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RELATIVE MV-ALGEBRAS AND RELATIVE HOMOMORPHISMS

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ABSTRACT. In this paper we define the notion of relative subalgebra of an MV-algebra A. A particular case of this notion is the notion of interval subalgebra of A; this has been already studied in the literature.

Applying these notions, two new categories denoted as $r\mathscr{MV}$ and $int\mathscr{MV}$ are introduced. In both cases the objects are MV-algebras, but the homomorphisms are defined by means of relative subalgebras or by interval subalgebras, respectively. The relations occurring between these categories and the category of all MV-algebras with usual homomorphisms are investigated. The main results of the paper deal with one-generated free MV-algebras in the variety generated by the finite chains S_i, $i \leqslant p$ (p varying over the set of all positive integers) and their relations to certain relative subalgebras of the cyclic free MV-algebra.

1. Introduction

Several times it happens that given an MV-algebra A, special subsets of A, which are MV-algebras but not MV-subalgebras of A, are considered, and that they help in getting information about A. Indeed the same happens in the theory of Boolean Algebras, where *relative* algebras are considered, see [9]. We recall that Sikorski [10] and Tarski [11] proved the following generalization of the Cantor-Bernstein theorem: For any two σ-complete Boolean algebras A and B and elements $a \in A$ and $b \in B$, if B is isomorphic to the interval $[0, a] \subseteq A$ and A is isomorphic to $[0, b] \subseteq B$, then A and B are isomorphic. It can be seen, then, that subsets of Boolean algebras which are Boolean algebras play a role.

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Generalizations of the above mentioned theorems to MV-algebras, say Cantor-Bernstein type theorems, involve a similar structure in MV-algebraic setting, i.e. the structure of *interval MV-algebra* subset of an MV-algebra, see for example [4], [5], [6].

We recall that in decomposing an MV-algebra A as a direct product sometime MV-algebras having, as the underlying set, a subset $(b]$ of A, are considered, where b is an idempotent element of A and $(b]$ is the principal ideal of A generated by b. The MV-algebraic structure on $(b]$ is defined in a canonical way, see [2] where Theorem 6.8.5 provides a decomposition of complete MV-algebras. It is worth to observe that in the MV-algebra A the MV-algebraic structure over $[0, b]$ is defined with the help of the map $h_b\colon A \to A$, just setting $h_b(x) = b \wedge x$ and $\neg_b x = b \wedge \neg x$. Then $((b], \oplus, \neg_b, 0)$ is an MV-algebra and h_b is a homomorphism of A onto $(b]$. Also a certain property of h_b can be trivially observed, actually the identity map $\delta\colon h_b(A) \to A$ is such that $h_b \circ \delta = ID_{h_b(A)}$, where $ID_{h_b(A)}$ denotes the identity map of $h_b(A)$. We mentioned such a trivial property because, as we shall see (Section 5), this property will assume more significance in a wider categorical context. Similar examples to above ones already shown can be found again in [2, Proposition 6.4.1, Proposition 6.4.3, Theorem 6.7.3]. In [1] the authors defined an MV-algebraic structure on the interval $[0, a]$ of a given MV-algebra A, with $a \in A \setminus \{0\}$. After denoting such algebra by A_a, they called it a *pseudo-subalgebra* of A. Then, it turns out that every MV-algebra A' is a pseudo-subalgebra of some perfect MV-algebra A, (see [1, Theorem 30]). An analogous construction was presented in [7] and [8] where a structure of MV-algebra has been defined over the interval $[a, b]$ of an arbitrary MV-algebra A, with $a, b \in A$.

