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A NOTE ON SLANT SUBMANIFOLDS OF NEARLY TRANS-SASAKIAN MANIFOLDS

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ABSTRACT. The geometry of slant submanifolds of a nearly trans-Sasakian manifold is studied when the tensor field Q is parallel. It is proved that Q is not parallel on the submanifold unless it is anti-invariant and thus the result of [CABRERIZO, J. L.—CARRIAZO, A.—FERNANDEZ, L. M.—FERNANDEZ, M.: *Slant submanifolds in Sasakian manifolds*, Glasg. Math. J. **42** (2000), 125–138] and [GUPTA, R. S.—KHURSHEED HAIDER, S. M.—SHARFUDIN, A.: *Slant submanifolds of a trans-Sasakian manifold*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **47** (2004), 45–57] are generalized.

1. Introduction

Geometry of submanifolds of Sasakian and Kenmotsu manifolds have been an active area of research since long (cf. [2], [13] etc). As the class of trans-Sasakian manifolds includes Sasakian and Kenmotsu manifolds both, the study of geometry of submanifolds of trans-Sasakian manifolds becomes more meaningful. The historical back ground of trans-Sasakian manifolds can be traced back to the classification of almost Hermitain manifolds by A. Gray and L. M. Hervella [9]. One of the classes that appears in this classification, denoted by W_4 is closely related with locally conformal Kaehler manifolds. An almost contact metric structure (ϕ, ξ, η, g) on a manifold $\overline{M}$ is called a trans-Sasakian structure if $\overline{M} \times R$ belongs to the class W_4 of almost Hermitain manifolds. D. Blair and J. A. Oubina [5] showed that an almost contact metric manifold $\overline{M}$ with structure tensor (ϕ, ξ, η, g) is a trans-Sasakian manifold if

$$(\overline{\nabla}_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{1.1}$$

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for any vector fields X, Y on $\overline{M}$, where α and β are smooth functions and $\overline{\nabla}$ is the Riemannian connection on $\overline{M}$. A trans-Sasakian structure described by (1.1) is termed as structure of type (α, β). Thus a trans-Sasakian structure of type $(0, 0)$ is cosymplectic, a trans-Sasakian structure of type $(0, \beta)$ is β-Kenmotsu and a trans-Sasakian structure of type $(\alpha, 0)$ is α-Sasakian.

Recently C. Gherghe [8] introduced a nearly trans-Sasakian structure, which generalizes trans-Sasakian structure in the same sense as nearly Sasakian structure generalizes Sasakian structure.

An almost contact metric structre (ϕ, ξ, η, g) on $\overline{M}$ is a nearly trans-Sasakian structure if

$$\begin{aligned}(\overline{\nabla}_X \phi)Y + (\overline{\nabla}_Y \phi)X &= \alpha[2g(X,Y)\xi - \eta(Y)X - \eta(X)Y] \\ &\quad -\beta[\eta(Y)\phi X + \eta(X)\phi Y]. \qquad (1.2)\end{aligned}$$

A trans-Sasakian structure is always a nearly trans-Sasakian. Moreover, a nearly trans-Sasakian structure of type (α, β) is nearly Sasakian or nearly Kenmotsu or nearly cosymplectic accordingly as $\beta = 0$ or $\alpha = 0$ or $\alpha = \beta = 0$.

J. S. Kim et.al [12] initiated the study of semi-invariant submanifolds of nearly trans-Sasakian manifolds and obtained many results on the extrinsic geometric aspects of these submanifolds, whereas slant submanifolds are studied in the setting of trans-Sasakian manifolds by R. S. Gupta et.al [10]. Extending the study, in the present note we have considered slant submanifolds of nearly trans-Sasakian manifolds.

2. Preliminaries

Let $\overline{M}$ be an almost contact metric manifold with almost contact metric structure (ϕ, ξ, η, g) i.e., ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and g is the compatible Riemannian metric such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \qquad (2.1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi). \qquad (2.2)$$

for each vector fields X, Y on $\overline{M}$. If there exist smooth functions α, β and the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$ such that (1.1) holds, then $\overline{M}$ is said to be a trans-Sasakian manifold. If however, a more general tensorial equation (1.2) is satisfied, then $\overline{M}$ is called a nearly trans-Sasakian manifold.

