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THEOREMS FOR GENERALIZED FAVARD-KANTOROVICH AND FAVARD-DURRMEYER OPERATORS IN EXPONENTIAL FUNCTION SPACES

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ABSTRACT. We consider the Kantorovich and the Durrmeyer type modifications of the generalized Favard operators and we prove some direct approximation theorems for functions f such that $w_\sigma f \in L_p(R)$, where $1 \le p \le \infty$ and $w_\sigma(x) = \exp(-\sigma x^2)$, $\sigma > 0$.

1. Introduction

Let $\gamma = (\gamma_n)_{n=1}^{\infty}$ be a positive sequence convergent to zero. For functions $f\colon R \to R$ the generalized Favard operatos are defined formally by

$$F_n f(x) = \sum_{k=-\infty}^{\infty} f(k/n) p_{n,k}(x;\gamma), \qquad (x \in \mathbb{R},\ \ n \in \mathbb{N}),$$

where

$$p_{n,k}(x;\gamma) = \frac{1}{n\gamma_n\sqrt{2\pi}} \exp\left(-\frac{1}{2\gamma_n^2}\left(\frac{k}{n} - x\right)^2\right)$$

(see [5]). In the case where $\gamma_n^2 = \vartheta/(2n)$ with a positive constant ϑ, F_n become the known Favard operators introduced by J. Favard [4] as discrete analogs of the singular Weierstrass integral. Some approximation properties of the clasical Favard operators for continuous functions f on $\mathbb{R}$ are presented in [1, 2], and for

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the generalized operators $F_n f$ are given e.g. in [5, 8]. For measurable functions f on $\mathbb{R}$ we introduce, also formally, the generalized Favard-Kantorovich operators

$$F_n^* f(x) = n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int\limits_{k/n}^{(k+1)/n} f(t)\,\mathrm{d}t, \qquad (x \in \mathbb{R},\ \ n \in \mathbb{N})$$

and the generalized Favard-Durrmeyer operators

$$\tilde{F}_n f(x) = n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int\limits_{-\infty}^{\infty} p_{n,k}(t;\gamma) f(t)\,\mathrm{d}t, \qquad (x \in \mathbb{R},\ \ n \in \mathbb{N})$$

where the integrals are understood in the sense of Lebesgue. Some estimates concerning the rate of pointwise convergence of operators $F_n^* f$ and $\tilde{F}_n f$ can be found in [6, 7].

In this paper we present estimates of the rate of the norm convergence of these operators in the weighted function spaces

$$L_{p,\sigma}(\mathbb{R}) = \Big\{ f:\ \|w_\sigma f\|_p < \infty \Big\} \qquad \text{for} \quad 1 \le p \le \infty,$$

where $w_\sigma(x) = \exp(-\sigma x^2)$, $\sigma > 0$ and

$$\|g\|_p = \left(\int\limits_{-\infty}^{\infty} |g(x)|^p\,\mathrm{d}x \right)^{1/p} \qquad \text{if} \quad 1 \le p < \infty,$$

$$\|g\|_\infty = \operatorname*{essup}_{x\in\mathbb{R}} |g(x)|.$$

We define the following weighted modulus of smoothness of the function $f \in L_{p,\sigma}(\mathbb{R})$ as

$$\omega_2(f;\delta)_{\sigma,p} = \sup_{|h|\le\delta} \left\| w_\sigma \Delta_h^2 f \right\|_p,$$

where

$$\Delta_h^2 f(x) = f(x+h) - 2f(x) + f(x-h), \qquad (x \in \mathbb{R}).$$

Throughout the paper, the symbols $K(c,\sigma,\sigma_1)$, $K_j(c,\sigma,\sigma_1)$ $(j = 1,2,\dots)$ will mean some positive constants, not necessarily the same at each occurrence, depending only on the parameters indicated in parentheses.

2. Auxiliary estimates

As is known ([7, pp. 104, 105]), for all $n \in \mathbb{N}$, $v \in \mathbb{N}$, $k \in \mathbb{Z} = \{0, \pm1, \pm2, \dots\}$

$$\int\limits_{-\infty}^{\infty} p_{n,k}(t;\gamma)\,\mathrm{d}t = \frac{1}{n}, \tag{1}$$

$$\int_{-\infty}^{\infty} \left|\frac{k}{n} - t\right|^{v} p_{n,k}(t;\gamma)\,\mathrm{d}t \leq \sqrt{(2v-1)!!}\,\frac{\gamma_n^v}{n}. \tag{2}$$

Further, let us observe that

$$\sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) = 1 + S_n(x;\gamma), \tag{3}$$

where

$$S_n(x;\gamma) = 2\sum_{j=1}^{\infty} \exp(-2\pi^2 j^2 n^2 \gamma_n^2)\cos(2\pi n j x)$$

(see [3, pp. 126, 204]; [5, p. 388]).

Thank's to [8, Lemma 2.1] and the known Schwarz inequality we have:

Lemma 2.1. ([8]) *Let $\gamma = (\gamma_n)_{n=1}^{\infty}$ be a positive sequence convergent to zero and let $n\gamma_n^2 \geq c$ for all $n \in \mathbb{N}$, with a positive absolute constant c. Then for $x \in \mathbb{R}$, $v \in \mathbb{N}$, $n \in \mathbb{N}$, we have*

$$|S_n(x;\gamma)| \leq 4A_c/n, \tag{4}$$

$$\sum_{k=-\infty}^{\infty} \left|\frac{k}{n} - x\right|^{v} p_{n,k}(x;\gamma) \leq 15A_c\left(\frac{2}{\mathrm{e}}\right)^{v/2}\sqrt{(2v)!}\,\gamma_n^v, \tag{5}$$

where $A_c = \max\left\{1, (2c\pi^2)^{-1}\right\}$.

