VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-010-0010-1
Math. Slovaca **60** (2010), No. 2, 257–264

GENERALIZED DIFFERENCE SEQUENCE SPACE DEFINED BY $|\bar{N}, p_k|$ SUMMABILITY AND AN ORLICZ FUNCTION IN SEMINORMED SPACE

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(*Communicated by Michal Zajac*)

ABSTRACT. The object of this paper is to introduce a new difference sequence space which arise from the notions of $|\bar{N}, p_k|$ summability and an Orlicz function in seminormed complex linear space. Various algebraic and topological properties and certain inclusion relations involving this space have been discussed. This study generalizes results: [ALTIN, Y.—ET, M.—TRIPATHY, B. C.: *The sequence space* $|\bar{N}_p|(M, r, q, s)$ *on seminormed spaces*, Appl. Math. Comput. **154** (2004), 423–430], [BHARDWAJ, V. K.—SINGH, N.: *Some sequence spaces defined by* $|\bar{N}, p_n|$ *summability*, Demonstratio Math. **32** (1999), 539–546] and [BHARDWAJ, V. K.—SINGH, N.: *Some sequence spaces defined by* $|\bar{N}, p_n|$ *summability and an Orlicz function*, Indian J. Pure Appl. Math. **31** (2000), 319–325].

1. Introduction

Let (s_k) denotes the sequence of partial sums of the infinite series $\sum_{k=0}^{\infty} a_k$. Denote by $(p_k)_{k\geq 0}$ a sequence of positive real numbers and write $P_k = \sum_{v=0}^{k} p_v$.

The series $\sum_{k=0}^{\infty} a_k$ (or the sequence (s_k)) is said to be summable $(\bar{N}, p_k)$ to the sum l (finite), if

$$t_k = \frac{1}{P_k}\sum_{v=0}^{k} p_v s_v \to l \qquad \text{as} \quad k \to \infty,$$

2000 Mathematics Subject Classification: Primary 40C05, 46A45.
Keywords: difference sequence space, Orlicz function, summability, topological linear space.

and is said to be absolutely summable $(\bar{N}, p_k)$, or summable $|\bar{N}, p_k|$, if the sequence $(t_k) \in \mathrm{BV}$, that is $\sum_k |t_k - t_{k-1}| < \infty$. Let $|\bar{N}_p|$ and $\bar{N}_p$, denote, respectively, the set of all sequences which are summable $|\bar{N}, p_k|$ and $(\bar{N}, p_k)$.

Given a sequence $a = (a_k)$, write for $k \geq 1$, $\phi_k(a) = t_k - t_{k-1}$. By an application of Abel's transformation we have

$$\phi_k(a) = t_k - t_{k-1} = \frac{p_k}{P_k P_{k-1}} \sum_{n=1}^{k} P_{n-1} a_n \qquad (k \geq 1).$$

Note that for any sequences a, b and scalar λ, we have

$$\phi_k(a+b) = \phi_k(a) + \phi_k(b) \quad \text{and} \quad \phi_k(\lambda a) = \lambda \phi_k(a)\,.$$

Lindenstrauss and Tzafriri [9] used the idea of an Orlicz function M to construct the sequence space l_M of all sequences of scalars (x_k) such that $\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty$ for some $\rho > 0$. The space l_M equipped with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

is a BK space ([6, p. 300]) usually called an Orlicz sequence space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x) = x^p$, $1 \leq p < \infty$. We recall [6], [9] that an Orlicz function M is a function from $[0, \infty)$ to $[0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ for all $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Note that an Orlicz function is always unbounded.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$. It is easy to see ([8]) that always $K > 2$. A simple example of an Orlicz function which satisfies the Δ_2-condition for all values of u is given by $M(u) = a|u|^\alpha$ $(\alpha > 1)$, since $M(2u) = a2^\alpha|u|^\alpha = 2^\alpha M(u)$. The Orlicz function $M(u) = \mathrm{e}^{|u|} - |u| - 1$ does not satisfy the Δ_2-condition.

The Δ_2-condition is equivalent to the inequality $M(lu) \leq K(l)M(u)$ which holds for all values of u, where l can be any number greater than unity.