Let M be a submanifold of $\overline{M}$. Then the induced Riemannian metric on M is denoted by the same symbol g and the induced Riemannian connection by ∇. Further, if $T\overline{M}$ and TM denote the tangent bundle on $\overline{M}$ and on M respectively and $T^{\perp}M$, the normal bundle on M, then the Gauss and Weingarten formulae are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.3}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \tag{2.4}$$

for each $X, Y \in TM$ and $N \in T^{\perp}M$, h and A_N denote respectively the second fundamental forms and the shape operator (corresponding to the normal vector field N) of the immersion of M into $\overline{M}$. The two are related as:

$$g(A_N X, Y) = g(h(X,Y), N). \tag{2.5}$$

Further, for each $x \in M$ and $X \in T_x(M)$, we decompose ϕX into tangential and normal parts respectively as:

$$\phi X = TX + FX. \tag{2.6}$$

Thus, T is an endomorphism and F is a normal valued 1-form on T_xM. Similarly decomposing ϕN, for $N \in T_x^{\perp}M$, into tangential and normal parts as:

$$\phi N = tN + fN, \tag{2.7}$$

we obtain a tangent valued 1-form t on $T_x^{\perp}M$ and an endomorphim f on $T_x^{\perp}M$. It is easy to observe that

$$FT + fF = 0. \tag{2.8}$$

Moreover, from (2.1) and (2.6),

$$g(T^2X, Y) = g(X, T^2Y), \tag{2.9}$$

for any $X, Y \in T_xM$. That shows that T^2 (which we subsequently denote by Q) is a self adjoint endomerphism on $T_x(M)$ for each $x \in M$. It is also easy to verify that the eigen values of Q belong to $[-1, 0]$ and that each non vanishing eigen value of Q has an even multiplicity. The $(1,1)$-tensor field on M determined by the endomorphism T and Q will be denoted by same letters. Their covariant derivatives are defined as:

$$(\overline{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{2.10}$$

$$(\overline{\nabla}_X Q)Y = \nabla_X QY - Q\nabla_X Y. \tag{2.11}$$

Similarly, the covariant derivatives of F, t and f are defined by the formulae

$$(\overline{\nabla}_X F)Y = \nabla_X^{\perp} FY - F\nabla_X Y, \tag{2.12}$$

$$(\overline{\nabla}_X t)N = \nabla_X tN - t\nabla_X^{\perp} N, \tag{2.13}$$

$$(\overline{\nabla}_X f)N = \nabla_X^{\perp} fN - f\nabla_X^{\perp} N, \tag{2.14}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$. Throughout, we assume that the structure vector field ξ is tangential to the submanifold M for other wise M is simply an anti-invariant submanifold [14].

For $x \in M$ and $X \in T_xM$ if X and ξ are linearly independent then the angle $\theta(X) \in [0, \frac{\pi}{2}]$ between ϕX and $T_x(M)$ is well defined. If $\theta(X)$ does not depend on the chocie of $x \in M$ and $X \in T_x(M)$, then M is said to be a slant submanifold of $\overline{M}$. In this case, the constant angle θ is called the slant angle of M. Invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\frac{\pi}{2}$ respectively. On a slant submanifold M of an almost contact metric manifold $\overline{M}$, the tangent bundle TM and the normal bundle $T^{\perp}M$ are decomposed as

$$TM = D \oplus \langle \xi \rangle \,, \tag{2.15}$$

and

$$T^{\perp}M = FD \oplus \mu, \tag{2.16}$$

where $\langle \xi \rangle$ denotes the 1-dimensional distribution spanned by the structure vector field ξ and μ is the subbundle of the normal bundle $T^{\perp}M$ invariant under ϕ. A slant submanifold is called trivial if $D = \{0\}$.

J. L. Cabrerizo et.al [6] obtained the following characterization for a submanifold of an almost contact metric manifold to be a slant submanifold.

Theorem 2.1. ([6]) *Let M be a submanifold of an almost contact metric manifold $\overline{M}$ such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$Q = -\lambda(I - \eta \otimes \xi). \tag{2.17}$$

Furthermore, in such a case if θ is the slant angle of M then $\lambda = \cos^2 \theta$.