In case $v = 1$ more precise calculation leads to the estimate

$$\left|\sum_{k=-\infty}^{\infty}\left(\frac{k}{n} - x\right) p_{n,k}(x;\gamma)\right| \leq \frac{2A_c}{n}. \tag{6}$$

Namely, from (3)

$$\sum_{k=-\infty}^{\infty} p'_{n,k}(x;\gamma) = -4\pi n \sum_{j=1}^{\infty} j\exp\left(-2\pi^2 j^2 n^2 \gamma_n^2\right)\sin\left(2\pi n j x\right).$$

Consequently

$$\left|\sum_{k=-\infty}^{\infty}\left(\frac{k}{n} - x\right) p_{n,k}(x;\gamma)\right| \leq 4\pi n\gamma_n^2 \sum_{j=1}^{\infty} j\exp\left(-2\pi^2 j n^2 \gamma_n^2\right).$$

From the identity

$$\sum_{j=1}^{\infty} j\exp(-ju) = \frac{1}{\exp(u)-1} + \frac{1}{(\exp(u)-1)^2}, \qquad u > 0,$$

we have

$$
\begin{aligned}
&\left|\sum_{k=-\infty}^{\infty}\left(\frac{k}{n}-x\right)p_{n,k}(x;\gamma)\right| \\
&\leq 4\pi n\gamma_n^2\left(\frac{1}{\exp(2\pi^2n^2\gamma_n^2)-1}+\frac{1}{\left(\exp(2\pi^2n^2\gamma_n^2)-1\right)^2}\right) \\
&\leq 4\pi n\gamma_n^2\left(\frac{1}{2\pi^2n^2\gamma_n^2}+\frac{1}{(2\pi^2n^2\gamma_n^2)^2}\right)\leq\frac{2A_c}{n}.
\end{aligned}
$$

Lemma 2.2. *If $f\in L_{p,\sigma}(\mathbb{R})$, $f'\in AC_{\mathrm{loc}}(\mathbb{R})$, $f''\in L_{p,\sigma}(\mathbb{R})$, $\sigma>0$, $1\leq p\leq\infty$, then for $\sigma_1>\sigma$*

$$
\|w_{\sigma_1}f'\|_p\leq 4\exp\left(\frac{\sigma\sigma_1}{\sigma_1-\sigma}\right)\left(\|w_\sigma f\|_p+\|w_\sigma f''\|_p\right). \tag{7}
$$

P r o o f. Let $a\in\mathbb{R}$, $b\in\mathbb{R}$, $a<b$. Using the Taylor expansion with the integral remainder we get

$$
f\left(x+(b-a)/2\right)=f(x)+(b-a)f'(x)/2+\int_0^{(b-a)/2}\left((b-a)/2-u\right)f''(x+u)\,\mathrm{d}u.
$$

Clearly, if $u\in[0;(b-a)/2]$, then $|(b-a)/2-u|\leq(b-a)/2$. So, for $1\leq p<\infty$ we have by the Minkowski inequality

$$
\begin{aligned}
&\|w_{\sigma_1}f'\|_{[a;(a+b)/2]} \\
&\leq\frac{2}{b-a}\left\{\left(\int_a^{(a+b)/2}\left(\exp(-\sigma_1x^2)|f(x+(b-a)/2|\right)^p\,\mathrm{d}x\right)^{1/p}\right. \\
&\quad+\left(\int_a^{(a+b)/2}\left(\exp(-\sigma_1x^2)|f(x)|\right)^p\,\mathrm{d}x\right)^{1/p} \\
&\quad\left.+\frac{b-a}{2}\left(\int_a^{(a+b)/2}\left(\int_0^{(b-a)/2}\exp(-\sigma_1x^2)|f''(x+u)|\,\mathrm{d}u\right)^p\,\mathrm{d}x\right)^{1/p}\right\},
\end{aligned}
$$

where the symbol $\|\cdot\|_{[\alpha;\beta]}$ means the usual norm in the space $L_p([\alpha,\beta])$.

Using the inequality $\exp\left(-\sigma_1 x^2 + \sigma\left(x + (b-a)/2\right)^2\right) \le \exp\left(\left(\frac{b-a}{2}\right)^2 \frac{\sigma\sigma_1}{\sigma_1-\sigma}\right)$ and substituting $t = (b-a)/2 + x$ we easily observe that

$$\left(\int_a^{(a+b)/2} \left(\exp(-\sigma_1 x^2)|f(x+(b-a)/2|)^p \,\mathrm{d}x\right)^{1/p}$$

$$\le \exp\left(\left(\frac{b-a}{2}\right)^2 \frac{\sigma\sigma_1}{\sigma_1-\sigma}\right)\left(\int_{(a+b)/2}^{b} \left(\exp(-\sigma t^2)|f(t|\right)^p \,\mathrm{d}t\right)^{1/p}.$$

