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TRANSLATABLE RADII OF AN OPERATOR IN THE DIRECTION OF ANOTHER OPERATOR II

Kallol Paul

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ABSTRACT. One of the couple of translatable radii of an operator in the direction of another operator introduced in earlier work [PAUL, K.: *Translatable radii of an operator in the direction of another operator*, Scientae Mathematicae **2** (1999), 119–122] is studied in details. A necessary and sufficient condition for a unit vector f to be a stationary vector of the generalized eigenvalue problem $Tf = \lambda Af$ is obtained. Finally a theorem of Williams ([WILLIAMS, J. P.: *Finite operators*, Proc. Amer. Math. Soc. **26** (1970), 129–136]) is generalized to obtain a translatable radius of an operator in the direction of another operator.

1. Introduction

Let T and A be two bounded linear operators on a complex Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Consider the generalized eigenvalue problem $Tf = \lambda Af$ where $f \in H$ and $\lambda \in C$, λ is called the eigenvalue of the above equation and f the corresponding eigenvector. The non-negative functional

$$M_T(f) = \left\| Tf - \frac{(Tf, Af)}{(Af, Af)} Af \right\|, \qquad \text{provided} \quad \|Af\| \neq 0,$$

gives the deviation of a unit vector f from being an eigenvector and

$$M_T(A) = \sup_{\|f\|=1} \left\{ \left\| Tf - \frac{(Tf, Af)}{(Af, Af)} Af \right\| \right\}, \qquad \text{provided} \quad 0 \notin \sigma_{\text{app}} A,$$

gives the supremum of all those deviations, where $\sigma_{\text{app}}A$ is the set of approximate eigenvalues of A.

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Geometrically $Tf - \frac{(Tf,Af)}{(Af,Af)}Af$ is the component of Tf perpendicular to Af. For $A = I$ problems related to the concepts considered here have been studied by Bjorck and Thomee [2], Garske [8], Prasanna [14], Fujii and Prasanna [6], Furuta, Izumino and Prasanna [7], Fujii and Nakamoto [5], Izumino [9], Nakamoto and Sheth [11], Mustafaev and Shulman [10] and many others.

Bjorck and Thomee [2] have shown that for a normal operator T,

$$M_T = \sup_{\|f\|=1} \{\|Tf - (Tf, f)f\|\} = R_T,$$

where R_T is the radius of the smallest circle containing the spectrum. Garske [8] improved on the result to prove that for any bounded linear operator T,

$$M_T = \sup_{\|f\|=1} \{\|Tf - (Tf, f)f\|\} \geq R_T.$$

Stampfli [15] proved that for a bounded linear operator T there exists a unique complex scalar c_T, defined as the center of mass of T such that

$$\|T - c_T I\|^2 + |\lambda|^2 \leq \|T - c_T I + \lambda I\|^2, \qquad \text{for all} \quad \lambda \in C.$$

With the help of Stampfli's result Prasanna [14] proved that $M_T = \|T - c_T I\|$. Later Fujii and Prasanna [6] improved on the inequality of Garske to show that $M_T \geq w_T$ where w_T is the radius of the smallest circle containing the numerical range.

In [12] we proved that for any two bounded linear operators T and A if $0 \notin \sigma_{\text{app}} A$ then there exists a unique complex scalar λ_0 such that $\|T - \lambda_0 A\| \leq \|T - \lambda A\|$ for all $\lambda \in C$. We defined $T - \lambda_0 A$ as the *minimal-norm translation of T in the direction of A* and proved that $\|T - \lambda_0 A\| = M_T(A)$. The equality of $\inf_\lambda \|T - \lambda A\| = M_T(A)$ was also studied by E. Asplund and V. Pták [1]. Then in [13] we introduced a couple of *translatable radii of an operator T in the direction of another operator A* as follows:

If 0 does not belong to the approximate point spectrum of A let

$$M_T(A) = \sup_{\|f\|=1} \left\{ \left\| Tf - \frac{(Tf, Af)}{(Af, Af)} Af \right\| \right\}$$

i.e.,

$$M_T(A) = \sup_{\|f\|=1} \left\{ \|Tf\|^2 - \frac{|(Tf, Af)|^2}{(Af, Af)} \right\}^{1/2}$$

and if $0 \notin \overline{W(A)}$, where $\overline{W(A)}$ stands for the closure of the numerical range of A, let

$$\tilde{M}_T(A) = \sup_{\|f\|=1} \left\{ \left\| Tf - \frac{(Tf, f)}{(Af, f)} Af \right\| \right\}.$$

We defined $M_T(A)$ and $\tilde{M}_T(A)$ as translatable radii of the operator T in the direction of A and proved in [13] that if $0 \notin \overline{W(A)}$ then $\tilde{M}_T(A) \geq M_T(A) \geq m_T(A)/\|A^{-1}\|$, where $m_T(A)$ is the radius of the smallest circle containing the set $W_T(A) = \{(Tf, Af)/(Af, Af) : \|f\| = 1\}$.

