

be true.

Then the boundary-value problem (1), (2) *has a unique solution.*

Remark 3. Theorem 3 generalizes [3, Theorem 2] and [14, Theorem 2].

Theorem 3 allows one to obtain the following two corollaries.

Corollary 2. *Let there exist certain positive linear operators* $g_i = (g_{ik})_{k=1}^n : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i = 1,2$, *which satisfy the condition* (17) *for arbitrary absolutely continuous functions* $u = (u_k)_{k=1}^n$ *and* $v = (v_k)_{k=1}^n$ *with the properties* (8) *and there exist linear functionals* $h_{ik} \colon D([a,b],\mathbb{R}^n) \to \mathbb{R}$, $i = 1,2$, $k = 1,2,\ldots,n$, *which satisfy* (10), *and, moreover, the inclusions*

$$g_1 \in \mathscr{S}_{a,h_1}([a,b],\mathbb{R}^n), \qquad -\frac{1}{2}g_2 \in \mathscr{S}_{a,\frac{1}{2}(h_1+h_2)}([a,b],\mathbb{R}^n), \tag{19}$$

hold.

Then the boundary-value problem (1), (2) *has a unique solution.*

Corollary 3. *Let there exist certain positive linear operators* $g_i = (g_{ik})_{k=1}^n : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i = 0,1$, *which satisfy the condition* (17) *for arbitrary absolutely continuous functions* $u = (u_k)_{k=1}^n$ *and* $v = (v_k)_{k=1}^n$ *with the properties* (8) *and there exist linear functionals* $h_{ik} \colon D([a,b],\mathbb{R}^n) \to \mathbb{R}$, $i = 1,2$, $k = 1,2,\ldots,n$, *which satisfy* (10), *and, moreover, the inclusions*

$$g_0 + \frac{1}{2}g_1 \in \mathscr{S}_{a,h_1}([a,b],\mathbb{R}^n), \qquad -\frac{1}{4}g_1 \in \mathscr{S}_{a,\frac{1}{2}(h_1+h_2)}([a,b],\mathbb{R}^n) \tag{20}$$

are true.

Then problem (1), (2) *has a unique solution.*

Some conditions sufficient for the fulfillment of inclusions of type (18), (19), and (20) can be derived, in particular, from the results of [11, 13]. In the case of the Cauchy problem, one can refer, e.g., to [2, 5, 6, 3, 16] and references therein.

5. Auxiliary statements

Let us consider the abstract operator equation

$$Fx = z, \tag{21}$$

where $F \colon E_1 \to E_2$ is a mapping, $\langle E_1, \|\cdot\|_{E_1}\rangle$ is a normed space, $\langle E_2, \|\cdot\|_{E_2}\rangle$ is a Banach space over the field $\mathbb{R}$, $K_i \subset E_i$, $i = 1,2$, are closed cones, and z is an arbitrary element from E_2.

The following statement is due to M. A. Krasnoselskii, E. A. Lifshits, Yu. V. Pokornyi, and V. Ya. Stetsenko [10, Theorem 7] (see also [9, Theorem 49.4]).

Theorem 4. *Let the cone K_2 be normal and reproducing. Furthermore, let $B_k\colon E_1 \to E_2$, $k = 1, 2$, be additive and homogeneous operators such that B_1^{-1} and $(B_1 + B_2)^{-1}$ exist and possess the properties*

$$B_1^{-1}(K_2) \subset K_1, \qquad (B_1 + B_2)^{-1}(K_2) \subset K_1, \tag{22}$$

and, furthermore, the relation

$$\{Fx - Fy - B_1(x - y),\ B_2(x - y) - Fx + Fy\} \subset K_2 \tag{23}$$

is satisfied for any pair $(x, y) \in E_1^2$ such that $x - y \in K_1$.

Then equation (21) has a unique solution $u \in E_1$ for an arbitrary element $z \in E_2$.

Let us recall that a cone $K \subset E$ in a Banach space $\langle E, \|\cdot\|_E\rangle$ is normal if and only if the relation

$$\inf\Big\{\gamma \in (0, +\infty) :\ (\forall x, y \in K)\big(y - x \in K \implies \|x\|_E \le \gamma\|y\|_E\big)\Big\} < +\infty$$

is true. By definition, the cone K is reproducing in E if and only if an arbitrary element x from E can be represented in the form $x = u - v$, where u and v belong to K (see, e.g., [8, 9]).

Lemma 1.