Looking at the above examples we can observe that in very special different ways such subsets MV-algebras are built up. Here we generalize the aforementioned constructions showing that one can uniformly define subsets of A which are MV-algebras. These algebras, described in the present paper, are called *relative MV-subalgebras*. The existence of relative MV-subalgebras pushes us to consider a new category of MV-algebras having as objects still MV-algebras, but different morphisms, morphisms which are more general than the MV-homomorphisms. Following this line we can define an intermediate category, still having MV-algebras as objects and, as morphisms between MV-algebras A and B, maps which are not MV-homomorphisms but, roughly speaking, preserving MV-algebras which are intervals in A and in B, respectively. This allows to express, for example, the Cantor-Bernstein type theorem, for Boolean algebras, above mentioned referring to Sikorski and Tarski, in categorical terms inside this new category.

Let $\mathscr{MV}$ be the variety of all MV-algebras, $\mathbb{N}$ be the set of all positive integers and $p \in \mathbb{N}$. Denote by $\mathscr{K}_p$ the locally finite subvariety of $\mathscr{MV}$ generated by the finite chains $S_i = \{0, \frac{1}{i}, \ldots, \frac{i-1}{i}, 1\}$, with $i \leqslant p$, i.e. $\mathscr{K}_p = V(\{S_1, \ldots, S_p\})$. Let $F_p(m)$ be the m-generated free MV-algebra in the variety $\mathscr{K}_p$ and $F(m)$ be the m-generated free MV-algebra in the variety $\mathscr{MV}$.

As we shall show, the new class of morphisms between MV-algebras helps in describing a hidden relationship between $F_p(1)$ algebras, p varying over the set of all positive integers, and $F(1)$.

Actually we show that:

1. up to isomorphism, every one-generated free $F_p(1)$ algebra is a relative MV-subalgebra of the cyclic free MV-algebra $F(1)$, for any p;
2. up to isomorphism, the set of one-generated free $F_p(1)$ algebras, p varying in the set of all positive integers, forms a directed system in the category of relative MV-algebras;
3. up to isomorphism, each one-generated free $F_p(1)$ algebra is a retractive subalgebra of $F(1)$, in the category of relative MV-algebras;
4. there is a family $\mathscr{D} = \{D_p\}_{p\in\mathbb{N}}$ of finite sequences of elements of $Q \cap [0,1]$ (sub-Farey sequences), such that each element $D_p \in \mathscr{D}$ allows us to cut out a relative MV-subalgebra of $F(1)$, which is isomorphic to $F_p(1)$.

We shall refer to [2] for any unexplained notion on MV-algebras and, for a better readability of the paper, we confine to Appendix the results, useful for our aims, which essentially concern with elementary properties of the integer numbers.

2. Relative MV-subalgebras

Let $A = (A, \oplus, ^*, 0)$ be a nontrivial MV-algebra, $1 = 0^*$ and $xy = (x^* \oplus y^*)^*$. Following the tradition, we consider the * operation more binding than any other operation, and the product more binding than the addition.

Let $a, b \in A$, with $a \leqslant b$.

Lemma 1. *For every $x, y \in [a,b]$, $x \oplus a^*y = a^*x \oplus y$.*

Proof. Since $x, y \geqslant a$, we have: $x \oplus a^*y = a \oplus a^*x \oplus a^*y = a^*x \oplus y$. □

We define two new operations in $[a,b]$:

1. for $x, y \in [a,b]$, $x \uplus y = (a \oplus a^*x \oplus a^*y) \wedge b = (x \oplus a^*y) \wedge b = (a^*x \oplus y) \wedge b$;
2. for $x \in [a,b]$, $\overline{x} = a \oplus x^*b$.

We call *relative MV-subalgebra* of A every nonempty subset $P_A(a,b)$ of $[a,b]$ closed with respect the above operations. If $a = b$ we say that the relative MV-subalgebra $P_A(a,b)$ is trivial. In the sequel, when there is no ambiguity, we shall drop the subscript A.

Proposition 2. *Let $P(a,b)$ be a relative MV-subalgebra of A. Then $(P(a,b), \uplus, ^{-}, a)$ is an MV-algebra, where $\overline{a} = b$ and $L(P(a,b))$ is a sublattice of $L(A)$.*

Proof. Let $x \in P(a,b)$. $x \uplus \overline{x} = (a^*x \oplus a \oplus x^*b) \wedge b = b \in P(a,b)$; moreover $\overline{b} = a \oplus b^*b = a \in P(a,b)$. Thus $a, b \in P(a,b)$ and $\overline{a} = a \oplus a^*b = a \vee b = b$.

