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ON LATTICE EMBEDDINGS OF A LATTICE INTO ITS INTERVALS

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ABSTRACT. In connection with his investigation of convexities generated by fractal lattices, Czédli formulated a conjecture concerning lattice embeddings of a lattice into its intervals. In the present note we modify the conditions from Czédli's conjecture; we consider only intervals having more than two elements. Further, we prove the validity of Czédli's conjecture.

1. Introduction

In connection with his investigation of convexities generated by fractal lattices, Czédli [1] expressed a conjecture concerning the existence of a bounded lattice L such that certain conditions dealing with lattice embeddings of L into intervals of L are satisfied.

Let $\mathcal{C}$ be the class of all lattices L having the least element 0_L and the greatest element 1_L with $L \neq \{0_L, 1_L\}$ such that whenever $[x, y]$ is an interval of L with $[x, y] \neq L$ and $\operatorname{card}[x, y] \geqq 2$, then

(i) there exists a lattice embedding $\varphi_1 \colon L \to [x, y]$ with $\varphi_1(0_L) = x$ and $\varphi_1(1_L) = y$;

(ii) there exists a lattice embedding $\varphi_2 \colon [x, y] \to L$ with $\varphi_2(x) = 0_L$ and $\varphi_2(y) = 1_L$;

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(iii) there exists and interval $[x_1, y_1]$ in L with $\operatorname{card}[x_1, y_1] \geqq 2$ such that this interval is not isomorphic to L.

Czédli conjectured that the class $\mathcal{C}$ is nonempty. We prove that there exists an infinite set of mutually non-isomorphic lattices which belong to the class $\mathcal{C}$.

Further, we will deal with a class $\mathcal{C}_1$ of lattices which is defined in the same way as the class $\mathcal{C}$ with the distinction that instead of the relation $\operatorname{card}[x, y] \geqq 2$ we consider now the relation $\operatorname{card}[x, y] > 2$; next, we assume that the condition (iii) is replaced by the condition

(iv) if $[x, y]$ is an interval in L with $x \neq y$ then this interval fails to be isomorphic to L.

We denote by $\mathcal{C}_{11}$ the class of all lattices $L \in \mathcal{C}_1$ such that L is a chain.

We prove that there exists a proper class of mutually non-isomorphic lattices belonging to $\mathcal{C}_{11}$. Hence, in particular, the class $\mathcal{C}_1$ is nonempty.

Dealing with $\mathcal{C}_{11}$ we use lexicographic products of linearly ordered sets.

From the fact that $\mathcal{C} \neq \emptyset$ we cannot immediately conclude that the class $\mathcal{C}_1$ is nonempty (since the condition (iv) is essentially stronger than the condition (ii)).

The notion of convexity of lattices is due to Fried (cf. [2]); convexities of lattices have been dealt with in [3] and [4].

2. The class $\mathcal{C}_{11}$

Let Q be the set of all rationals with the natural linear order and let $Q_0 = [-1, 1]$ be the interval of Q. Each subset of a linearly ordered set is linearly ordered by the induced order.

Let α be an infinite cardinal and let β be the first ordinal with $\operatorname{card} \beta = \alpha$. Then we have $\operatorname{cof} \beta = \beta$, where $\operatorname{cof} \beta$ is the cofinality of β. The collection of all ordinals having this property is a proper class. For each $i \in \beta$ let $Q_i = Q_0$. We denote by H the lexicographic product of the system $(Q_i)_{i \in \beta}$. The elements of H are written in the form $x = (x_i)_{i \in \beta}$ with $x_i \in Q_i$ for each $i \in \beta$.

Let L_β be the set of all $x \in H$ such that there exists an index $i(x) \in \beta$ having the property that $x_i = x_{i(x)}$ for each $i \in \beta$ with $i > i(x)$. Then we have

(1) $\operatorname{card} L_\beta = \operatorname{card} \beta$.

We say that L_β is a restricted lexicographic product of the system $(Q_i)_{i \in I}$.

Let $i_1 \in \beta$ and $t \in L_\beta$. We denote

$$L_\beta(i_1, t) = \{x \in L_\beta :\ x_i = t_i \text{ for each } i \in \beta \text{ with } i \leqq i_1\}.$$

From the relation $\operatorname{cof} \beta = \beta$ we obtain:

Lemma 2.1. *$L_\beta(i_1, t) \simeq L_\beta$ for each $i_1 \in \beta$ and each $t \in L_\beta$.*

Let u and v be elements of L_β such that

$$u_{i_1} < t_{i_1} < v_{i_1},$$

$$u_i = t_i = v_i \qquad \text{for each} \quad i < i_1.$$

Then we have:

Lemma 2.2. *$u < x < v$ for each $x \in L_\beta(i_1, t)$.*

We consider two distinct elements p and q which do not belong to L_β. We put

$$L_\beta^0 = \{p, q\} \cup L_\beta.$$

We define a binary relation $<_0$ on the set L_β^0 as follows:

(i) we put $p <_0 q$ and $p <_0 x$, $x <_0 q$ for each $x \in L_\beta$;

(ii) for $x, y \in L_\beta$, the relation $x <_0 y$ is equivalent with $x < y$.