Das [4] introduced the concept of stationary distance vectors while studying the eigenvalue problem $Tf = \lambda f$. Following the ideas of Das we here use the concept of stationary distance vectors to study the generalized eigenvalue problem $Tf = \lambda Af$ and the translatable radius $M_T(A)$. We investigate the structure of the vectors for which the translatable radius $M_T(A)$ is attained and prove that if $M_T(A)$ is attained at a vector f then $M_{T^*}(A^*)$ is attained at the vector $h/\|h\|$, where $h = Tf - (Tf, Af)/(Af, Af)Af$. We also show that if g is a state (normalized positive functional) on the Banach algebra $B(H, H)$ of all bounded linear operators on H then

$$M_T(A) = \sup\left\{ g(T^*T) - \frac{|g(A^*T)|^2}{g(A^*A)} : \ g \ \text{is a state and} \ g(A^*A) \neq 0 \right\}.$$

The last result mentioned here is a generalization of a theorem of Williams [16].

2. Stationary distance vectors of the generalized eigenvalue problem $Tf = \lambda Af$

In this section we study the following:

"For any two bounded linear operators T and A what are the vectors that are nearest to or farthest from being eigenvectors of the equation $Tf = \lambda Af$ in the sense that $\|Tf - (Tf, Af)/(Af, Af)Af\|$ with unit f is minimum or maximum?"

We give a necessary and sufficient condition that a unit vector f is at a stationary distance from being an eigenvector. We call such f's the stationary distance vectors and the corresponding $\lambda = (Tf, Af)/(Af, Af)$ the stationary distance value of the eigenvalue problem $Tf = \lambda Af$. We use the concept of stationary vectors the definition of which is given below:

Definition 1 (Stationary vector). Let φ be a functional defined on the unit sphere of H. Then a unit vector f is said to be a stationary vector and φ is said to have a stationary value at f iff the function $w_g(t)$ of a real variable t, defined as

$$w_g(t) = \varphi\left(\frac{f + tg}{\|f + tg\|}\right)$$

has a stationary value at $t = 0$, i.e., $w_g'(0) = 0$ for an arbitrary but fixed vector $g \in H$, e.g., if $\varphi(f) = \|Tf - (Tf, Af)/(Af, Af)Af\|^2$ then a stationary vector f

of functional φ is called the stationary distance vector of the eigenvalue problem $Tf = \lambda Af$.

We assume that 0 does not belong to the approximate point spectrum of A and prove the following theorem:

Theorem 1. *The necessary and sufficient condition for a unit vector f to be a stationary distance vector of the generalized eigenvalue problem $Tf = \lambda Af$ is that it satisfies the following*

$$(T^* - \bar{\lambda}A^*)(T - \lambda A)f = \|h\|^2 f$$

where $h = Tf - \lambda Af$ and $\lambda = \frac{(Tf,Af)}{(Af,Af)}$.

P r o o f. Consider $M_T(f) = \|Tf - (Tf, Af)/(Af, Af)Af\|$. Define the function $w_g(t)$ of a real variable t as follows

$$w_g(t) = M_T^2\left(\frac{f+tg}{\|f+tg\|}\right) = \frac{\|T(f+tg)\|^2}{\|f+tg\|^2} - \frac{|(T(f+tg), A(f+tg))|^2}{(A(f+tg), A(f+tg))\|f+tg\|^2},$$

where g is arbitrary but fixed vector in H.

At a stationary vector f we have $w_g'(0) = 0$ and so

$$\begin{aligned}
&2\operatorname{Re}(T^*Tf, g) - 2\|Tf\|^2\operatorname{Re}(f, g)\\
&- \frac{\|Af\|^2}{\|Af\|^4}\Big[(Tf, Af)\{\overline{(Tf, Ag) + (Tg, Af)}\} + \overline{(Tf, Af)}\{(Tf, Ag) + (Tg, Af)\}\Big]\\
&+ \frac{|(Tf, Af)|^2}{\|Af\|^4}\{2\|Af\|^2\operatorname{Re}(f, g) + 2\operatorname{Re}(A^*Af, g)\} = 0.
\end{aligned}$$

Since g is arbitrary we get,

$$T^*Tf - \|Tf\|^2 f - \lambda T^*Af - \bar{\lambda}A^*Tf + \|Af\|^2\lambda^2 f + \lambda^2 A^*Af = 0,$$

where $\lambda = (Tf, Af)/(Af, Af)$.