1. $\uplus$ is associative.

Indeed $(x \uplus y) \uplus z = (((x \oplus a^*y) \wedge b) \oplus a^*z) \wedge b = ((x \oplus a^*y \oplus a^*z) \wedge (b \oplus a^*z)) \wedge b = (x \oplus a^*y \oplus a^*z) \wedge b$.

On other hand $x \uplus (y \uplus z) = x \uplus ((y \oplus a^*z) \wedge b) = (a^*x \oplus ((y \oplus a^*z) \wedge b)) \wedge b = (a^*x \oplus y \oplus a^*z) \wedge b$.

The thesis follows from Lemma 1.

2. $\uplus$ is commutative; it follows by definition.

3. $x \uplus a = (x \oplus a^*a) \wedge b = x$.

4. $x \uplus b = (a^*x \oplus b) \wedge b = b$.

5. $\overline{(\overline{x} \uplus y)} \uplus y = \overline{(\overline{y} \uplus x)} \uplus x = x \vee y$.

Indeed, set $\alpha = \overline{x} \uplus y$, $\alpha = (a \oplus x^*b \oplus a^*y) \wedge b = (x^*b \oplus y) \wedge b$ and $\overline{\alpha} = a \oplus [(x^*b \oplus y) \wedge b]^*b = a \oplus [(x^*b \oplus y)^* \vee b^*]b = a \oplus (x^*b \oplus y)^*b = a \oplus (x \wedge b)y^*$.

Hence $\overline{\alpha} \uplus y = (a \oplus (x \wedge b)y^* \oplus a^*y) \wedge b = (y \oplus (x \wedge b)y^*) \wedge b = y \vee (x \wedge b) = y \vee x$.

Exchanging the roles of x and y, we get $\overline{(\overline{y} \uplus x)} \uplus x = x \vee y$. Thus the equality 5 is proved. □

Given an MV-algebra A, if $P(a,b) = [a,b]$, then the relative MV-subalgebra $P(a,b)$ of A will be called *interval algebra of A* or simply *interval algebra*.

Example. We shall now exhibit an example of relative subalgebra which is not an interval algebra.

Let $F(1)$ be the MV-algebra of McNaughton functions with one variable. Let $a = \underline{0}$, the function identically zero on $[0,1]$, $b = (x \vee x^*)^2$, $f = x^2$ and $g = (x^*)^2$, where x is the generator of $F(1)$. Set $P(a,b) = \{a, b, f, g\}$, we get that $(P(a,b), \uplus, ^{-}, a)$ is a relative subalgebra of $F(1)$, which is not an interval subalgebra of $F(1)$.

Beginning from the MV-algebra $(P_A(a,b), \uplus, ^{-}, a)$ and two elements $c, d \in P_A(a,b)$ with $c < d$, we can construct a relative MV-subalgebra $P_{P_A(a,b)}(c,d)$ of $P_A(a,b)$, defining two new operations † and ¬:

for $x, y \in [c, d]$, $x \dagger y = (x \uplus \overline{c} \circ y) \wedge d$,

for $x \in [c, d]$, $\neg x = c \uplus \overline{c} \circ d$,

where $x \circ y = \overline{(\overline{x} \uplus \overline{y})}$.

The next proposition shows that every relative MV-subalgebra of $P_A(a, b)$ is a relative MV-subalgebra of A. Indeed we have:

Proposition 3. *For $x, y \in [c, d]$*

1. $x \dagger y = (x \oplus c^* y) \wedge d$;
2. $\neg x = c \oplus x^* d$.

