VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-010-0004-z
Math. Slovaca **60** (2010), No. 2, 179–188

ON IDEMPOTENT MODIFICATIONS OF GENERALIZED MV-ALGEBRAS

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(*Communicated by Jiří Rachůnek*)

ABSTRACT. The notion of idempotent modification of an algebra was introduced by Ježek; he proved that the idempotent modification of a group is always subdirectly irreducible. In the present note we show that the idempotent modification of a generalized MV-algebra having more than two elements is directly irreducible if and only if there exists an element in $\mathcal{A}$ which fails to be boolean. Some further results on idempotent modifications are also proved.

1. Introduction

The notion of idempotent modification $\mathcal{A}'$ of an algebra $\mathcal{A}$ was introduced in [8]. It is defined as follows.

Assume that $\mathcal{A} = (A; F)$ is an algebra with the underlying set A and with the set F of basic operations. The underlying set of $\mathcal{A}'$ is equal to A; the system F' of basic operations of $\mathcal{A}'$ consists of operations f', where $f \in F$ and

1) if f is a nulary or a unary operation, then $f' = f$;
2) if f is an n-ary operation with $n > 1$ and if $a_1, a_2, \dots, a_n \in A$, then

$$f'(a_1, a_2, \dots, a_n) = \begin{cases} a_1 & \text{if} \quad a_1 = a_2 = \dots = a_n \\ f(a_1, a_2, \dots, a_n) & \text{otherwise.} \end{cases}$$

2000 Mathematics Subject Classification: Primary 06D35; Secondary 06F20.
Keywords: generalized MV-algebra, lattice ordered group, direct irreducibility, subdirect irreducibility, boolean element.
Supported by VEGA Agency grant 1/0539/08.
This work was supported by the Slovak Research and Development Agency under the contract No. APVV-0071-06.
This work has been partially supported by the Slovak Academy of Sciences via the project Center of Excellence – Physics of Information (grant I/2/2005).

The main result of [8] says that if $\mathcal{V}_1$ is the variety of all groups, and $G \in \mathcal{V}_1$, then G' is subdirectly irreducible.

It is also remarked in [8] that it would be interesting to find other varieties having the mentioned property.

In [7] it was shown that there exist infinitely many such varieties.

Let $\mathcal{A}$ be an MV-algebra; a result concerning subdirect irreducibility of $\mathcal{A}'$ was proved in [7].

In the present note we prove that the idempotent modification $\mathcal{A}'$ of a generalized MV-algebra $\mathcal{A}$ having more than two elements is directly irreducible if and only if there exists an element in $\mathcal{A}$ which is not boolean.

We also show that if G is a lattice ordered group then G' is subdirectly irreducible. Some further results concerning idempotent modifications are also proved.

2. Preliminaries

The notion of generalized MV-algebra was introduced independently in [3], [4] and in [9] (in [3], [4] the term 'pseudo MV-algebra' was applied).

For a generalized MV-algebra $\mathcal{A}$ we denote by A its underlying set. Using the operations of $\mathcal{A}$ we can define a partial order $\leqq$ on the set A such that $(A; \leqq)$ turns out to be a distributive lattice. Therefore, without loss of generality, the lattice operations $\vee$ and $\wedge$ can be included into the set of basic operations of $\mathcal{A}$.

In this sense, a generalized MV-algebra $\mathcal{A}$ is considered as an algebra $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1, \vee, \wedge)$ of type (2,1,1,0,0,2,2) such that

1) the axioms (M1)–(M8) from [3] are satisfied;
2) for each $x, y \in A$, $x \wedge y = x$ iff $\neg x \oplus y = 1$;
3) $(A; \vee, \wedge)$ is a distributive lattice with the least element 0 and the greatest element 1.

(Cf. [3], [9].)

If the operation $\oplus$ is commutative, then $\mathcal{A}$ is an MV-algebra. (Cf. [2].)