(1) *$D([a, b], \mathbb{R}^n)$ is a normed space with the norm*

$$D([a, b], \mathbb{R}^n) \ni u \longmapsto \int_a^b \|u'(\xi)\|\,\mathrm{d}\xi + \|u(a)\|.$$

(2) *The set $D^+([a, b], \mathbb{R}^n)$ is a cone in the space $D([a, b], \mathbb{R}^n)$.*

(3) *The set $D^{++}([a, b], \mathbb{R}^n)$ is a normal and reproducing cone in the space $D([a, b], \mathbb{R}^n)$.*

P r o o f. The assertions of Lemma 1 follow immediately from the definitions of the sets $D^+([a, b], \mathbb{R}^n)$ and $D^{++}([a, b], \mathbb{R}^n)$ (see formulae (3) and (4) in Section 2). □

The next lemma establishes the relation between the property described by Definition 2 and the positive invertibility of a certain operator.

Lemma 2. *If $p = (p_k)_{k=1}^n\colon D([a, b], \mathbb{R}^n) \to L_1([a, b], \mathbb{R}^n)$ is a linear operator such that*

$$p \in \mathscr{S}_{a,h}([a, b], \mathbb{R}^n), \tag{24}$$

then the operator $V_{p,h}\colon D([a, b], \mathbb{R}^n) \to D([a, b], \mathbb{R}^n)$ given by the formula

$$D([a, b], \mathbb{R}^n) \ni u \longmapsto V_{p,h}u := u - \int_a^{\cdot} (pu)(\xi)\,\mathrm{d}\xi - h(u) \tag{25}$$

is invertible and, moreover, its inverse $V_{p,h}^{-1}$ *satisfies the inclusion*

$$V_{p,h}^{-1}(D^{++}([a,b],\mathbb{R}^n)) \subset D^{+}([a,b],\mathbb{R}^n). \tag{26}$$

P r o o f. Let the mapping p belong to the set $\mathscr{S}_{a,h}([a,b],\mathbb{R}^n)$. Given an arbitrary function $y = (y_k)_{k=1}^n \in D([a,b],\mathbb{R}^n)$, consider the equation

$$V_{p,h}u = y. \tag{27}$$

We need to show that equation (27) has a unique solution u for any y and, moreover, its solution u has property (7) if

$$\min_{t\in[a,b]} y_k(t) \ge 0, \qquad \operatorname{vrai\,min}_{t\in[a,b]} y_k'(t) \ge 0 \tag{28}$$

for all $k = 1,2,\dots,n$.

Indeed, it is easy to see from notation (25) that an absolutely continuous function u satisfies equation (27) if, and only if

$$u_k'(t) = (p_k u)(t) + y_k'(t), \qquad t\in[a,b],\ \ k=1,2,\dots,n, \tag{29}$$

$$u_k(a) = h_k(u) + y_k(a), \qquad k=1,2,\dots,n. \tag{30}$$

Recalling Definition 2 and taking assumption (24) into account, we conclude that the linear inhomogeneous boundary-value problem (29), (30) is uniquely solvable for arbitrary absolutely continuous functions y_k, $k = 1,2,\dots,n$. Moreover, condition (24) implies that all the components of the solution of (29), (30) are non-negative for y satisfying (28). □

The following statement is an obvious consequence of formula (25).

Lemma 3. *For arbitrary linear operators* $p_i\colon D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i = 1,2$, *the identity*

$$V_{p_1,h_1} + V_{p_2,h_2} = 2V_{\frac{1}{2}(p_1+p_2),\frac{1}{2}(h_1+h_2)} \tag{31}$$

is true.

6. Proofs

6.1. Proof of Theorem 1

Consider the boundary-value problem (1), (2). It is obvious that an absolutely continuous vector function $u = (u_k)_{k=1}^n\colon [a,b] \to \mathbb{R}^n$ is a solution of (1), (2) if, and only if it satisfies the equation

$$u(t) = \int_a^t (fu)(s)\,\mathrm{d}s + \varphi(u), \qquad t\in[a,b]. \tag{32}$$

Let us define the mapping $F\colon E \to E$ by setting $E := E_1 := E_2 := D([a,b],\mathbb{R}^n)$ and

$$(Fu)(t) := u(t) - \int_a^t (fu)(s)\,\mathrm{d}s - \varphi(u), \qquad t \in [a,b], \tag{33}$$

for any u from $D([a,b],\mathbb{R}^n)$. Then equation (32) takes form (21) with $z = 0$. It is obvious that $F = (F_k)_{k=1}^n$ is an operator acting in the Banach space E.