It is easy to see that $M_1 + M_2$ is an Orlicz function when M_1 and M_2 are Orlicz functions, and that the function M^v (v is a positive integer), the composition of an Orlicz function M with itself v times, is also an Orlicz function. If an Orlicz function M satisfies the Δ_2-condition, then so does the composite Orlicz function M^v.

By w we shall denote the space of all scalar sequences. l_∞, c and c_0 denote the spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$. The notion of difference sequence spaces was introduced by Kizmaz [7]. It was generalized by Et and Colak [4] as follows:

Let m be a non-negative integer. Then

$$X(\Delta^m) = \{x = (x_k) : \ (\Delta^m x_k) \in X\},$$

for $X = l_\infty, c, c_0$; where $\Delta^0 x = (x_k)$ and $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ for all $k \in \mathbb{N}$. The sequence spaces $X(\Delta^m)$ are BK spaces normed by $\|x\|_\Delta = \sum\limits_{i=1}^{m} |x_i| + \|\Delta^m x_k\|_\infty$, $X \in \{l_\infty, c, c_0\}$. Et and Nuray [5] defined a more general space $\Delta^m(X) = \big\{x = (x_k) : \ (\Delta^m x_k) \in X\big\}$, where $m \in \mathbb{N}$ and X is any sequence space.

Throughout the paper X denotes a seminormed complex linear space with seminorm q, M is an Orlicz function, $s \geq 0$ is a real number and $r = (r_k)$ is a bounded sequence of strictly positive real numbers. The symbol $w(X)$ denotes the space of all X-valued sequences.

We now introduce the following generalized difference X-valued sequence space using Orlicz function M.

$$|\bar{N_p}|(\Delta^m, M, r, q, s) = \left\{ a \in w(X) : \ \sum_{k=1}^{\infty} k^{-s} \left[M\left(q\left(\frac{\Delta^m \phi_k(a)}{\rho} \right) \right) \right]^{r_k} < \infty \right. \\ \left. \text{for some } \ \rho > 0 \right\},$$

where $\Delta^m \phi_k(a) = \Delta^{m-1}\phi_k(a) - \Delta^{m-1}\phi_{k+1}(a)$.

Some well-known spaces are obtained by specializing X, q, M, m, p, r and s.

(i) If $X = \mathbb{C}$, $q(x) = |x|$, $m = 0$, $M(x) = x$ and $s = 0$, then $|\bar{N_p}|(\Delta^m, M, r, q, s) = |\bar{N_p}|(r)$ (Bhardwaj and Singh [2]).

(ii) If $X = \mathbb{C}$, $q(x) = |x|$, $m = 0$ and $s = 0$, then $|\bar{N_p}|(\Delta^m, M, r, q, s) = |\bar{N_p}|(M, r)$ (Bhardwaj and Singh [3]).

(iii) If $m = 0$, then $|\bar{N_p}|(\Delta^m, M, r, q, s) = |\bar{N_p}|(M, r, q, s)$ (Altin et al. [1]).

(iv) If $X = \mathbb{C}$, $q(x) = |x|$, $m = 0$, $M(x) = x$, $s = 0$ and $p_k = 1$ for all k, then $|\bar{N_p}|(\Delta^m, M, r, q, s) = |C_1|(r)$ (Nanda and Mohanty [11]).

(v) If $X = \mathbb{C}$, $q(x) = |x|$, $m = 0$, $M(x) = x$, $s = 0$ and $r_k = 1$ for all k, then $|\bar{N_p}|(\Delta^m, M, r, q, s) = |\bar{N_p}|$.

(vi) If $X = \mathbb{C}$, $q(x) = |x|$, $m = 0$, $M(x) = x$, $s = 0$ and $p_k = r_k = 1$ for all k, then $|\bar{N_p}|(\Delta^m, M, r, q, s) = |C_1|$.

We denote $|\bar{N_p}|(\Delta^m, M, r, q, s)$ by $|\bar{N_p}|(\Delta^m, r, q, s)$ when $M(x) = x$ and by $|\bar{N_p}|(\Delta^m, M, r, q)$ when $s = 0$.

In this paper, we propose to study the linear topological structure of the sequence space $|\bar{N_p}|(\Delta^m, M, r, q, s)$. Certain inclusion relations between these spaces have been discussed. The composite space $|\bar{N}_p|(\Delta^m, M^v, r, q, s)$ using composite Orlicz function M^v has also been studied.