For lattice ordered groups we apply the notation and terminology as in [5] with the distinction that the group operation is written additively; the commutativity of this operation is not assumed to be valid.

Let G be the lattice ordered group with a strong unit u and let A be the interval $[0, u]$ of G. For $x, y \in A$ we put

$$x \oplus y = (x + y) \wedge u, \quad \neg x = u - x, \quad \sim x = -x + u, \quad 1 = u.$$

Then $\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1, \vee, \wedge)$ is a generalized MV-algebra; it will be denoted by $\Gamma(G, u)$.

According to [2], for each generalized MV-algebra $\mathcal{A}$ there exists a lattice ordered group G with a strong unit u such that $\mathcal{A} = \Gamma(G, u)$.

In what follows, when speaking about a generalized MV-algebra $\mathcal{A}$ we always suppose that G and u are as above.

3. Direct products

Assume that $\mathcal{B}_1$ and $\mathcal{B}_2$ are algebras of the same type. The direct product $\mathcal{B}_1 \times \mathcal{B}_2$ is defined in the usual way.

An algebra $\mathcal{B}$ is *directly reducible* if there exist algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ such that

1) $\operatorname{card} \mathcal{B}_1 > 1$ and $\operatorname{card} \mathcal{B}_2 > 1$;

2) there exists an isomorphism $\varphi\colon \mathcal{B} \to \mathcal{B}_1 \times \mathcal{B}_2$.

In this case we say that φ determines a direct product decomposition of $\mathcal{B}$.

Direct product decompositions of generalized MV-algebras were investigated in [6].

In the present section we assume that

$$\mathcal{A} = (A; \oplus, \neg, \sim, 0, 1, \vee, \wedge)$$

is a generalized MV-algebra. We put $\ell(\mathcal{A}) = (A; \vee, \wedge)$ and we say that $\ell(\mathcal{A})$ is the *underlying lattice* of $\mathcal{A}$.

An element a of A is called *boolean* if it possesses a complement in the lattice $\ell(\mathcal{A})$.

Below we will apply the following fact (cf. [6]): If a is a boolean element and b is its complement, then $\varphi\colon A \to [0, a] \times [0, b]$ defined by $\varphi(x) = (x \wedge a, x \wedge b)$ is an isomorphism of $\mathcal{A} = (A, \oplus, \neg, \sim, 0, 1)$ onto the direct product of $([0, a], \oplus, \neg_a, \sim_a, 0, a)$ and $([0, b], \oplus, \neg_b, \sim_b, 0, b)$ where $\neg_p x = \neg x \wedge p$ and $\sim_p x = \sim x \wedge p$ (for $p \in \{a, b\}$).

We obviously have:

Lemma 3.1. *Suppose that* $\operatorname{card} A > 2$ *and that all elements of A are boolean. Then the algebra $\mathcal{A}'$ is directly reducible.*

Lemma 3.2. *Assume that the algebra $\mathcal{A}'$ is directly reducible. Then* $\operatorname{card} A > 2$ *and each element of A is boolean.*

Proof. From the definition of direct reducibility we obtain that $\operatorname{card} A > 2$. By way of contradiction, suppose there exists an element x in A such that x fails to be boolean.

In view of the assumption, there exist algebras

$$\mathcal{B}_i = (B_i; \oplus_i, \neg_i, \sim_i, 0_i, 1_i, \vee_i, \wedge_i) \qquad (i = 1, 2)$$

of type $(2,1,1,0,0,2,2)$, and an isomorphism

$$\varphi\colon \mathcal{A}' \to \mathcal{B}_1 \times \mathcal{B}_2. \tag{1}$$

We have

$$\ell(\mathcal{A}') = (A; \vee, \wedge), \quad \ell(\mathcal{B}_i) = (B_i; \vee_i, \wedge_i) \qquad (i = 1, 2).$$

From (1) we obtain that φ determines a direct product decomposition

$$\varphi\colon \ell(\mathcal{A}') \to \ell(\mathcal{B}_1) \times \ell(\mathcal{B}_2). \tag{2}$$

Since $\ell(\mathcal{A}') = \ell(\mathcal{A})$, the mapping φ also determines a direct product decomposition of $\ell(\mathcal{A})$.