On a slant submanifold M of an almost contact metric manifold $\overline{M}$, as an immiadiate consequence of formula (2.17), we have

$$g(TX, TY) = \cos^2 \theta (g(X, Y) - \eta(X)\eta(Y)), \tag{2.18}$$

and

$$g(FX, FY) = \sin^2 \theta (g(X, Y) - \eta(X)\eta(Y)), \tag{2.19}$$

for each $X, Y \in TM$.

Now, denoting by $\mathscr{P}_X Y$ and $\mathscr{Q}_X Y$, the tangential and normal parts of $(\overline{\nabla}_X \phi)Y$ and making use of equations (2.6), (2.7), the Gauss and Weingarten formulae we may derive

$$\mathscr{P}_X Y = (\overline{\nabla}_X T)Y - A_{FY}X - th(X, Y), \tag{2.20}$$

and

$$\mathscr{Q}_X Y = (\overline{\nabla}_X F)Y + h(X, PY) - fh(X, Y). \tag{2.21}$$

3. Parallelisim of Q on a slant submanifold

We first make use of Theorem (2.2) to see the impact of parallelism of the $(1,1)$ tensor field Q on a slant submanifold of an almost contact metric manifold.

Theorem 3.1. *Let M ba a slant submanifold of an almost contact metric manifold $\overline{M}$. Then Q is parallel if and only if at least one of the following is true*

(i) *M is anti-invariant*

(ii) $\dim(M) \geq 3$.

(iii) *M is trivial.*

P r o o f. Let θ be the slant angle of M in $\overline{M}$, then for any $X, Y \in T(M)$, by equation (2.17)

$$Q(\triangledown_X Y) = \cos^2\theta(-\triangledown_X Y + \eta(\triangledown_X Y)\xi), \tag{3.1}$$

and

$$QY = \cos^2\theta(-Y + \eta(Y)\xi).$$

Differentiating the last equation covariantly with respect to X, we get

$$\triangledown_X QY = \cos^2\theta(-\triangledown_X Y + \eta(\triangledown_X Y))\xi + g(Y, \triangledown_X \xi)\xi + \eta(Y)\triangledown_X \xi). \tag{3.2}$$

From equations (2.11), (3.1) and (3.2), it follows that

$$(\triangledown_X Q)Y = \cos^2\theta(g(Y, \triangledown_X \xi)\xi + \eta(Y)\triangledown_X \xi). \tag{3.3}$$

Now, if Q is parallel, then from (3.3), it follows that either $\cos\theta = 0$ i.e. M is anti-invariant which accounts for case (i), or else we have

$$g(Y, \triangledown_X \xi)\xi + \eta(Y)\triangledown_X \xi = 0. \tag{3.4}$$

The above equation has a solution if and only if $\triangledown_X \xi = 0$ and therefore either $D = \{0\}$ or we can pick at least two linearly independent vectors X and PX (belonging to a unique non zero eigen value of Q) to span D. In this case, the eigen value is necessarily non zero as $\theta = \frac{\pi}{2}$ has already been taken care off. Hence, the $\dim(M) \geq 3$.

Now, we obtain some usefull implications of formulae (1.2) and (3.3) in order to study the parallelism of Q on a slant submanifold of a nearly tran-Sasakian manifold $\overline{M}$. Throughout the section, we denote by M a slant submanifold of a nearly trans-Sasakian manifold $\overline{M}$ such that ξ is tangential to M. In view of the decomposition (2.16), we may write

$$h(X, \xi) = h_{FD}(X, \xi) + h_\mu(X, \xi), \tag{3.5}$$

for any $X \in TM$, where $h_{FD}(X, \xi) \in FD$ and $h_\mu(X, \xi) \in \mu$.

Further, by equation (1.2)

$$\mathscr{P}_X Y + \mathscr{P}_Y X = \alpha(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y) - \beta(\eta(Y)TX + \eta(X)TY), \tag{3.6}$$

and

$$\mathscr{Q}_X Y + \mathscr{Q}_Y X = -\beta(\eta(Y)FX + \eta(X)FY). \tag{3.7}$$

In particular, for $Y = \xi$, the above equations yield

$$\mathscr{P}_X \xi + \mathscr{P}_\xi X = \alpha(\eta(X)\xi - X) - \beta TX, \tag{3.8}$$

$$\mathscr{Q}_X \xi + \mathscr{Q}_\xi X = -\beta FX. \tag{3.9}$$

□

Lemma 3.1. *Let M be a proper slant submanifold of a nearly trans-Sasakian manifold $\overline{M}$ with $\xi \in TM$. Then for any $X \in D$*

$$h_{FD}(X, \xi) = -\alpha \csc^2 \theta FX$$

where θ is the slant angle of M in $\overline{M}$.