The following inequalities (see, e.g., [10, p. 190]) are needed throughout the paper.

Let $r = (r_k)$ be a bounded sequence of strictly positive real numbers. If $H = \sup_k r_k$, then for any complex a_k and b_k,

$$|a_k + b_k|^{r_k} \le C(|a_k|^{r_k} + |b_k|^{r_k}), \tag{1}$$

where $C = \max(1, 2^{H-1})$. Also for any complex λ,

$$|\lambda|^{r_k} \le \max(1, |\lambda|^H). \tag{2}$$

2. Linear topological structure of $|\bar{N}_p|(\Delta^m, M, r, q, s)$ space and inclusion theorems

In this section we examine various algebraic and topological properties of the space $|\bar{N}_p|(\Delta^m, M, r, q, s)$ and investigate some inclusion relations.

Theorem 2.1. *For any Orlicz function M, $|\bar{N}_p|(\Delta^m, M, r, q, s)$ is a linear space over the complex field $\mathbb{C}$.*

The proof is a routine verification by using standard techniques and hence is omitted.

Theorem 2.2. *For any Orlicz function M, $|\bar{N}_p|(\Delta^m, M, r, q, s)$ is a topological linear space, paranormed by*

$$g_\Delta(a) = \inf\left\{\rho^{r_n/G} : \left(\sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{\Delta^m\phi_k(a)}{\rho}\right)\right)\right]^{r_k}\right)^{\frac{1}{G}} \le 1,\ \ n = 1, 2, \dots\right\}$$

where $G = \max\Big(1, \sup_k r_k\Big)$.

The proof uses ideas similar to those used (e.g.) in [1, p. 427] and the fact that every paranormed space is a topological linear space [12, p. 37].

Remark 2.3. g_Δ need not be total, e.g., if $p_k = 1$ for all k and $a = (a_k)$ is any non-zero constant sequence then $\phi_k(a)$ is constant for all k and hence $g_\Delta(a)$ is zero for $m \ge 1$.

Theorem 2.4. *Let M, M_1, M_2 be Orlicz functions, then*

(i) *If there is a positive constant β such that $M(t) \le \beta t$ for all $t \ge 0$, then* $|\bar{N}_p|(\Delta^m, M_1, r, q, s) \subseteq |\bar{N}_p|(\Delta^m, M \circ M_1, r, q, s)$,

(ii) $|\bar{N_p}|(\Delta^m, M_1, r, q, s) \bigcap |\bar{N_p}|(\Delta^m, M_2, r, q, s) \subseteq |\bar{N_p}|(\Delta^m, M_1 + M_2, r, q, s)$.

Proof.

(i) Let $a \in |\bar{N_p}|(\Delta^m, M_1, r, q, s)$ so that $\sum\limits_{k=1}^{\infty} k^{-s}\left[M_1\left(q\left(\frac{\Delta^m \phi_k(a)}{\rho}\right)\right)\right]^{r_k} < \infty$ for some $\rho > 0$. Since $M(t) \leq \beta t$ for all $t \geq 0$, we have by inequality (2)

$$\sum_{k=1}^{\infty} k^{-s}[M(u_k)]^{r_k} \leq \max(1, \beta^H) \sum_{k=1}^{\infty} k^{-s}[u_k]^{r_k}$$

where $u_k = M_1\left(q\left(\frac{\Delta^m \phi_k(a)}{\rho}\right)\right)$ and hence $a \in |\bar{N_p}|(\Delta^m, M \circ M_1, r, q, s)$.

(ii) The proof is immediate using (1). □

Theorem 2.5. *For an Orlicz function M, if $\lim\limits_{u\to\infty} \frac{M(u/\rho)}{(u/\rho)} > 0$ for some $\rho > 0$, then $|\bar{N_p}|(\Delta^m, M, r, q, s) \subseteq |\bar{N_p}|(\Delta^m, r, q, s)$.*