Moreover, by the Hölder inequality

$$\left(\int_a^{(a+b)/2}\left(\int_0^{(b-a)/2} \exp(-\sigma_1 x^2)|f''(x+u)|\,\mathrm{d}u\right)^p \mathrm{d}x\right)^{1/p}$$

$$= \left(\int_a^{(a+b)/2}\left(\exp(-\sigma_1 x^2)\int_x^{x+(b-a)/2} \exp(\sigma t^2)\exp(-\sigma t^2)|f''(t)|\,\mathrm{d}t\right)^p \mathrm{d}x\right)^{1/p}$$

$$\le \exp\left(\left(\frac{b-a}{2}\right)^2 \frac{\sigma\sigma_1}{\sigma_1-\sigma}\right)\left(\frac{b-a}{2}\right)^{2-1/p}\left(\int_a^b \left(\exp(-\sigma t^2)|f''(t)|\right)^p \,\mathrm{d}t\right)^{1/p}.$$

Hence

$$\|w_{\sigma_1} f'\|_{[a;(a+b)/2]}$$
$$\le \frac{2}{b-a}\exp\left(\left(\frac{b-a}{2}\right)^2 \frac{\sigma\sigma_1}{\sigma_1-\sigma}\right)\left(\|w_\sigma f\|_{[a;b]} + \left(\frac{b-a}{2}\right)^{3-1/p}\|w_\sigma f''\|_{[a;b]}\right).$$

Consequently

$$\begin{aligned}\|w_{\sigma_1} f'\|_p^p &= \sum_{k=-\infty}^{\infty} \|w_{\sigma_1} f'\|_{[k;k+1]}^p \\ &\le \sum_{k=-\infty}^{\infty} 2^p \exp\left(\frac{\sigma\sigma_1}{\sigma_1-\sigma}p\right)\left(\|w_\sigma f\|_{[k;k+2]}^p + \|w_\sigma f''\|_{[k;k+2]}^p\right) \\ &= 2^{p+1}\exp\left(\frac{\sigma\sigma_1}{\sigma_1-\sigma}p\right)\left(\|w_\sigma f\|_p^p + \|w_\sigma f''\|_p^p\right).\end{aligned}$$

Therefore for $1 \le p < \infty$ inequality (7) is now evident. If $p = \infty$, then this inequality follows by simple calculation from the Taylor formula

$$f'(x) = f(x+1) - f(x) - \int_0^1 (1-u) f''(x+u)\,\mathrm{d}u.$$

□

Lemma 2.3. *Let $\gamma = (\gamma_n)_{n=1}^{\infty}$ be as in* Lemma 1 *and let $f \in L_{p,\sigma}(\mathbb{R})$, $\sigma > 0$, $1 \le p \le \infty$. Then for $\sigma_1 > \sigma$*

$$\|w_{\sigma_1} F_n^* f\|_p \le 3\sqrt{2} A_c \exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1 - \sigma}\right) \|w_\sigma f\|_p\,, \tag{8}$$

for all $n \in \mathbb{N}$ such that $\gamma_n^2 \le \frac{\sigma_1-\sigma}{4\sigma(\sigma+\sigma_1)}$,

$$\left\|w_{\sigma_1} \tilde{F}_n f\right\|_p \le 5 A_c \|w_\sigma f\|_p\,, \tag{9}$$

for all $n \in \mathbb{N}$ such that $\gamma_n \le \max\left\{\frac{\sigma_1-\sigma}{2\sqrt{\sigma}(\sigma+\sigma_1)}; \frac{\sqrt{\sigma_1-\sigma}}{\sqrt{2}(\sigma+\sigma_1)}\right\}$.

P r o o f. In view of the definition of the Favard-Kantorovich operators

$$\begin{aligned}
&\exp(-\sigma_1 x^2)|F_n^* f(x)| \\
&\le n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \exp(-\sigma_1 x^2) \exp\left(\frac{\sigma \tilde{k}^2}{n^2}\right) \int_{k/n}^{(k+1)/n} \exp(-\sigma t^2)|f(t)|\,\mathrm{d}t,
\end{aligned}$$

where $\tilde{k} = k$ or $\tilde{k} = k+1$, $\tilde{k}$ is that $\exp\left(\sigma \tilde{k}^2/n^2\right) = \max_{j=k,k+1}\left\{\exp(\sigma j^2/n^2)\right\}$. Using the inequality

$$(u+v)^2 \le \frac{\sigma+\sigma_1}{2\sigma} u^2 + \frac{\sigma+\sigma_1}{\sigma_1-\sigma} v^2 \qquad (u \in \mathbb{R},\ \ v \in \mathbb{R}) \tag{10}$$

we can easily observe, that

$$\begin{aligned}
&p_{n,k}(x;\gamma) \exp(-\sigma_1 x^2) \exp\left(\sigma\left(\frac{k+1}{n}\right)^2\right) \\
&\le p_{n,k}(x;\gamma) \exp(-\sigma_1 x^2) \exp\left(\sigma \frac{\sigma+\sigma_1}{2\sigma}\left(\frac{1}{n}+x\right)^2 + \sigma\frac{\sigma+\sigma_1}{\sigma_1-\sigma}\left(\frac{k}{n}-x\right)^2\right) \\
&= p_{n,k}(x;\gamma) \exp\left(\sigma\frac{\sigma+\sigma_1}{\sigma_1-\sigma}\left(\frac{k}{n}-x\right)^2 - \frac{\sigma_1-\sigma}{2}\left(x - \frac{\sigma+\sigma_1}{\sigma_1-\sigma}\frac{1}{n}\right)^2\right. \\
&\qquad\qquad \left. + \frac{1}{n^2}\left(\frac{(\sigma+\sigma_1)^2}{2(\sigma_1-\sigma)} + \frac{\sigma+\sigma_1}{2}\right)\right)
\end{aligned}$$