Hence, applying the same argument as in the proof of Lemma 3.1 (based on [6]) we conclude that there exists a direct product decomposition

$$\varphi\colon \mathcal{A} \to \mathcal{A}_1 \times \mathcal{A}_2, \tag{3}$$

where $\ell(\mathcal{A}_1) = \ell(\mathcal{B}_1)$ and $\ell(\mathcal{A}_2) = \ell(\mathcal{B}_2)$.

Let x be as above and $\varphi(x) = (x_1, x_2)$. If x_1 is boolean in $\mathcal{A}_1$ and x_2 is boolean in $\mathcal{A}_2$, then according to (3) we get that x is boolean in $\mathcal{A}$, which is a contradiction. Thus without loss of generality we can suppose that x_1 fails to be boolean in $\mathcal{B}_1$. This yields that $x_1 \oplus x_1 > x_1$.

We have $\operatorname{card} B_2 > 1$, hence there exists $y_2 \in B_2$ with $y_2 \neq x_2$. Put $y = \varphi^{-1}(x_1, y_2)$. Then in view of (3),

$$\varphi(x \oplus y) = (x_1 \oplus x_1, x_2 \oplus y_2).$$

Since $x \neq y$, we get $x \oplus' y = x \oplus y$, whence

$$\varphi(x \oplus' y) = \varphi(x \oplus y).$$

At the same time, we can consider the direct product decomposition (1). We get

$$\varphi(x \oplus' x) = (x_1 \oplus_1 x_1, x_2 \oplus_2 x_2).$$

Since $x \oplus' x = x$, we obtain

$$x_1 \oplus_1 x_1 = x_1. \tag{4}$$

Further, in view of (1) we have

$$\varphi(x \oplus' y) = (x_1 \oplus_1 x_1, x_2 \oplus_2 y_2).$$

Hence according to (4),

$$\varphi(x \oplus' y) = (x_1, x_2 \oplus_2 y_2).$$

This yields $x_1 = x_1 \oplus x_1$; we arrived at a contradiction. $\square$

It is clear that if $\mathcal{A}$ is a generalized MV-algebra with $\operatorname{card} A \leqq 2$, then all elements of A are boolean and the algebra $\mathcal{A}'$ is directly irreducible.

Theorem 3.3. *Let $\mathcal{A}$ be a generalized MV-algebra with* $\operatorname{card} A > 2$. *Then the following conditions are equivalent:*

(i) *There exists an element in A which is not boolean.*

(ii) *The algebra $\mathcal{A}'$ is directly irreducible.*

Proof. This is a consequence of 3.1 and 3.2. □

4. Some further results

In the present section we deal with idempotent modifications of lattice ordered groups and of some algebras which are related to MV-algebras.

For any algebra $\mathcal{C}$ we denote by $\operatorname{con}\mathcal{C}$ the system of all congruence relations on $\mathcal{C}$.

From [8, Theorem 10] we immediately obtain:

Proposition 4.1. *Let $\mathcal{G} = (G; +, \vee, \wedge)$ be a lattice ordered group. Then its idempotent modification $\mathcal{G}'$ is subdirectly irreducible.*

In the remaining part of this section we assume that $\mathcal{A}$ is an MV-algebra. We also suppose that G is a lattice ordered group with a strong unit u such that $\mathcal{A} = \Gamma(G, u)$ and that the lattice $\ell(\mathcal{A})$ is linearly ordered. Then the underlying lattice of G is linearly ordered as well.