It is easy to see, that function u is the unique solution of the boundary-value problem (1), (2) if and only if it satisfies the equation

$$Fu = 0. \tag{34}$$

Thus, it is sufficient to show that equation (34) has a unique solution in the space $D([a,b],\mathbb{R}^n)$.

Relation (9) is equivalent to inequalities

$$\begin{aligned} -p_{1k}(u-v)(t) &\le -(f_k u)(t) + (f_k v)(t) \\ &\le -p_{2k}(u-v)(t), \qquad k = 1,2,\dots,n, \end{aligned} \tag{35}$$

for any pair $\{u,v\}$ from $D([a,b],\mathbb{R}^n)$ with properties (8). Integrating (35) we obtain that the inequalities

$$\begin{aligned} -\int_a^t p_{1k}(u-v)(s)\,\mathrm{d}s &\le -\int_a^t (f_k u)(s)\,\mathrm{d}s + \int_a^t (f_k v)(s)\,\mathrm{d}s \\ &\le -\int_a^t p_{2k}(u-v)(s)\,\mathrm{d}s, \qquad k = 1,2,\dots,n \end{aligned} \tag{36}$$

hold for any pair $\{u,v\}$ from $D([a,b],\mathbb{R}^n)$ with property (8). Therefore, for all such pairs, the following inequalities are fulfilled:

$$\begin{aligned} &u_k(t) - v_k(t) - \int_a^t p_{1k}(u-v)(s)\,\mathrm{d}s - \varphi_k(u) + \varphi_k(v) \\ &\le u_k(t) - v_k(t) - \int_a^t (f_k u)(s)\,\mathrm{d}s + \int_a^t (f_k v)(s)\,\mathrm{d}s - \varphi_k(u) + \varphi_k(v) \\ &\le u_k(t) - v_k(t) - \int_a^t p_{2k}(u-v)(s)\,\mathrm{d}s - \varphi_k(u) + \varphi_k(v) \end{aligned} \tag{37}$$

for all $k = 1,2,\dots,n$ and $t \in [a,b]$. Taking into account (10) and (33) we get that for all functions u and v from $D([a,b],\mathbb{R}^n)$ with properties (8) the following estimates hold:

$$\begin{aligned} & u_k(t) - v_k(t) - \int_a^t p_{1k}(u-v)(s)\,\mathrm{d}s - h_{1k}(u-v) \\ & \le (F_k u)(t) - (F_k v)(t) \qquad\qquad (38) \\ & \le u_k(t) - v_k(t) - \int_a^t p_{2k}(u-v)(s)\,\mathrm{d}s - h_{2k}(u-v), \qquad k = 1, 2, \ldots, n. \end{aligned}$$

Let us define the linear mappings $B_{ik}\colon D([a,b],\mathbb{R}^n) \to D([a,b],\mathbb{R})$, $i = 1,2$, $k = 1, 2, \ldots, n$, by putting

$$B_{ik}u := u_k(\cdot) - h_{ik}(u) - \int_a^{\cdot} (p_{ik}u)(\xi)\,\mathrm{d}\xi \tag{39}$$

for an arbitrary u from $D([a,b],\mathbb{R}^n)$ and construct the corresponding mappings $B_i\colon D([a,b],\mathbb{R}^n) \to D([a,b],\mathbb{R}^n)$, $i = 1,2$, according to the formula

$$D([a,b],\mathbb{R}^n) \ni u \longmapsto B_i u := \begin{pmatrix} B_{i1}u \\ B_{i2}u \\ \vdots \\ B_{in}u \end{pmatrix}, \qquad i = 1, 2. \tag{40}$$

This fact that $p_1 \in S_{a,h_1}$ means that V_{p_1,h_1} is invertible and the inclusion

$$V^{-1}_{p_1,h_1}(K_2) \subset K_1 \tag{41}$$

is true. We assume that $\frac{1}{2}(p_1 + p_2)$ belong to the set $S_{a,\frac{1}{2}(h_1+h_2)}([a,b],\mathbb{R}^n)$. Taking into account Lemmas 2 and 3 we get that the operator

$$\frac{1}{2} V^{-1}_{\frac{1}{2}(p_1+p_2),\frac{1}{2}(h_1+h_2)}$$

exists and coincides with the inverse operator to $V_{p_1,h_1} + V_{p_2,h_2}$. Moreover, this operator is positive in the sense that

$$\bigl(V_{p_1,h_1} + V_{p_2,h_2}\bigr)^{-1}(K_2) \subset K_1. \tag{42}$$

The last property means that mapping (39) satisfies condition (23) with K_1 and K_2 defined by the formulae

$$K_1 = D^{+}([a,b],\mathbb{R}^n), \qquad K_2 = D^{++}([a,b],\mathbb{R}^n). \tag{43}$$

By virtue of Lemma 1, the set K_1 forms a cone in the normed space $D([a,b],\mathbb{R}^n)$, whereas the set K_2 is a normal and reproducing cone in the Banach space $D([a,b],\mathbb{R}^n)$.