$$\le p_{n,k}(x;\gamma)\exp\left(\sigma\frac{\sigma+\sigma_1}{\sigma_1-\sigma}\left(\frac{k}{n}-x\right)^2\exp\left(\frac{1}{n^2}\left(\frac{(\sigma+\sigma_1)^2}{2(\sigma_1-\sigma)}+\frac{\sigma+\sigma_1}{2}\right)\right)\right.$$

$$=\frac{1}{n\gamma_n\sqrt{2\pi}}\exp\left(-\left(\frac{k}{n}-x\right)^2\left(\frac{1}{2\gamma_n^2}-\frac{\sigma(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\exp\left(\frac{1}{n^2}\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\right..$$

If $\gamma_n^2 \le \frac{\sigma_1-\sigma}{4\sigma(\sigma+\sigma_1)}$, then $\frac{1}{2\gamma_n^2}-\frac{\sigma(\sigma+\sigma_1)}{\sigma_1-\sigma}\ge\frac{1}{4\gamma_n^2}=\frac{1}{2(\sqrt{2}\gamma_n)^2}$.

Consequently

$$\begin{aligned}p_{n,k}(x;\gamma)\exp(-\sigma_1x^2)\exp\left(\sigma\left(\frac{k+1}{n}\right)^2\right)&\\ \le\sqrt{2}\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)p_{n,k}(x;\sqrt{2}\gamma),&\end{aligned}\tag{11}$$

where the symbol $\sqrt{2}\gamma$ means the sequence $(\sqrt{2}\gamma_n)_{n=1}^{\infty}$.

Analogously

$$p_{n,k}(x;\gamma)\exp(-\sigma_1x^2)\exp\left(\sigma\left(\frac{k}{n}\right)^2\right)\le\sqrt{2}p_{n,k}(x;\sqrt{2}\gamma).\tag{12}$$

Therefore

$$\begin{aligned}&\exp(-\sigma_1x^2)|F_n^*f(x)|\\ &\le\sqrt{2}\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)n\sum_{k=-\infty}^{\infty}p_{n,k}(x;\sqrt{2}\gamma)\int_{k/n}^{(k+1)/n}\exp(-\sigma t^2)|f(t)|\,\mathrm{d}t.\end{aligned}$$

From (1), we have

$$\|w_{\sigma_1}F_n^*f\|_1\le\sqrt{2}\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\|w_\sigma f\|_1\,.$$

Instead for $p=\infty$, from (3) and (4) it follows that

$$\|w_{\sigma_1}F_n^*f\|_\infty\le\sqrt{2}\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\|w_\sigma f\|_\infty\sup_{x\in\mathbb{R}}\left(\sum_{k=-\infty}^{\infty}p_{n,k}(x;\sqrt{2}\gamma)\right)$$

$$\begin{aligned}\sqrt{2}\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\|w_\sigma f\|_\infty\sup_{x\in\mathbb{R}}\left(1+\left|S_n(x;\sqrt{2}\gamma)\right|\right)&\\ \le3\sqrt{2}A_c\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\|w_\sigma f\|_\infty.&\end{aligned}$$

Finally by the Riesz-Thorin theorem we have (8).

In view of the definition of the Favard-Durrmeyer operators

$$\exp(-\sigma_1 x^2)|\tilde{F}_n f(x)| \le n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\exp(-\sigma_1 x^2) \int_{-\infty}^{\infty} p_{n,k}(t;\gamma)|f(t)|\,\mathrm{d}t.$$

Using (10) by simple calculation we have

$$\begin{aligned}
&p_{n,k}(x;\gamma)p_{n,k}(t;\gamma)\exp(-\sigma_1 x^2)\exp(\sigma t^2)\\
&\le p_{n,k}(x;\gamma)p_{n,k}(t;\gamma)\exp\left(-\sigma_1 x^2 + \frac{\sigma+\sigma_1}{2}x^2\right)\cdot\\
&\quad\cdot\exp\left(\frac{\sigma(\sigma+\sigma_1)}{\sigma-\sigma_1}\left(\frac{\sigma+\sigma_1}{2\sigma}\left(\frac{k}{n}-t\right)^2+\frac{\sigma+\sigma_1}{\sigma_1-\sigma}\left(\frac{k}{n}-x\right)^2\right)\right)\\
&\le \frac{1}{n\gamma_n\sqrt{2\pi}}\exp\left(-\frac{1}{2\gamma_n^2}\left(\frac{k}{n}-x\right)^2\right)\exp\left(\frac{\sigma(\sigma+\sigma_1)^2}{(\sigma_1-\sigma)^2}\left(\frac{k}{n}-x\right)^2\right)\cdot\\
&\quad\cdot\frac{1}{n\gamma_n\sqrt{2\pi}}\exp\left(-\frac{1}{2\gamma_n^2}\left(\frac{k}{n}-t\right)^2\right)\exp\left(\frac{(\sigma+\sigma_1)^2}{2(\sigma_1-\sigma)}\left(\frac{k}{n}-t\right)^2\right).
\end{aligned}$$