Proposition 4.2. (Cf. [7].) *If $\mathcal{A}$ is semisimple, then the algebra $\mathcal{A}'$ is simple.*

Let us consider the algebra $\mathcal{A}_0 = (A; \oplus, \vee, \wedge)$. We will deal with its idempotent modification $\mathcal{A}_0' = (A; \oplus', \vee', \wedge')$. Since the operations $\vee$ and $\wedge$ are idempotent, $\vee'$ and $\wedge'$ coincide with $\vee$ or $\wedge$, respectively.

We show that the result analogous to that in Proposition 4.2 does not in general hold for algebra $\mathcal{A}_0'$. (Cf. Proposition 4.8 below.)

For $a \in A$ and $n \in \mathbb{N}$ we denote

$$na = a + \cdots + a \quad (n\text{-times})$$

$$n \cdot a = a \oplus \cdots \oplus a \quad (n\text{-times}).$$

By applying the induction, we obtain:

Lemma 4.3. *Let $a \in A$. Then $n \cdot a = (na) \wedge u$.*

For $a, b \in A$ we put $a \ll b$ if $na < b$ for each $n \in \mathbb{N}$. Further, let $\overline{a}$ be the set of all $a_1 \in A$ such that neither $a \ll a_1$ nor $a_1 \ll a$ is valid. In other words, a_1 belongs to $\overline{a}$ iff there are positive integers n_1 and n_2 such that $n_1 a \geqq a_1$ and $n_2 a_1 \geqq a$.

We set

$$S = \{\overline{a} : \ a \in A\}.$$

If $a, b \in A$, $a < b$ and $\overline{a} \neq \overline{b}$, then $a_1 < b_1$ for each $a_1 \in \overline{a}$ and $b_1 \in \overline{b}$. In such a case we put $\overline{a} < \overline{b}$. Thus the relation $<$ determines a linear order on the set S. Clearly, $\overline{0} = \{0\}$.

The definition of semisimplicity yields:

Lemma 4.4. *Assume that* $\operatorname{card} A \geqq 2$. *Then* $\mathcal{A}$ *is semisimple iff* $\operatorname{card} S = 2$.

Lemma 4.5. *Let* $a \in A$.

(i) $\overline{a}$ *is a convex subset of* $\ell(\mathcal{A})$*;*

(ii) $\overline{a}$ *is a subalgebra of* $\mathcal{A}'_0$.

Proof. The assertion (i) is obvious; thus $\overline{a}$ is closed with respect to the operations $\vee$ and $\wedge$.

Let $n \in \mathbb{N}$. Put $b = na \wedge u$. Hence $b \geqq a \wedge u = a$. For $n_1 \in \mathbb{N}$ with $n_1 > n$ we have $n_1 a \geqq na \geqq b$. Thus $b \in \overline{a}$.

Let $x, y \in \overline{a}$. If $x = y$, then $x \oplus' y = x$, hence $x \oplus' y \in \overline{a}$. Assume that $x \neq y$. Without loss of generality we can suppose that $x < y$. We get

$$x \oplus' y = x \oplus y = (x + y) \wedge u,$$
$$y \leqq (x + y) \wedge u \leqq 2y \wedge u.$$

Since $2y \wedge u \in \overline{a}$, in view of (i) we obtain $x \oplus' y \in \overline{a}$. □

Let $a \in A$. For $x, y \in A$ we put $x \,\rho^a\, y$ iff either

(i) $\overline{x} = \overline{y}$ and $x \geqq a$, $y \geqq a$
or
(ii) $x = y$.

Lemma 4.6. ρ^a *is a congruence relation of the algebra* $\mathcal{A}'_0$.

Proof. In view of the assertion (i) of 4.5 we conclude that ρ^a is a congruence relation with respect to the operations $\vee$ and $\wedge$. We have to verify that ρ^a is a congruence relation with respect to the operation $\oplus'$.