Finally, according to equalities (39), we have $B_i = V_{p_i,h_i}$, $i = 1, 2$. Therefore, by virtue of assumption (16) and Lemma 2, we conclude that the inverse operator B^{-1} and $(B_1 + B_2)^{-1}$ exist and possess properties (22) with respect to cones (43). Applying Theorem 4, we establish the unique solvability of the initial value problem (1), (2).

Theorem 1 is proved.

6.2. Proof of Corollary 1

It is easy to see that inequalities (14) are equivalent to the conditions

$$-l_k(u - v)(t) \le -(f_k u)(t) + (f_k v)(t) \le 0, \qquad t \in [a, b], \quad k = 1, 2, \dots, n, \tag{44}$$

and inequalities (13) are equivalent to the relations

$$-h_k(u - v) \le -\varphi_k(u) + \varphi_k(v) \le 0, \qquad k = 1, 2, \dots, n, \tag{45}$$

for u and v satisfying (8). Let us put

$$p_1 := l, \qquad p_2 := 0. \tag{46}$$

Considering (44), we find that the operator f admits estimate (9) with the operators p_1 and p_2 defined by formulae (46). Therefore, it remains only to note that assumption (12) ensures the validity of inclusions (11). Thus, applying Theorem 1 we arrive at the assertion of the corollary.

6.3. Proof of Theorem 2

One can verify that condition (16) is equivalent to the relation

$$\begin{aligned} -l_{2k}&(u - v)(t) + l_{1k}(u - v)(t) \\ &\le (f_k u)(t) - (f_k v)(t) \\ &\le l_{2k}(u - v)(t) + l_{1k}(u - v)(t), \qquad t \in [a, b], \end{aligned} \tag{47}$$

for arbitrary functions $u = (u_k)_{k=1}^n$ and $v = (v_k)_{k=1}^n$ from $D([a, b], \mathbb{R}^n)$ with property (8).

Let us put

$$(p_{ik} x)(t) := (l_{1k} x)(t) - (-1)^i (l_{2k} x)(t), \qquad t \in [a, b], \quad i = 1, 2, \tag{48}$$

for any x from $D([a, b], \mathbb{R}^n)$ and $k = 1, 2, \dots, n$. Considering relations (47), we find that the operator f admits the estimate (9) with operators l_1 and l_2 defined by formulae (48). Therefore, it remains only to note that assumption (15) ensures the validity of inclusions (11). Application of Theorem 1 thus leads us to the assertion of Theorem 2.

6.4. Proof of Theorem 3

One can check that, under conditions (17), (18), the operators $l_i : D([a,b],\mathbb{R}^n) \to L_1([a,b],\mathbb{R}^n)$, $i = 1,2$, defined by the formulae

$$l_1 := -\theta g_2, \qquad l_2 := g_1 + (1-\theta)g_2, \tag{49}$$

satisfy conditions (15), (16) of Theorem 2. Indeed, estimate (17), the assumption $\theta \in (0,1)$, and the positivity of the operator g_2 imply that, for any absolutely continuous functions $u = (u_k)_{k=1}^n$ and $v = (v_k)_{k=1}^n$ with properties (8), the relations

$$\begin{aligned}
&|(f_k u)(t) - (f_k v)(t) + \theta g_2(u-v)(t)| \\
&= |(f_k u)(t) - (f_k v)(t) + g_{2k}(u-v)(t) - (1-\theta)g_{2k}(u-v)(t)| \\
&\le g_{1k}(u-v)(t) + |(1-\theta)g_{2k}(u-v)(t)| \\
&= g_{1k}(u-v)(t) + (1-\theta)(g_{2k}(u-v)(t)), \qquad t \in [a,b], \;\; k = 1,2,\dots,n,
\end{aligned}$$

are true. This means that f admits estimate (16) with the operators p_1 and p_2 defined by formulae (49). Therefore, it remains only to note that assumption (18) ensures the validity of inclusions (15) for operators (49). Applying Theorem 2, we arrive at the required assertion.