Consequently if $\frac{1}{2\gamma_n^2}-\frac{(\sigma+\sigma_1)^2\sigma}{(\sigma_1-\sigma)^2}\ge\frac{1}{4\gamma_n^2}$ and $\frac{1}{2\gamma_n^2}-\frac{(\sigma+\sigma_1)^2}{2(\sigma_1-\sigma)}\ge\frac{1}{4\gamma_n^2}$, then

$$p_{n,k}(x;\gamma)p_{n,k}(t;\gamma)\exp(-\sigma_1 x^2)\exp(\sigma t^2)\le 2p_{n,k}(x;\sqrt{2}\gamma)p_{n,k}(t;\sqrt{2}\gamma),$$

for all $n\in\mathbb{N}$ such that $\gamma_n\le\max\left\{\frac{\sigma_1-\sigma}{2\sqrt{\sigma}(\sigma+\sigma_1)};\frac{\sqrt{\sigma_1-\sigma}}{\sqrt{2}(\sigma+\sigma_1)}\right\}$.

Therefore

$$\exp(-\sigma_1 x^2)|\tilde{F}_n f(x)|\le n\sum_{k=-\infty}^{\infty}p_{n,k}(x;\sqrt{2}\gamma)\int_{-\infty}^{\infty}p_{n,k}(t;\sqrt{2}\gamma)\exp(-\sigma t^2)|f(t)|\,\mathrm{d}t.$$

From (1), (3) and (4), we have (9) for $p=1$ and for $p=\infty$. By the Riesz-Thorin Theorem (9) is obvious. □

3. Approximation results

Theorem 3.1. *Let $\gamma=(\gamma_n)_{n=1}^{\infty}$ be as in* Lemma 1. *If $f\in L_{p,\sigma}(\mathbb{R})$, $\sigma>0$, $1\le p\le\infty$ have the derivatives $f'\in AC_{\mathrm{loc}}(\mathbb{R})$, $f''\in L_{p,\sigma}(\mathbb{R})$ then for $\sigma_1>\sigma$*

$$\left\|w_{\sigma_1}\left(F_n^* f-f\right)\right\|_p\le K(c,\sigma,\sigma_1)\gamma_n^2\left(\|w_\sigma f\|_p+\|w_\sigma f''\|_p\right),\tag{13}$$

for all $n\in\mathbb{N}$ such that $\gamma_n^2\le\frac{\sigma_1-\sigma}{4\sigma(\sigma+\sigma_1)}$,

$$\left\|w_{\sigma_1}\left(\tilde{F}_n f-f\right)\right\|_p\le K(c,\sigma,\sigma_1)\gamma_n^2\left(\|w_\sigma f\|_p+\|w_\sigma f''\|_p\right)\tag{14}$$

for all $n\in\mathbb{N}$ such that $\gamma_n\le\max\left\{\frac{\sigma_1-\sigma}{2\sqrt{\sigma}(\sigma+\sigma_1)};\frac{\sqrt{\sigma_1-\sigma}}{\sqrt{2}(\sigma+\sigma_1)}\right\}$.

P r o o f. Using the Taylor expansion with the integral remainder for function f we have

$$f(t) = f(x) + (t-x)f'(x) + R_{f,x}(t), \tag{15}$$

where

$$R_{f,x}(t) = \int_x^t (t-v)f''(v)\,\mathrm{d}v.$$

Simple calculations, (15) and (3) give us

$$\begin{aligned} &F_n^* f(x) - f(x) \\ &= f(x)S_n(x) + f'(x)n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\frac{1}{2}\left(\frac{2}{n}\left(\frac{k}{n}-x\right)+\frac{1}{n^2}\right) + F_n^* R_{f,x}(x). \end{aligned}$$

Further, using (4) and (6) we get

$$|F_n^* f(x) - f(x)| \le \frac{4A_c}{n}|f(x)| + \frac{9A_c}{2n}|f'(x)| + |F_n^* R_{f,x}(x)|. \tag{16}$$

Obviously,

$$\begin{aligned} |F_n^* R_{f,x}(x)| &\le n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma) \int_{k/n}^{(k+1)/n} |t-x| \left| \int_x^t |f''(v)|\,\mathrm{d}v \right| \mathrm{d}t \\ &\le \sum_{k=-\infty}^{\infty} p_{n,k}(x;\gamma)\left(\left|\frac{k}{n}-x\right| + \frac{1}{n}\right)\left|\int_x^{k^*/n} |f''(v)|\,\mathrm{d}v\right|, \end{aligned}$$

where $k^* = k$ or $k^* = k+1$, k^* is that $\left|\int_x^{k^*/n} |f''(v)|\,\mathrm{d}v\right| = \max_{j=k,k+1}\left|\int_x^{j/n} |f''(v)|\,\mathrm{d}v\right|$.