Proof. If $\lim\limits_{u\to\infty} \frac{M(u/\rho)}{(u/\rho)} > 0$ then there exists a number $\alpha > 0$ such that $M(u/\rho) \geq \alpha(u/\rho)$ for all $u > 0$ and some $\rho > 0$. Let $a \in |\bar{N_p}|(\Delta^m, M, r, q, s)$ so that $\sum\limits_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{\Delta^m \phi_k(a)}{\rho}\right)\right)\right]^{r_k} < \infty$ for some $\rho > 0$.

$$\sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{\Delta^m \phi_k(a)}{\rho}\right)\right)\right]^{r_k} \geq \max\left(1, \left(\frac{\alpha}{\rho}\right)^H\right) \sum_{k=1}^{\infty} k^{-s}[q(\Delta^m \phi_k(a))]^{r_k}.$$

Hence $a \in |\bar{N_p}|(\Delta^m, r, q, s)$. □

Theorem 2.6. *Let M be an Orlicz function which satisfies Δ_2-condition, q, q_1, q_2 be seminorms and s, s_1, s_2 be non-negative real numbers. Then*

(i) $|\bar{N_p}|(\Delta^m, M, r, q_1, s) \cap |\bar{N_p}|(\Delta^m, M, r, q_2, s) \subseteq |\bar{N_p}|(\Delta^m, M, r, q_1 + q_2, s)$.

(ii) *If there exists a constant $L > 1$ such that $q_2(x) \leq Lq_1(x)$ for all $x \in X$, then $|\bar{N_p}|(\Delta^m, M, r, q_1, s) \subseteq |\bar{N_p}|(\Delta^m, M, r, q_2, s)$.*

(iii) *If $s_1 \leq s_2$, then $|\bar{N_p}|(\Delta^m, M, r, q, s_1) \subseteq |\bar{N_p}|(\Delta^m, M, r, q, s_2)$.*

Proof. The proof of (i) is straightforward using (1).

(ii) Let $a \in |\bar{N}_p|(\Delta^m, M, r, q_1, s)$. Then as M satisfies Δ_2-condition,

$$\begin{aligned}
&\sum_{k=1}^{\infty} k^{-s}\left[M\left(q_2\left(\frac{\Delta^m \phi_k(a)}{\rho}\right)\right)\right]^{r_k}\\
&\leq \sum_{k=1}^{\infty} k^{-s}\left[M\left(Lq_1\left(\frac{\Delta^m \phi_k(a)}{\rho}\right)\right)\right]^{r_k}\\
&\leq \sum_{k=1}^{\infty} k^{-s}\left[K(L)M\left(q_1\left(\frac{\Delta^m \phi_k(a)}{\rho}\right)\right)\right]^{r_k}\\
&\leq \max\left(1,(K(L))^H\right)\sum_{k=1}^{\infty} k^{-s}\left[M\left(q_1\left(\frac{\Delta^m \phi_k(a)}{\rho}\right)\right)\right]^{r_k} < \infty.
\end{aligned}$$

Hence $a \in |\bar{N}_p|(\Delta^m, M, r, q_2, s)$.

The proof of (iii) is trivial. □

Theorem 2.7. *Let $m \geq 1$, then $|\bar{N}_p|(\Delta^{m-1}, M, q, s) \subseteq |\bar{N}_p|(\Delta^m, M, q, s)$.*

Proof. Let $a \in |\bar{N}_p|(\Delta^{m-1}, M, q, s)$ then $\sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{\Delta^{m-1}\phi_k(a)}{\rho}\right)\right)\right] < \infty$ for some $\rho > 0$. Let $\rho_1 = 2\rho$ then as q is seminorm and M is non-decreasing and convex, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{\Delta^m \phi_k(a)}{\rho_1}\right)\right)\right]\\
&\leq \sum_{k=1}^{\infty} k^{-s}\left[\frac{1}{2}M\left(q\left(\frac{\Delta^{m-1} \phi_k(a)}{\rho}\right)\right)+\frac{1}{2}M\left(q\left(\frac{\Delta^{m-1} \phi_{k+1}(a)}{\rho}\right)\right)\right]\\
&< \sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{\Delta^{m-1} \phi_k(a)}{\rho}\right)\right)\right]+\sum_{k=1}^{\infty} k^{-s}\left[M\left(q\left(\frac{\Delta^{m-1} \phi_{k+1}(a)}{\rho}\right)\right)\right]
\end{aligned}$$

whence $a \in |\bar{N}_p|(\Delta^m, M, q, s)$.