Let $x, y, z \in A$ and $x \,\rho^a\, y$. For verifying the validity of the relation $(x \oplus' z) \,\rho^a\, (y \oplus' z)$ it suffices to consider the case $x < y$. For $z \in \overline{x}$, the mentioned relation holds according to 4.5. Assume that z does not belong to $\overline{x}$. Hence $x \neq z \neq y$. Then

$$x \oplus' z = (x + z) \wedge u, \quad y \oplus' z = (y + z) \wedge u.$$

Since $x \geqq a$ and $y \geqq a$ we have $x \oplus' z \geqq a$ and $y \oplus' z \geqq a$. We distinguish two cases.

a) Assume that $z < x$. Then we also have $z < y$. Thus

$$x \leqq (x + z) \wedge u \leqq (x + y) \wedge u = x \oplus' y.$$

In view of 4.5 we obtain $x \oplus' z \in \overline{x}$. Further,

$$y \leqq (y+z) \wedge u \leqq (y+x) \wedge u = x \oplus' y,$$

whence $y \oplus' z \in \overline{y} = \overline{x}$ and so $(x \oplus' z)\, \rho^a\, (y \oplus' z)$.

b) Now assume that $x < z$. If $z \leqq y$, then in view of 4.5(i) we would have $z \in \overline{x}$, which is a contradiction. Thus $y < z$. We also have $x \notin \overline{z}$, $y \notin \overline{z}$. Therefore by applying similar steps as in a) we obtain

$$x \oplus' z \in \overline{z}, \quad y \oplus' z \in \overline{z}.$$

Hence $(x \oplus' z)\, \rho^a\, (y \oplus' z)$. □

Proposition 4.7. *Let $\mathcal{A}$ be a linearly ordered MV-algebra. Suppose that the corresponding set S has more than two elements. Then the algebra $\mathcal{A}'_0$ is not simple.*

Proof. This is a consequence of 4.6. □

Let Z be the additive group of all integers with the natural linear order. Put $u = 2$; then u is a strong unit of the linearly ordered group Z. Consider the MV-algebra $\mathcal{A}_1 = \Gamma(Z, u)$. Let $\mathcal{A}_{10}$ be defined analogously as $\mathcal{A}_0$ above.

Proposition 4.8. *The idempotent modification $\mathcal{A}'_{10}$ of $\mathcal{A}_{10}$ is subdirectly reducible.*

Proof. We denote by A_1 the underlying set of $\mathcal{A}_1$; hence $A_1 = \{0, 1, 2\}$. Let us deal with the partitions

$$\rho_1^0 = \{\{0\}, \{1, 2\}\}, \quad \rho_2^0 = \{\{0, 1\}, \{2\}\}$$

of the set A_1. For $i \in \{1, 2\}$, let ρ_i be the equivalence relation corresponding to ρ_i^0.

It is obvious that ρ_1 and ρ_2 are congruence relations with respect to the operations $\vee'$ and $\wedge'$. For showing that ρ_1 is a congruence with respect to $\oplus'$ we have to verify that for each $x \in A_1$, the relation $(x \oplus' 1)\, \rho_1\, (x \oplus' 2)$ is valid. Put $f_1(x) = x \oplus' 1$, $f_2(x) = x \oplus' 2$. We get

$$f_1(0) = 1, \quad f_2(0) = 2, \quad f_1(1) = 1, \quad f_2(1) = 2, \quad f_1(2) = 2,$$

as desired.

For considering the equivalence ρ_2 we denote $f_0(x) = x \oplus' 0$. We get $f_0(0) = 0$, $f_0(1) = 1$, $f_0(2) = 2$; in view of the values of f_1 we obtain that ρ_2 is a congruence relation with respect to the operation $\oplus'$.

Let $\rho_{\min}$ be the least element of $\operatorname{con} \mathcal{A}'_{10}$. Since $\rho_1 \neq \rho_{\min} \neq \rho_2$ and $\rho_1 \wedge \rho_2 = \rho_{\min}$ we conclude that $\mathcal{A}'_{10}$ is subdirectly reducible. □

Again, let $\mathcal{A}$ be as above. We denote by A^0 the set $\{x \in A : x \neq u\}$; this set is partially ordered by the relation of partial order induced from $\ell(\mathcal{A})$.