We introduce the notation

$$D(l,n,x) = \left\{k \in Z:\ l\gamma_n \le \left|\frac{k}{n}-x\right| < (l+1)\gamma_n\right\}, \qquad \text{for} \quad l \in N_0. \tag{17}$$

From there

$$|F_n^* R_{f,x}(x)| \le \sum_{l=0}^{\infty} \sum_{k \in D(l,n,x)} p_{n,k}(x;\gamma)\left(\left|\frac{k}{n}-x\right| + \frac{1}{n}\right)\left|\int_x^{k^*/n} |f''(v)|\,\mathrm{d}v\right|.$$

From this inequality, (11), (12) and for $n \in \mathbb{N}$ such that $\gamma_n^2 \le \frac{\sigma_1-\sigma}{4\sigma(\sigma+\sigma_1)}$, we have

$$\begin{aligned}&\exp(-\sigma_1 x^2)\,|F_n^* R_{f,x}(x)|\\ &\le \sqrt{2}\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\sum_{l=0}^{\infty}\sum_{k\in D(l,n,x)} p_{n,k}\big(x;\sqrt{2}\gamma\big)\cdot\\ &\quad\cdot\left(\left|\frac{k}{n}-x\right|+\frac{1}{n}\right)\left|\int_x^{k^*/n}\exp(-\sigma v^2)|f''(v)|\,\mathrm{d}v\right|.\end{aligned}$$

But, using (5) and the condition $n\gamma_n^2 \ge c$ we have for $l \ge 1$

$$\begin{aligned}&\sum_{k\in D(l,n,x)} p_{n,k}\big(x;\sqrt{2}\gamma\big)\left(\left|\frac{k}{n}-x\right|+\frac{1}{n}\right)\\ &\le \frac{1}{l^3\gamma_n^3}\sum_{k=-\infty}^{\infty} p_{n,k}\big(x;\sqrt{2}\gamma\big)\left(\left|\frac{k}{n}-x\right|^4+\frac{1}{n}\left|\frac{k}{n}-x\right|^3\right)\le K(c)\gamma_n\frac{1}{(l+1)^3}.\end{aligned}$$

Analogously, for $l=0$

$$\sum_{k\in D(0,n,x)} p_{n,k}(x;\sqrt{2}\gamma)\left(\left|\frac{k}{n}-x\right|+\frac{1}{n}\right)\le K(c)\gamma_n,$$

and consequently,

$$\begin{aligned}&\exp(-\sigma_1 x^2)\,|F_n^* R_{f,x}(x)|\\ &\le K(c)\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\gamma_n\sum_{l=0}^{\infty}\frac{1}{(l+1)^3}\left|\int_{|v-x|\le(l+1)\gamma_n+1/n}\exp(-\sigma v^2)|f''(v)|\,\mathrm{d}v\right|.\end{aligned}$$

Hence

$$\begin{aligned}&\|w_{\sigma_1}F_n^* R_f\|_1\\ &\le K(c)\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\gamma_n\sum_{l=0}^{\infty}\frac{1}{(l+1)^3}\int_{-\infty}^{\infty}\left|\int_{|v-x|\le(l+1)\gamma_n+1/n}\exp(-\sigma v^2)|f''(v)|\,\mathrm{d}v\right|\mathrm{d}x\\ &= K(c)\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\gamma_n\sum_{l=0}^{\infty}\frac{1}{(l+1)^3}\int_{-\infty}^{\infty}\left(\int_{v-(l+1)\gamma_n+1/n}^{v+(l+1)\gamma_n+1/n}\exp(-\sigma v^2)|f''(v)|\,\mathrm{d}x\right)\mathrm{d}v\\ &\le K(c)\exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right)\gamma_n\,\|w_\sigma f''\|_1\sum_{l=0}^{\infty}\frac{1}{(l+1)^3}2\left((l+1)\gamma_n+1/n\right).\end{aligned}$$

Because $n\gamma_n^2 \geq c$, so

$$\left\|w_{\sigma_1} F_n^* R_f\right\|_1 \leq K_1(c) \exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right) \gamma_n^2 \left\|w_\sigma f''\right\|_1 .$$

Analogously, for $p=\infty$

$$\begin{aligned}
&\left\|w_{\sigma_1} F_n^* R_f\right\|_\infty \leq K(c) \exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right) \gamma_n \cdot \\
&\quad \cdot \operatorname*{essup}_{x\in\mathbb{R}} \sum_{l=0}^{\infty} \frac{1}{(l+1)^3} \left| \int\limits_{|v-x|\leq (l+1)\gamma_n+1/n} \exp(-\sigma v^2)|f''(v)|\,\mathrm{d}v \right| \\
&\leq K_2(c) \exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right) \gamma_n^2 \left\|w_\sigma f''\right\|_\infty .
\end{aligned}$$

From these inequalities and the Riesz–Thorin theorem, we have

$$\left\|w_{\sigma_1} F_n^* R_f\right\|_p \leq K_2(c) \exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right) \gamma_n^2 \left\|w_\sigma f''\right\|_p$$

for $1 \leq p \leq \infty$. Now, in view of (16),

$$\begin{aligned}
&\left\|w_{\sigma_1}(F_n^* f - f)\right\|_p \\
&\leq \frac{4A_c}{n}\left\|w_{\sigma_1} f\right\|_p + \frac{9A_c}{2n}\left\|w_{\sigma_1} f'\right\|_p + K_2(c) \exp\left(\frac{\sigma_1(\sigma+\sigma_1)}{\sigma_1-\sigma}\right) \gamma_n^2 \left\|w_\sigma f''\right\|_p .
\end{aligned}$$

Finally by (7) and the condition $n\gamma_n^2 \geq c$, the desired estimate for (13) is established.