In general $|\bar{N}_p|(\Delta^i, M, q, s) \subseteq |\bar{N}_p|(\Delta^m, M, q, s)$ for all $i = 1, 2, \dots, m-1$ and the inclusion is strict. To show that the inclusion is strict, consider the following example. □

Example 2.8. Let $X = \mathbb{C}$, $q(x) = |x|$, $M(x) = x$, $s = 0$ and $p_k = 1$ for all k. Let $a = (a_k)$ be defined by $a_k = 3k^2 - 3k + 1$, then $a \notin |\bar{N}_p|(\Delta^2, M, q, s)$ but $a \in |\bar{N}_p|(\Delta^3, M, q, s)$.

Theorem 2.9. *If $t = (t_k)$ and $r = (r_k)$ are bounded sequences of positive real numbers with $0 < t_k \leq r_k < \infty$ for each k, then for any Orlicz function M, $|\bar{N}_p|(\Delta^m, M, t, q) \subseteq |\bar{N}_p|(\Delta^m, M, r, q)$.*

Proof. Let $a \in |\bar{N}_p|(\Delta^m, M, t, q)$. Then there exists some $\rho > 0$ such that $\sum_{k=1}^{\infty} \left[M\left(q\left(\frac{\Delta^m \phi_k(a)}{\rho}\right)\right)\right]^{t_k} < \infty$. This implies that $M\left(q\left(\frac{\Delta^m \phi_i(a)}{\rho}\right)\right) \leq 1$ for sufficiently large values of i, say $i \geq k_0$ for some fixed $k_0 \in \mathbb{N}$. Since M is non-decreasing, we get

$$\sum_{k \geq k_0}^{\infty} \left[M\left(q\left(\frac{\Delta^m \phi_k(a)}{\rho}\right)\right)\right]^{r_k} \leq \sum_{k \geq k_0}^{\infty} \left[M\left(q\left(\frac{\Delta^m \phi_k(a)}{\rho}\right)\right)\right]^{t_k} < \infty.$$

Hence $a \in |\bar{N}_p|(\Delta^m, M, r, q)$. □

3. Composite space $|\bar{N}_p|(\Delta^m, M^v, r, q, s)$ using composite Orlicz function M^v

Taking Orlicz function M^v instead of M in the space $|\bar{N}_p|(\Delta^m, M, r, q, s)$, we can define the composite space $|\bar{N}_p|(\Delta^m, M^v, r, q, s)$ as follows:

Definition 3.1. For a fixed natural number v, we define

$$|\bar{N}_p|(\Delta^m, M^v, r, q, s) = \left\{a \in w(X) : \sum_{k=1}^{\infty} k^{-s} \left[M^v\left(q\left(\frac{\Delta^m \phi_k(a)}{\rho}\right)\right)\right]^{r_k} < \infty \right.$$
$$\left. \text{for some } \rho > 0\right\}.$$

Theorem 3.2. *For any Orlicz function M and $v \in \mathbb{N}$,*

(i) $|\bar{N}_p|(\Delta^m, M^v, r, q, s) \subseteq |\bar{N}_p|(\Delta^m, r, q, s)$ *if there exists a constant $\alpha \geq 1$ such that $M(t) \geq \alpha t$ for all $t \geq 0$.*

(ii) *Suppose there exists a constant β, $0 < \beta \leq 1$, such that $M(t) \leq \beta t$ for all $t \geq 0$ and let $n, v \in \mathbb{N}$ be such that $n < v$, then $|\bar{N}_p|(\Delta^m, r, q, s) \subseteq |\bar{N}_p|(\Delta^m, M^n, r, q, s) \subseteq |\bar{N}_p|(\Delta^m, M^v, r, q, s)$.*

The proof follows from Theorem 2.5 and Theorem 2.4(i) and hence is omitted.

Example 3.3. $M_1(t) = e^t - 1 \geq t$ and $M_2(t) = \frac{t^2}{1+t} \leq t$ for all $t \geq 0$ satisfy the conditions given in Theorem 3.2(i),(ii) respectively.

Acknowledgement. We wish to thank the referee for his/her several valuable suggestions that have improved the presentation of the paper.

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Received 9. 2. 2008
Accepted 27. 3. 2009

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