Hence L_β^0 turns out to be a chain with the least element p and the greatest element q.

We denote by p_0 the element of L_β such that $(p_0)_i = -1$ for each $i \in I$. Similarly, q_0 is defined to be the element of L_β with $(q_0)_i = 1$ for each $i \in I$. Thus p_0 is the least element of L_β and q_0 is the greatest element of L_β. From this and from the definition of $<_0$ we obtain:

Lemma 2.3. *The intervals $[p, p_0]$ and $[q_0, q]$ of the linearly ordered set L_β^0 are prime intervals.*

Lemma 2.4. *Let $x, y \in L_\beta$, $x < y$. Then there is $z \in L_\beta$ with $x < z < y$.*

Proof. There exists $i_0 \in \beta$ such that $x_{i_0} < y_{i_0}$ and $x_i = y_i$ for each $i \in \beta$ with $i < i_0$. Further, there is $t \in Q_{i_0}$ such that $x_{i_0} < t < y_{i_0}$. Also, there exists $z \in L_\beta$ with $z_{i_0} = t$, $z_i = x_i$ for $i < i_0$. Then $x < z < y$. □

We verified that if $[x, y]$ is an interval in L_β, then it cannot be a prime interval. As a consequence we obtain:

Lemma 2.5. *Let $[x, y]$ be an interval in L_β^0, $[x, y] \neq L_\beta^0$. Then $[x, y]$ fails to be isomorphic to L_β^0.*

Lemma 2.6. *Let $[x, y]$ be an interval in L_β^0 with $\operatorname{card}[x, y] > 2$. Then there exists a lattice embedding $\varphi_1 \colon L_\beta^0 \to [x, y]$ such that $\varphi_1(p) = x$ and $\varphi_1(q) = y$.*

P r o o f. From the relation $\operatorname{card}[x, y] > 2$ we conclude that there exist $z, z' \in [x, y]$ with $z, z' \in L_\beta$, $z < z'$, $z, z' \notin \{p_0, q_0\}$.

There is $i_0 \in \beta$ such that $z_{i_0} < z'_{i_0}$ and $z_i = z'_i$ for $i < i_0$. Further, there is $c \in Q_{i_0}$ with $z_{i_0} < c < z'_{i_0}$. Also, there exists $t \in L_\beta$ such that $t_{i_0} = c$ and $t_i = z_i$ for $i < i_0$. Then $z < t < z'$, whence $t \in [x, y]$.

Moreover, according to Lemma 2.2, we have $L_\beta(i_0, t) \subseteq [x, y]$.

According to Lemma 2.1 there exists an isomorphism φ of L_β onto $L_\beta(i_0, t)$. Let us consider a mapping $\varphi_1 \colon L^0_\beta \to [x, y]$ which is defined as follows: we put $\varphi_1(p) = x$, $\varphi_1(q) = y$ and $\varphi_1(s) = \varphi(s)$ for each $s \in L_\beta$. Then applying Lemma 2.1 again we conclude that φ_1 has the desired properties. □

Lemma 2.7. *Let $[x, y]$ be an interval in L^0_β with $\operatorname{card}[x, y] > 2$. Then there exists a lattice embedding $\varphi_2 \colon [x, y] \to L^0_\beta$ such that $\varphi_2(x) = p$ and $\varphi_2(y) = q$.*

P r o o f. As $[x, y] \subseteq L^0_s$, it suffices to put $\varphi_2(x) = p$, $\varphi_2(y) = q$ and $\varphi_2(z) = z$ for all $z \in [x, y]$, $z \notin \{x, y\}$. □

In view of Lemmas 2.5, 2.6 and 2.7 we obtain:

Lemma 2.8. *The linearly ordered set L^0_β belongs to the class $\mathcal{C}_{11}$.*

If α' is an infinite cardinal with $\alpha' \neq \alpha$ and if β' is defined analogously as β above, $L^0_{\beta'}$ is not isomorphic to L^0_β. Thus we have:

Theorem 2.9. *There exists a proper class of mutually non-isomorphic lattices belonging to $\mathcal{C}_{11}$.*

3. On Czédli's conjecture

We recall some definitions (cf. [1], [2]).