Now, we will prove (14). Simple calculations, (3), (4), (6) and (15) give us

$$\left|\tilde{F}_n f(x) - f(x)\right| \leq \frac{4A_c}{n}|f(x)| + \frac{2A_c}{n}|f'(x)| + |\tilde{F}_n R_{f,x}(x)|. \tag{18}$$

Further

$$\begin{aligned}
&\exp(-\sigma_1 x^2)|\tilde{F}_n R_{f,x}(x)| \\
&\leq n \left| \sum_{k=-\infty}^{\infty} \exp(-\sigma_1 x^2) p_{n,k}(x;\gamma) \int\limits_{-\infty}^{\infty} p_{n,k}(t;\gamma)|t-x| \int\limits_x^t |f''(v)|\,\mathrm{d}v\,\mathrm{d}t \right| \\
&\leq n \sum_{k=-\infty}^{\infty} \exp(-\sigma_1 x^2) p_{n,k}(x;\gamma) \cdot \\
&\quad \cdot \int\limits_{-\infty}^{\infty} p_{n,k}(t;\gamma) \left(\left|\frac{k}{n}-x\right| + \left|\frac{k}{n}-t\right| \right) \left(\left| \int\limits_x^{k/n} |f''(v)|\,\mathrm{d}v \right| + \left| \int\limits_{k/n}^{t} |f''(v)|\,\mathrm{d}v \right| \right) \mathrm{d}t
\end{aligned}$$

$$\leq n \sum_{k=-\infty}^{\infty} \exp(-\sigma_1 x^2) p_{n,k}(x;\gamma) \int_{-\infty}^{\infty} p_{n,k}(t;\gamma) \left(\left| \frac{k}{n} - x \right| + \left| \frac{k}{n} - t \right| \right) \exp(\sigma \tilde{k}^2) \cdot$$
$$\cdot \left(\left| \int_x^{k/n} \exp(-\sigma v^2) |f''(v)| \, \mathrm{d}v \right| + \left| \int_{k/n}^{t} \exp(-\sigma v^2) |f''(v)| \, \mathrm{d}v \right| \right) \mathrm{d}t,$$

where $\tilde{k} = \max\{|x|, |t|, |k/n|\}$.

In the same way, as (11) and (12), we can show

$$p_{n,k}(x;\gamma) p_{n,k}(t;\gamma) \exp(-\sigma_1 x^2) \exp(\sigma \tilde{k}^2) \leq 2 p_{n,k}(x;\sqrt{2}\gamma) p_{n,k}(t;\sqrt{2}\gamma),$$

for all $n \in \mathbb{N}$ such that $\gamma_n \leq \max\left\{ \frac{\sigma_1 - \sigma}{2\sqrt{\sigma}(\sigma + \sigma_1)}; \frac{\sqrt{\sigma_1 - \sigma}}{\sqrt{2}(\sigma + \sigma_1)} \right\}$.

Therefore

$$\exp(-\sigma_1 x^2) |\tilde{F}_n R_{f,x}(x)|$$
$$\leq 2n \sum_{k=-\infty}^{\infty} p_{n,k}(x;\sqrt{2}\gamma) \int_{-\infty}^{\infty} p_{n,k}(t;\sqrt{2}\gamma) \left(\left| \frac{k}{n} - x \right| + \left| \frac{k}{n} - t \right| \right) \cdot$$
$$\cdot \left(\left| \int_x^{k/n} \exp(-\sigma v^2) |f''(v)| \, \mathrm{d}v \right| + \left| \int_{k/n}^{t} \exp(-\sigma v^2) |f''(v)| \, \mathrm{d}v \right| \right) \mathrm{d}t.$$

From (2), we have

$$\left\| w_{\sigma_1} \tilde{F}_n R_f \right\|_1$$
$$\leq 2 \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} p_{n,k}(x;\sqrt{2}\gamma) \left(\left| \frac{k}{n} - x \right| + \gamma_n \right) \left| \int_x^{k/n} \exp(-\sigma v^2) |f''(v)| \, \mathrm{d}v \right| \mathrm{d}x$$
$$+ 2 \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} p_{n,k}(t;\sqrt{2}\gamma) \left(\left| \frac{k}{n} - t \right| + \gamma_n \right) \left| \int_{k/n}^{t} \exp(-\sigma v^2) |f''(v)| \, \mathrm{d}v \right| \mathrm{d}t$$
$$= 4 \sum_{l=0}^{\infty} \sum_{k \in D(l,n,x)} \int_{-\infty}^{\infty} p_{n,k}(x;\sqrt{2}\gamma) \left(\left| \frac{k}{n} - x \right| + \gamma_n \right) \left| \int_{k/n}^{x} \exp(-\sigma v^2) |f''(v)| \, \mathrm{d}v \right|,$$

where $D(l,n,x)$ is defined by (17). Further, in the same way as for $\|w_{\sigma_1} F_n^* R_f\|$, we have

$$\|w_{\sigma_1} F_n^* R_f\|_1 \leq K_1(c) \exp\left(\frac{\sigma_1(\sigma + \sigma_1)}{\sigma_1 - \sigma} \right) \gamma_n^2 \, \|w_\sigma f''\|_1 \, .$$