6.5. Proof of Corollary 2

This statement follows from Theorem 3 in the case where one puts $\theta := \frac{1}{2}$.

6.6. Proof of Corollary 3

It is sufficient to apply Theorem 3 with $\theta := \frac{1}{4}$.

REFERENCES

[1] AZBELEV, N.—MAKSIMOV, V.—RAKHMATULLINA, L.: *Introduction to the Theory of Linear Functional Differential Equations*, World Federation Publishers Company, Atlanta, GA, 1995.

[2] BRAVYI, E.—HAKL, R.—LOMTATIDZE, A.: *Optimal conditions for unique solvability of the Cauchy problem for first order linear functional differential equations*, Czechoslovak Math. J. **52(127)** (2002), 513–530.

[3] DILNA, N.: *On the unique solvability of the initial-value problem for non-linear functional-differential equations*, Mem. Differential Equations Math. Phys. (To appear).

[4] DIL'NAYA, N. Z.—RONTO, A. N.: *Some new conditions for the solvability of the Cauchy problem for systems of linear functional-differential equations*, Ukrainian Math. J. **56** (2004), 1033–1053.

[5] DILNA, N. Z.—RONTO, A. N.: *General conditions for the unique solvability of the initial-value problem for nonlinear functional-differential equations*, Ukrainian Math. J. **60** (2008), No. 2, 167–172.

[6] DILNA, N. Z.—RONTO, A. N.: *General conditions guaranteeing the solvability of the Cauchy problem for functional differential equations*, Math. Bohem. **133** (2008), 435–445.

[7] HAKL, R.—LOMTATIDZE, A.—PŮŽA, B.: *On a boundary value problem for first-order scalar functional differential equations*, Nonlinear Anal. **53** (2003), 391–405.

[8] KRASNOSELSKII, M. A.: *Positive Solutions of Operator Equations*, Wolters-Noordhoff Scientific Publications Ltd., Groningen, 1964.

[9] KRASNOSELSKII, M. A.—ZABREIKO, P. P.: *Geometrical Methods of Nonlinear Analysis*, Springer, Berlin-New York, 1984.

[10] KRASNOSELSKII, M. A. — LIFSHITS, E. A. — POKORNYI, YU. V. — STETSENKO, V. YA.: *Positively invertible linear operators and the solvability of nonlinear equations*, Dokl. Akad. Nauk Tadzhik. SSR **17** (1974), 12–14.

[11] LOMTATIDZE, A.—OPLUŠTIL, Z.: *On nonnegative solutions of a certain boundary value problem for first order linear functional differential equations*, Electron. J. Qual. Theory Differ. Equ. **16** (2004), 1–21 (Proc. 7th Coll. Qualitative Theory of Differ. Equations).

[12] LOMTATIDZE, A.—OPLUŠTIL, Z.—ŠREMR, J.: *On a nonlocal boundary value problem for first order linear functional differential equations*, Mem. Differential Equations Math. Phys. **41** (2007), 69–85.

[13] OPLUŠTIL, Z.: *New solvability conditions for a nonlocal boundary-value problem for nonlinear functional differential equations*, Nonlinear Oscil. **11** (2008), 384–406.

[14] PYLYPENKO, V.—RONTO, A.—DILNA, N.: *On the unique solvability of a non-local boundary value problem for linear functional differential equations*, Math. Model. Anal. **13** (2008), 241–250.

[15] ŠREMR, J.: *On the Cauchy type problem for systems of functional differential equations*, Nonlinear Anal. **67** (2007), 3240–3260.

[16] ŠREMR, J.: *Solvability conditions of the Cauchy problem for two-dimensional systems of linear functional differential equations with monotone operators*, Math. Bohem. **132** (2007), 263–295.

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* *Institute of Mathematics*
National Academy of Sciences of Ukraine
Tereshchenkivska 3 St.
01601 Kiev
UKRAINE

Current address:
Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49 St.
SK–814 73 Bratislava
SLOVAKIA

E-mail: dilna@imath.kiev.ua
nataliya.dilna@mat.savba.sk
URL: http://www.mat.savba.sk/˜dilna/

** *Institute of Mathematics*
Academy of Sciences of the Czech Republic
Žižkova 22
CZ–616 62 Brno
CZECH REPUBLIC

E-mail: ronto@ipm.cz