Let $h = Tf - \lambda Af$, then

$$(h, Af) = 0 \qquad \text{and} \qquad \|h\|^2 = \|Tf\|^2 - |(Tf, Af)|^2/(Af, Af).$$

So we get $(T^* - \bar{\lambda}A^*)(T - \lambda A)f = \|h\|^2 f$. Thus the theorem is proved. □

We now prove the following corollary:

Corollary 1. *If $M_T(A)$ is attained at f then $M_{T^*}(A^*)$ is also attained at $h/\|h\|$ where $h = Tf - (Tf, Af)/(Af, Af)Af$.*

P r o o f. Suppose $M_T(A)$ is attained at a vector f and $\lambda = \frac{(Tf,Af)}{(Af,Af)}$. Then f is a stationary distance vector and so we get

$$\begin{aligned} & (T^* - \bar{\lambda}A^*)(T - \lambda A)f = \|h\|^2 f \\ \implies \quad & (T^* - \bar{\lambda}A^*)h = \|h\|^2 f \\ \implies \quad & (T^*h, A^*h) = \bar{\lambda}(A^*h, A^*h) \\ \implies \quad & \bar{\lambda} = \frac{(T^*h, A^*h)}{(A^*h, A^*h)}. \end{aligned}$$

Now

$$\begin{aligned} & T^*h = \bar{\lambda}A^*h + \|h\|^2 f \\ \implies \quad & \|T^*h\|^2 = |\bar{\lambda}|^2 \|A^*h\|^2 + \|h\|^4 \\ \implies \quad & \|T^*h\|^2 = \|h\|^2 \left\{ \|Tf\|^2 - \frac{|(Tf,Af)|^2}{(Af,Af)} \right\} + \frac{|(Tf,Af)|^2}{(Af,Af)} \cdot \frac{\|A^*h\|^2}{\|Af\|^2}. \end{aligned}$$

If the minimal-norm translation of T in the direction of A is T itself then the minimal-norm translation of T^* in the direction of A^* is also T^*. So if $M_T(A) = \|T\|$, then $M_{T^*}(A^*) = \|T^*\|$. Let $M_T(A) = \|T\| = \|Tf\|$, $(Tf,Af)/(Af,Af) = 0$. Then $M_{T^*}(A^*) = \|T^*\| = \|T\| = \|T^*h\|/\|h\|$, since $(Tf,Af)/(Af,Af) = 0$. This completes the proof. □

Next we prove the following theorem:

Theorem 2. *Suppose T and A are two selfadjoint operators and f be a unit stationary distance vector such that (Tf,Af) is real, then f can be expressed as the linear combination of two eigenvectors of the problem $Tf = \lambda Af$.*

P r o o f. As both T and A are selfadjoint and f is a stationary distance vector with (Tf,Af) real we get from the last theorem

$$(T - \lambda A)^2 f = \|h\|^2 f.$$

So we get

$$\begin{aligned} & (T - \lambda A)^2 f \pm \|h\|h = \|h\|^2 f \pm \|h\|h \\ \implies \quad & T(Tf - \lambda Af \pm \|h\|f) = (\lambda A \pm \|h\|)(Tf - \lambda Af \pm \|h\|f). \end{aligned}$$

Let

$$g_1 = Tf - \lambda Af + \|h\|f$$

and

$$g_2 = Tf - \lambda Af - \|h\|f.$$

Then we get

$$Tg_1 = (\lambda A + \|h\|)g_1 \qquad \text{and} \qquad Tg_2 = (\lambda A - \|h\|)g_2$$

so that

$$(T - \lambda A)g_1 = \|h\|g_1 \qquad \text{and} \qquad (T - \lambda A)g_2 = -\|h\|g_2.$$

Thus $f = (g_1 - g_2)/(2\|h\|)$ completes the proof. □

3. On the attainment of $M_T(A)$

Suppose $\{f_n\}$ be a sequence of unit vectors such that

$$\|Tf_n\|^2 - \frac{|(Tf_n, Af_n)|^2}{(Af_n, Af_n)} \longrightarrow M_T(A)^2.$$

As the unit sphere in H is weakly compact without loss of generality we may assume that $\{f_n\}$ converges weakly to f, i,e, $f_n \rightharpoonup f$.