3. The category of relative MV-algebras

Definition 4. Let A and B be MV-algebras. We call *relative homomorphism* from A to B a map $h\colon A \to B$ such that, for every relative MV-subalgebra $D = P(a, b)$ of A, $h(D)$ is a relative MV-subalgebra of B and the restriction of h to D is an MV-homomorphism from D to $h(D)$. If h is an injective map, we shall say that h is a *relative isomorphism*.

Proposition 5. *Every relative homomorphism from A to B is an order preserving map.*

Proof. Let $a, b \in A$, $a \leqslant b$ and $D = P(a, b) = \{a, b\}$. By hypothesis $\{h(a), h(b)\}$ is a relative MV-subalgebra of B and the restriction of h to D is an MV-homomorphism from D to $h(D)$. Thus $h(a) \leqslant h(b)$. □

Proposition 6. *Every homomorphism h from A to B is a relative homomorphism.*

Proof. Let $D = P(a, b)$ be a relative subalgebra of A. By hypothesis $h(a) \leqslant h(x) \leqslant h(b)$, for every $x \in D$; thus $h(D) \subseteq [h(a), h(b)]$. Moreover $h(x \uplus y) = h((x \oplus a^* y) \wedge b) = (h(x) \oplus h(a)^* h(y)) \wedge h(b) = h(x) \uplus h(y)$; $h(\overline{x}) = h(a \oplus x^* b) = h(a) \oplus h(x)^* h(b) = \overline{h(x)}$. □

Corollary 7. *The identity 1_A, defined on the MV-algebra A is a relative homomorphism.*

There are relative homomorphisms which are not homomorphisms. As an example consider the two finite MV-chains $S_2 = \{0, \frac{1}{2}, 1\}$, $S_5 = \{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$ and the map h from S_2 to S_5, defined as $h(0) = \frac{1}{5}$, $h(\frac{1}{2}) = \frac{2}{5}$, $h(1) = \frac{3}{5}$. The mapping h is a relative homomorphism from S_2 to S_5, but is not a homomorphism from S_2 to S_5.

Theorem 8. *The class r$\mathscr{MV}$, whose objects are MV-algebras and whose morphisms are the relative homomorphisms, is a category.*

Proof. By Corollary 7, every object A has the identity.

Let us consider, as categorical composition, the ordinary composition of functions. Then it is immediate to show that $f \circ g$ is a relative homomorphism and that $(f \circ g) \circ k = f \circ (g \circ k)$, for every triplet f, g, k of relative homomorphisms. □

Definition 9. Let A and B be MV-algebras. We call *interval homomorphism* from A to B a map $h\colon A \to B$ such that, for every interval MV-subalgebra $D = [a, b]$ of A, $h(D)$ is an interval MV-subalgebra of B and the restriction of h to D is an MV-homomorphism from D to $h(D)$. If h is an injective map, we will say that h is an *interval isomorphism*

Proposition 10. *Every homomorphism h from A onto B is an interval homomorphism.*

Proof. We shall limit ourselves to verifying that if $D = [a, b]$ is an interval of A, then $h(D) = [h(a), h(b)]$. Being h a homomorphism, $h(D) \subseteq [h(a), h(b)]$. Let now $y \in [h(a), h(b)]$; by surjectivity of h, there is $x \in A$, such that $h(x) = y$. Thus $z = a \vee (x \wedge b) \in [a, b]$ and $h(z) = y$. □

Corollary 11. *The identity 1_A, defined on the MV-algebra A is an interval homomorphism.*

Theorem 12. *The class int$\mathscr{MV}$, whose objects are MV-algebras and whose morphisms are the interval homomorphisms, is a category.*

Proof. Analogous to the proof of Theorem 8. □

Theorem 13. *The category $\mathscr{MV}$ is a subcategory of int$\mathscr{MV}$, and int$\mathscr{MV}$ is a subcategory of r$\mathscr{MV}$.*