Proposition 4.9. *Let $\mathcal{A}$ be a linearly ordered MV-algebra. Assume that A^0 does not have the greatest element. Then the algebra $\mathcal{A}_0'$ is subdirectly reducible.*

P r o o f. For each $a \in A^0$ with $a \neq u$ we consider the congruence relation ρ^a on $\mathcal{A}_0'$ defined as above; then $\rho^a \neq \rho_{\min}$. Put

$$\bigwedge_{a\in A^0} \rho^a = \alpha.$$

It suffices to verify that $\alpha = \rho_{\min}$.

By way of contradiction, assume that $\alpha \neq \rho_{\min}$. Hence there exist $x, y \in A$ such that $x < y$ and $x \,\alpha\, y$. If $y = u$, then there is $y' \in A$ with $x < y' < y$. We get $x \,\alpha\, y'$. Thus without loss of generality we can suppose that both x and y belong to A^0.

There exists $a_1 \in A^0$ with $x < a_1$ and $y < a_1$. In view of the definition of ρ^{a_1}, the relation $x \,\rho^{a_1}\, y$ fails to be valid. Thus the relation $x \,\alpha\, y$ does not hold; we arrived at a contradiction. □

Lemma 4.10. *Let $x, y \in A$, $0 < x < y$. Assume that ρ is a congruence relation of $\mathcal{A}_0'$ such that $x \,\rho\, y$. Then for each $n \in \mathbb{N}$, $n \cdot y \,\rho\, y$.*

P r o o f. If $2 \cdot y = y$, then clearly $n \cdot y = y$ for each $n \in \mathbb{N}$. Hence it suffices to consider the case $2 \cdot y \neq y$. In this case we have $y < n \cdot y$ for each $n \in \mathbb{N}$, $n > 1$.

In view of $x \,\rho\, y$ we get $(x \oplus' x) \,\rho\, (x \oplus' y)$, thus $x \,\rho\, (x \oplus y)$. This yields

$$(x \oplus' y) \,\rho\, ((x \oplus y) \oplus' y).$$

We put

$$(x \oplus y) \oplus' y = t.$$

By calculating t we must distinguish the cases $x \oplus y = y$ and $x \oplus y \neq y$.

First assume that $x \oplus y = y$ is valid. We get

$$(x + y) \wedge u = y.$$

Since $\ell(\mathcal{A})$ is linearly ordered, we obtain that $\ell(G)$ is linearly ordered as well and thus the elements $x + y$ and u are comparable in G.

If $x + y \geqq u$, then $(x + y) \wedge u = u$, whence $y = u$ and so $2 \cdot y = y$, which is a contradiction.

If $x + y < u$, then $(x + y) \wedge u = x + y$, hence $x + y = y$ and so $x = 0$; again, we arrived at a contradiction.

Therefore we must have $x \oplus y \neq y$. Thus

$$t = (x \oplus y) \oplus y = x \oplus (y \oplus y) = x \oplus 2 \cdot y \geqq 2 \cdot y > y.$$

We obtain $y \,\rho\, (2 \cdot y)$.

Assume that n is a positive integer, $n \geqq 2$, and that $y\,\rho\,(n \cdot y)$. Then $(y \oplus' y)\,\rho\,(y \oplus' (n \cdot y))$. Since $y \neq n \cdot y$, we get

$$y \oplus' (n \cdot y) = y \oplus (n \cdot y) = (n+1) \cdot y,$$

thus $y\,\rho\,(n+1) \cdot y$. Therefore $(n \cdot y)\,\rho\,y$ for each $n \in \mathbb{N}$. □

Now let us suppose that the set A^0 has a greatest element which will be denoted by a_0. If p and q are elements of A such that the interval $[p, q]$ is a two-element set, then we express this fact by writing $p \prec q$.