Let L be a bounded lattice. If for each $a_1 < b_1 \in L$ and each $a_2 < b_2 \in L$ there is a lattice embedding $\psi \colon [a_1, b_1] \to [a_2, b_2]$ with $\psi(a_1) = a_2$ and $\psi(b_1) = b_2$ then we say that L is a quasi-fractal lattice or shortly a quasi-fractal. If L is isomorphic to each of its nontrivial intervals then L will be called a fractal lattice or a fractal.

Czédli [1] expressed the conjecture that there exists a quasi-fractal lattice which fails to be fractal.

Let $\mathcal{C}$ be as in Section 1. It is easy to verify that for a lattice L the following conditions are equivalent:

(i) L belongs to the class $\mathcal{C}$;

(ii) L is a quasi-fractal and it fails to be a fractal.

We denote by $\mathbb{R}$ the system of all real numbers with the usual linear order. Each nonempty subset of $\mathbb{R}$ is considered to be linearly ordered under the order induced from $\mathbb{R}$.

Let $a, b \in \mathbb{R}$, $a < b$. Further, let $x_i \in [a, b]$ $(i = 0, 1, 2, \ldots, n+1)$ where $a = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = b$. We put

$$L_0(n) = [a, b] \backslash \{x_0, x_1, \ldots, x_{n+1}\}, \qquad L(n) = [a, b] \backslash \{x_1, x_2, \ldots, x_n\}.$$

(For any sets A and B we denote by $A \backslash B$ the set of all elements of A which do not belong to B.)

If a', b' belong to $L(n)$, then we have to distinguish between the interval $[a', b']$ in $\mathbb{R}$ and the interval in $L(n)$ having the endpoints a' and b'; the latter will be denoted by $[a', b']_{L(n)}$. Evidently, $[a', b']_{L(n)} = [a', b'] \cap L(n)$ and $[a', b']_{L(n)} = [a', b']$ iff $x_i \notin [a', b']$ for each $i \in \{1, \ldots, n\}$.

The validity of the following lemma is obvious.

Lemma 3.1. *Let $a_1, b_1 \in L(n)$, $a_1 < b_1$ and let $[a_1, b_1]_{L(n)} = [a_1, b_1]$. Then there exists a lattice embedding φ of $L_0(n)$ into $[a_1, b_1]$.*

Lemma 3.2. *Let $x, y \in L(n)$, $x < y$. Then there exists a lattice embedding ψ_1 of $L(n)$ into the interval $[x, y]_{L(n)}$ with $\psi_1(a) = x$, $\psi_1(b) = y$.*

Proof. It is easy to see that there exist $a_1, b_1 \in L(n)$ such that $x \leqq a_1 < b \leqq y$, $[a_1, b_1]_{L(n)} = [a_1, b_1]$. Now it suffices to put $\psi_1(a) = x$, $\psi_1(b) = y$ and $\psi_1(t) = \varphi(t)$ for $t \in L_0(n)$, where φ is as in Lemma 3.1. □

Lemma 3.3. *Let $x, y \in L(n)$, $x < y$. Then there exists a lattice embedding ψ_2 of $[x, y]_{L(n)}$ into $L(n)$ such that $\psi_2(x) = a$, $\psi_2(y) = b$.*

Proof. As $[x, y]_{L(n)} \subseteq L(n)$, it suffices to put $\psi_2(x) = a$, $\psi_2(y) = b$, $\psi_2(z) = z$ for all $z \in L(n)$ with $x < z < y$. □

In view of Lemma 3.2 and Lemma 3.3 we conclude:

Lemma 3.4. *$L(n)$ is a quasi-fractal lattice.*

Again, let $[a_1, b_1]$ be as in Lemma 3.1. Hence $[a_1, b_1] = [a_1, b_1]_{L(n)}$. The lattice $[a_1, b_1]$ is complete. On the other hand, the lattice $L(n)$ fails to be complete for $n \geqq 1$. Thus $[a_1, b_1]$ is not isomorphic to $L(n)$. We obtain:

Lemma 3.5. *$L(n)$ fails to be a fractal lattice.*

If m and n are distinct positive integers, then $L(m)$ is not isomorphic to $L(n)$. Thus according to Lemma 3.4 and Lemma 3.5 we have:

THEOREM 3.6. *There exists an infinite set of mutually nonisomorphic chains such that*

(i) *each of these chains is a quasi-fractal but it fails to be a fractal;*

(ii) *each of these chains is a subset of* $\mathbb{R}$.

The following questions remain open:

a) Does there exist a proper class of mutually nonisomorphic chains belonging to the class $\mathcal{C}$?

b) Does there exist a denumerable lattice belonging to the class $\mathcal{C}$?

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