We notice that an example of interval homomorphism is given by the mapping $h_b\colon A \to A$ defined in the introduction. Furthermore, given the MV-algebras A and B and a map $h\colon A \to B$ such that $A \cong h(A) = [0, b]$ for some $b \in B$ and $[0, b]$ interval subalgebra of B, then h is an interval homomorphism from A to B. Hence genuine morphisms of the full subcategory of *int$\mathscr{MV}$* made by Boolean algebras are involved in the claim of a theorem of Cantor-Bernstein type already mentioned in the introduction. More precisely we have:

Theorem 14. *For any two σ-complete Boolean algebras A and B and elements $a \in A$, $b \in B$ and interval homomorphisms $\varphi\colon A \to B$ and $\psi\colon B \to A$ such that $\varphi(A)$ is MV-isomorphic to the interval algebra $[0, b]$, $\psi(B)$ is MV-isomorphic to the interval algebra $[0, a]$, then there is an interval isomorphism (actually an MV-isomorphism) between A and B.*

Similar translations can be obtained for other MV-algebraic generalizations of the Cantor-Bernstein theorem.

4. Free MV-algebras

Set $\varphi(1) = \{0,1\}$. For $n \in \mathbb{N} \setminus \{1\}$, we shall denote by $\varphi(n)$ the set of all $c \in \mathbb{N}$ such that $c < n$ and $\gcd(c,n) = 1$.

On the set of positive integers $\mathbb{N}$ we define the function $v_m(x)$ as follows: $v_m(1) = 2^m$, $v_m(2) = 3^m - 2^m$, $\ldots$, $v_m(p) = (p+1)^m - (v_m(n_1) + \cdots + v_m(n_{k-1}))$, where $n_1(=1), \ldots, n_{k-1}$ are all the divisors of p distinct from p. Then (see [3, Lemma 2.2])

$$F_p(m) \cong S_1^{v_m(1)} \times \cdots \times S_p^{v_m(p)}.$$

If we consider the case $m = 1$ and set $\varphi(1) = \{0,1\}$, then we have

$$F_p(1) \cong S_1^{v_1(1)} \times \cdots \times S_p^{v_1(p)}$$

where $v_1(i) = |\varphi(i)|$, for every $i = 1, 2, \ldots, p$.

It is known that:

Lemma 15. $\left|\bigcup\limits_{i=1}^{p} S_i\right| = \sum\limits_{i=1}^{p} |\varphi(i)| = p^2 \frac{3}{\pi^2} + p^2 \mathscr{O}\left(\frac{\lg p}{p}\right) + 1.$

Proof. For $p = 1$ the thesis is true. Then we proceed by induction. $\bigcup\limits_{i=1}^{p} S_i = \bigcup\limits_{i=1}^{p-1} S_i \cup \left\{\frac{k}{p} :\ k \in \varphi(p)\right\}$. Hence $\left|\bigcup\limits_{i=1}^{p} S_i\right| = \sum\limits_{i=1}^{p-1} |\varphi(i)| + |\varphi(p)| = \sum\limits_{i=1}^{p} |\varphi(i)|$. □

So every $f \in F_p(1)$ is a map $f\colon \bigcup\limits_{i=1}^{p} S_i \to \bigcup\limits_{i=1}^{p} S_i$, such that $f\left(\frac{p}{q}\right) \in S_q$, where $\frac{p}{q}$ is in irreducible form.

In the sequel, for every $p \in \mathbb{N}$, T_p will denote the increasing ranging of the elements of $\bigcup\limits_{i=1}^{p} S_i$.

Definition 16. Let X be a finite subset of $[0,1]$ and $x \in [0,1[$. We shall call *subsequent element* of x in X the smallest element of $\{y \in X :\ y > x\}$.

Analogously:

Definition 17. Let X be a finite subset of $[0,1]$ and $x \in]0,1]$. We shall call *previous element* of x in X the greatest element of $\{y \in X :\ y < x\}$.

If $u \geqslant 1$ is a positive real number, $[u]$ will denote the integer part of u, that is $[u] = \max\{n \in \mathbb{N} : \ n \leqslant u\}$.