Thus $a_0 \prec u$. From this we immediately obtain that for each $x \in A^0$ there exists a uniquely determined element $x' \in A$ such that $x \prec x'$.

Lemma 4.11. *Let* $\operatorname{card} A > 3$. *Assume that* $\rho_{\min} \neq \rho \in \operatorname{con} \mathcal{A}'_0$. *Then* $a_0\,\rho\,u$.

Proof. Since $\rho \neq \rho_{\min}$ there exist $x, y \in A$ such that $x < y$ and $x\,\rho\,y$. Further, there exists a uniquely determined element $v \in A$ with $v + y = u$.

If $v = 0$, then $y = u$ whence $x \leqq a_0 < y$ and then $a_0\,\rho\,u$. Assume that $v \neq 0$; then $v > 0$.

a) Suppose that $x \neq v \neq y$. Then $x + v \in A$, hence

$$\begin{aligned} &x + v = x \oplus v = x \oplus' v, \quad x + v < y + v,\\ &u = y + v = y \oplus v = y \oplus' v, \end{aligned}$$

thus $(x+v)\,\rho\,u$. Further, $x + v \leqq a_0 < u$. This yields $a_0\,\rho\,u$.

b) Suppose that $y = v$. Then $u = 2y$. If $x = 0$ and $x \prec y$, then $y \prec u$, whence $\operatorname{card} A = 3$, which is a contradiction; thus there exists $z \in A$ with $0 < z < y$. In such a case we can take the element z instead of x. Thus without loss of generality we can assume that $0 < x$.

For each $t \in A$ we put $\overline{t} = \{t_1 \in A : \ t_1\,\rho\,t\}$.

First, suppose that there exists a positive integer n_0 such that $n_0 y \geqq u$. Then in view of 4.3 we have $n_0 \cdot y = u$. According to 4.10, $n \cdot y \in \overline{y}$ for each $n \in \mathbb{N}$. Thus $u \in \overline{y}$ and also $a_0 \in \overline{y}$. Hence $a_0\,\rho\,u$.

Further, suppose that $ny < u$ for each positive integer n. Then $n \cdot y = ny$ for each $n \in \mathbb{N}$, hence $n_1 \cdot y \neq n_2 \cdot y$ whenever n_1 and n_2 are distinct elements of $\mathbb{N}$. Also, all $n \cdot y$ belong to $\overline{y}$. For each $n \in \mathbb{N}$ there is a uniquely determined element v_n of A such that $n \cdot y + v_n = u$. If $n_1, n_2 \in \mathbb{N}$, $n_1 \neq n_2$, then $v_{n_1} \neq v_{n_2}$. Thus there is $n_1 \in \mathbb{N}$ such that $v_{n_1} \neq x$ and $v_{n_1} \neq n_1 \cdot y$. Now let us consider the pair $(x, n_1 \cdot y)$ instead of the pair (x, y). According to a) we then conclude that $a_0\,\rho\,u$.

c) Suppose that $x = v$. We can apply the same argument concerning n_0 and n_1 as in b). Again, we obtain the relation $a_0\,\rho\,u$. □

Proposition 4.12. *Let $\mathcal{A}$ be a linearly ordered MV-algebra. Assume that the set A^0 has a greatest element.*

(i) *If* $\operatorname{card} A = 3$*, then the algebra* $\mathcal{A}'_0$ *is subdirectly reducible.*

(ii) *If* $\operatorname{card} A \neq 3$*, then the algebra* $\mathcal{A}'_0$ *is subdirectly irreducible.*

Proof. If $\operatorname{card} A = 3$, then $\mathcal{A} \simeq \mathcal{A}_1$, where $\mathcal{A}_1$ is as in Proposition 4.8; hence (i) is valid.

The assertion (ii) is a consequence of Lemma 4.11. □

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Received 21. 4. 2008
Accepted 13. 6. 2008

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