P r o o f. By equations (3.8) and (2.20), we obtain

$$(\overline{\nabla}_X T)\xi + (\overline{\nabla}_\xi T)X - A_{FX}\xi - 2th(X, \xi) = -(\alpha X + \beta TX),$$

which on making use of (2.10) and (2.1) simplifies as

$$-T\nabla_X \xi + \nabla_\xi TX - T\nabla_\xi X - A_{FX}\xi - 2th(X, \xi) = -(\alpha X + \beta TX). \tag{3.10}$$

As M is proper slant, by Theorem (3.1),

$$\nabla_X \xi = 0.$$

Making use of this fact while taking product with X in (3.10), we get

$$g(h(X, \xi), FX) = -\alpha g(X, X).$$

Using the polarization identity on the above, we obtain

$$g(h(X, \xi), FY) = -\alpha g(X, Y),$$

for each $X, Y \in D$.

The above relation in view of formulae (2.19) and (3.5) proves the assertion.

□

Theorem 3.2. *Let M be a slant submanifold of a nearly trans-Sasakian manifold $\overline{M}$ with structure vector field ξ tangential to M, then Q is parallel on M if and only if M is either anti-invariant or a trivial submanifold of $\overline{M}$.*

P r o o f. Let θ be the slant angle of M. Then from equation (3.3) it follows that either M is anti-invariant or

$$g(Y, \nabla_X(\xi)\xi + \eta(Y)\nabla_X \xi = 0. \tag{3.11}$$

However, (3.11) holds if and only if

$$\nabla_X \xi = 0. \tag{3.12}$$

Now taking $X \in D$ and writting $\mathcal{Q}_X\xi + \mathcal{Q}_\xi X$ by formula (2.21), we obtain

$$\mathcal{Q}_X\xi + \mathcal{Q}_\xi X = (\overline{\nabla}_X F)\xi + (\overline{\nabla}_\xi F)X + h(TX,\xi) - 2fh(X,\xi).$$

Substituting the value of $\mathcal{Q}_X\xi + \mathcal{Q}_\xi X$ from equation (3.9) into the above equation and taking product with FX, it is deduced that

$$\begin{aligned} g(F\nabla_X\xi, FX) - g(\nabla^\perp_\xi FX, FX) + g(F\nabla_\xi X, FX) & \\ -g(h(TX,\xi), FX) + 2g(fh(X,\xi), FX) &= \beta g(FX, FX). \end{aligned}$$

Making use of equations (3.12), (2.19) and (2.8), the above equation yields,

$$g(h(TX,\xi), FX) + 2g(h(X,\xi), FTX) = \beta \sin^2\theta g(X,X).$$

The two terms in the left hand side of the above equation vanish due to Lemma 3.1 and thus, we conclude that either

$$\theta = 0 \quad \text{or} \quad D = \{0\}.$$

On the other hand, differentiating the identity $g(\phi X, \xi) = 0$ covariantly with respect to $X \in D$, we get

$$g(\overline{\nabla}_X \phi X, \xi) + g(\phi X, \overline{\nabla}_X \xi) = 0,$$

or,

$$g((\overline{\nabla}_X\phi)X + \phi\overline{\nabla}_X X, \xi) + g(\phi X, \overline{\nabla}_X\xi) = 0. \tag{3.13}$$

If $\theta = 0$, then $\phi X = TX$. Making use of this fact and equations (1.2) and (3.12), it can be deduced from (3.13) that

$$\alpha\|X\|^2 = 0,$$

which means if $\theta = 0$, then either $\alpha = 0$ on M or $D = \{0\}$. This rules out the possibility of slant angle being zero as long as M is a non trivial slant submanifold of $\overline{M}$ with $\nabla Q = 0$. In other words, M cannot be invariant. Hence, if $\nabla_X\xi = 0$ then M is anti-invariant.

This proves the theorem completely. □

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