For $p = \infty$

$$\left\|w_{\sigma_1}\tilde{F}_n R_f\right\|_\infty$$

$$\leq 2n \operatorname*{essup}_{x\in\mathbb{R}} \sum_{k=-\infty}^{\infty} p_{n,k}(x;\sqrt{2}\gamma) \int_{-\infty}^{\infty} p_{n,k}(t;\sqrt{2}\gamma) \left(\left|\frac{k}{n}-x\right| + \left|\frac{k}{n}-t\right|\right) \cdot$$

$$\cdot \left(\left|\int_x^{k/n} \|w_\sigma f''\|_\infty \,\mathrm{d}v\right| + \left|\int_{k/n}^{t} \|w_\sigma f''\|_\infty \,\mathrm{d}v\right|\right) \mathrm{d}t$$

$$= 2n \|w_\sigma f''\|_\infty \operatorname*{essup}_{x\in\mathbb{R}} \sum_{k=-\infty}^{\infty} p_{n,k}(x;\sqrt{2}\gamma) \int_{-\infty}^{\infty} p_{n,k}(t;\sqrt{2}\gamma) \left(\left|\frac{k}{n}-x\right| + \left|\frac{k}{n}-t\right|\right)^2 \mathrm{d}t$$

$$\leq K(c) \|w_\sigma f''\|_\infty \gamma_n^2.$$

These inequalities, the Riesz-Thorin theorem and (18), (7) yield (14) immediately. □

Theorem 3.2. *Let $\gamma = (\gamma_n)_{n=1}^{\infty}$ be a positive null sequence satisfying $n\gamma_n^2 \geq c$ for all $n \in \mathbb{N}$, with some $c > 0$. If $f \in L_{p,\sigma}(\mathbb{R})$, $\sigma > 0$, $1 \leq p \leq \infty$ then for $\sigma_1 > \sigma$*

$$\|w_{\sigma_1}(F_n^* f - f)\|_p \leq K(c,\sigma,\sigma_1)\left(\omega_2(f;\gamma_n)_{\sigma,p} + \gamma_n^2 \|w_\sigma f\|_p\right),$$

for all $n \in \mathbb{N}$ such that $\gamma_n^2 \leq \frac{\sigma_1-\sigma}{4\sigma(\sigma+\sigma_1)}$,

$$\left\|w_{\sigma_1}\left(\tilde{F}_n f - f\right)\right\|_p \leq K(c,\sigma,\sigma_1)\left(\omega_2(f;\gamma_n)_{\sigma,p} + \gamma_n^2 \|w_\sigma f\|_p\right),$$

for all $n \in \mathbb{N}$ such that $\gamma_n \leq \max\left\{\frac{\sigma_1-\sigma}{2\sqrt{\sigma}(\sigma+\sigma_1)}; \frac{\sqrt{\sigma_1-\sigma}}{\sqrt{2}(\sigma+\sigma_1)}\right\}$.

P r o o f. We define

$$f_{\gamma_n}(x) = \gamma_n^{-2} \int_{-\gamma_n/2}^{\gamma_n/2} \int_{-\gamma_n/2}^{\gamma_n/2} f(x+t_1+t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2.$$

It is easy to see that

$$\|w_\sigma(f_{\gamma_n} - f)\|_p \leq \omega_2(f;\gamma_n)_{\sigma,p},$$

and

$$\left\|w_\sigma f''_{\gamma_n}\right\|_p \leq \gamma_n^{-2}\omega_2(f;\gamma_n)_{\sigma,p}.$$

Then

$$\|w_{\sigma_1}(F_n^* f - f)\|_p$$
$$\leq \|w_{\sigma_1}F_n^*(f - f_{\gamma_n})\|_p + \|w_{\sigma_1}(f - f_{\gamma_n})\|_p + \|w_{\sigma_1}(F_n^* f_{\gamma_n} - f_{\gamma_n})\|_p.$$

Using (8) and (10), we have

$$
\begin{aligned}
&\left\| w_{\sigma_1} \left(F_n^* f - f \right) \right\|_p \\
&\leq 5\sqrt{2} \exp\left(\frac{\sigma\sigma_1}{\sigma_1 - \sigma} \right) \left\| w_\sigma (f - f_{\gamma_n}) \right\|_p + \left\| w_\sigma \left(f - f_{\gamma_n} \right) \right\|_p \\
&\quad + K(c, \sigma, \sigma_1) \gamma_n^2 \left(\left\| w_\sigma f_{\gamma_n} \right\|_p + \left\| w_\sigma f''_{\gamma_n} \right\|_p \right) \\
&\leq K_1(c, \sigma, \sigma_1) \gamma_n^2 \left(\left\| w_\sigma (f - f_{\gamma_n}) \right\|_p + \gamma_n^2 \left\| w_\sigma (f - f_{\gamma_n}) \right\|_p \right. \\
&\quad \left. + \gamma_n^2 \left\| w_\sigma f'' \right\|_p + \gamma_n^2 \left\| w_\sigma f_{\gamma_n} \right\|_p \right) \\
&\leq K_2(c, \sigma, \sigma_1) \gamma_n^2 \left(\omega_2(f; \gamma_n)_{\sigma,p} + \gamma_n^2 \left\| w_\sigma f \right\|_p \right).
\end{aligned}
$$

In the same way, using (9) and (14) we get the inequality for $\left\| w_{\sigma_1} \left(\tilde{F}_n f - f \right) \right\|_p$, and the proof is completed. □

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