We now prove the following theorem:

Theorem 3. *Suppose $\{f_n\}$ be a weakly convergent sequence of unit vectors such that*

$$\|Tf_n\|^2 - \frac{|(Tf_n, Af_n)|^2}{(Af_n, Af_n)} \longrightarrow M_T(A)^2.$$

If the weak limit f is non-zero then $M_T(A)$ is attained for the vector $f/\|f\|$. If the supremum is not attained then all such sequences must tend weakly to zero.

Proof. Since $M_T(A)$ is translation invariant in the direction of A so without any loss of generality we may assume that the minimal-norm translation of T in the direction of A is T itself i,e, $M_T(A) = \|T\|$.

So there exists a sequence $\{f_n\}$, $f_n \in H$, $\|f_n\| = 1$ such that $\|Tf_n\| \longrightarrow \|T\|$ and $(Tf_n, Af_n) \longrightarrow 0$. Considering the positive operator $\|T\|^2 I - T^*T$ we have

$$\begin{aligned} & (\|T\|^2 f_n - T^*Tf_n, f_n) \longrightarrow 0 \\ \Longrightarrow \quad & \|T\|^2 f_n - T^*Tf_n \longrightarrow 0, \quad \text{by property of positive operators.} \end{aligned}$$

If $f \neq 0$ we have

$$\|T\|^2 (f_n, f) - (T^*Tf_n, f) \longrightarrow 0.$$

Since $f_n \rightharpoonup f$ and weak limit f is unique we get

$$\|T\|^2 = \frac{\|Tf\|^2}{\|f\|^2}.$$

The result that “if $f_n \rightharpoonup f$, $\|Tf_n\| \to \|T\|$ and $f \neq 0$ then $\|T\|$ is attained at $f/\|f\|$” follows directly from [3, Corollary 1] of Das. As $M_T(A) = \|T\|$ the theorem is proved. □

4. On generalization of a theorem of Williams

Let B denote the set of all normalized positive linear functionals (states) on B(H,H) i.e.,

$$B = \{g : \ g \in L(B(H,H),C) \ \text{ and } \ g(I) = 1 = \|g\|\}.$$

Clearly B is *weak** compact. Let $P = \{g : \ g \in B \ \text{ and } \ g(A^*A) \neq 0\}$.

Williams [16] proved that for any bounded linear operator T, $\|T\| \leq \|T - \lambda I\|$ for all $\lambda \in$ C iff there exists a state f such that $f(T^*T) = \|T^*T\|$ and $f(T) = 0$. We here show that if for two bounded linear operators T and A, $\|T\| \leq \|T - \lambda A\|$ for all $\lambda \in C$ then $\|T\|^2 = \sup\{g(T^*T) - \frac{|g(A^*T)|^2}{g(A^*A)} : g$ is a state and $g(A^*A) \neq 0\}$.

We now prove the following theorem:

Theorem 4. $[M_T(A)]^2 = \sup\left\{g(T^*T) - \frac{|g(A^*T)|^2}{g(A^*A)} : \textit{g is a state and } g(A^*A) \neq 0\right\}$.

P r o o f. Let $[S_T(A)]^2 = \sup\{ g(T^*T) - \frac{|g(A^*T)|^2}{g(A^*A)} : g$ is a state and $g(A^*A) \neq 0\}$.

Clearly $S_{T+\lambda A}(A) = S_T(A)$ and $M_{T+\lambda A}(A) = M_T(A)$ so that both are translation invariant in the direction of A. Without loss of generality we assume that $M_T(A) = \|T\|$.

Now for each $x \in H$, $\|x\| = 1$, let $g_x : B(H,H) \to C$ be defined as $g_x(U) = (Ux, x)$ for all $U \in B(H,H)$. Then g_x is a state and $g_x(A^*A) \neq 0$.

So

$$\begin{aligned} \|T\| &= \sup_{g_x} \left\{ g_x(T^*T) - \frac{|g_x(A^*T)|^2}{g_x(A^*A)} \right\}^{1/2} \\ &\leq \sup_{g \in P} \left\{ g(T^*T) - \frac{|g(A^*T)|^2}{g(A^*A)} \right\}^{1/2} \\ &\leq \sup_{g \in P} \{g(T^*T)\}^{1/2} \\ &= \|T\| \end{aligned}$$

This completes the proof. □

Note. For $A = I$ the result of Williams follows easily from Theorem 4.

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Reader in Mathematics
Department of Mathematics
Jadavpur University
Kolkata 700032
INDIA
E-mail: kalloldada@yahoo.co.in
kpaul@math.jdvu.ac.in