Now, with the help of the results proved in Appendix, we are going to characterize the previous and subsequent element of a given element of T_p.

Proposition 18. *Let $\frac{k}{n} \in T_p$, then*

1. *the subsequent element of $\frac{k}{n}$ in T_p is the rational number $\frac{h}{m}$ such that $(h,m) \in S^+(k,n)$* (see Section 8) *and $m = \max\{n_0 + tn : \ n_0 + tn \leqslant p\} = n_0 + t_0 n$, $t_0 = \left[\frac{p-n_0}{n}\right]$;*
2. *the previous element of $\frac{k}{n}$ in T_p is the rational number $\frac{h}{m}$ such that $(h,m) \in S^-(k,n)$* (see Section 8) *and $m = \max\{tn - n_0 : \ tn - n_0 \leqslant p\} = t_1 n - n_0$, $t_1 = \left[\frac{p+n_0}{n}\right]$.*

P r o o f.

1. By Proposition 37, 2 and 4, $\frac{h}{m} > \frac{k}{n}$. Let $\frac{k}{n} < \frac{r}{s} < \frac{h}{m}$. Then by Lemma 39, $s \geqslant n + m = n + n_0 + t_0 n = n_0 + (t_0+1)n > p$. Since every element of T_p has a positive integer less than p as denominator, then $\frac{r}{s} \notin T_p$ and $\frac{h}{m}$ is the subsequent element of $\frac{k}{n}$ in T_p.

2. Analogous to 1, using Lemma 40. □

Consider the following sequences of elements of $[0,1]$:

$$D_1 = \left\{0, \tfrac{1}{2}, 1\right\}$$
$$D_2 = \left\{0, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, 1\right\}$$
$$D_3 = \left\{0, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, 1\right\}$$
$$D_4 = \left\{0, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{2}{7}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{5}{7}, \tfrac{3}{4}, \tfrac{4}{5}, 1\right\}$$
$$\vdots$$

Thus D_p is obtained by T_p and by inserting, between any two consecutive elements $\frac{a}{b}, \frac{c}{d} \in T_p$, their *mediant* $\frac{a+c}{b+d}$.

Remark 19. With the notations of Proposition 18,

(i) if $\frac{k}{n} \in T_p$ and $\frac{h}{m}$ is its subsequent element in T_p, then the mediant between $\frac{k}{n}$ and $\frac{h}{m}$ is the rational number $\frac{k_0+(t_0+1)k}{n_0+(t_0+1)n}$. Thus $(k+h, n+m) \in S^+(k,n)$ (see Section 8),

(ii) if $\frac{k}{n} \in T_p$ and $\frac{h}{m}$ is its previous element in T_p, then the mediant between $\frac{h}{m}$ and $\frac{k}{n}$ is the rational number $\frac{(t_0+1)k-k_0}{(t_0+1)n-n_0}$. Thus $(k+h, n+m) \in S^-(k,n)$ (see Section 8).

From (i) and (ii) it follows:

(iii) Let $\frac{k}{n} \in T_p$, $\frac{h}{m}$ and $\frac{h'}{m'}$ be the previous and subsequent element of $\frac{k}{n}$ in T_p, respectively. Moreover, let $\frac{a}{b}$ and $\frac{c}{d}$ be the mediants between $\frac{h}{m}$ and $\frac{k}{n}$ and between $\frac{k}{n}$ and $\frac{h'}{m'}$, respectively. Then $\frac{a+c}{b+d} = \frac{k}{n}$.

Analogous finite sequences of elements of $[0,1] \cap Q$ (the *Farey* partitions) are considered by the authors in [2], with the purpose to give a proof of McNaughton's theorem in the one-variable case. Any sequence D_p will be called *sub-Farey sequence* and in particular *sub-Farey$_p$ sequence*.

For $p = 1, 2, 3$, the sub-Farey$_p$ sequence and Farey$_p$ partition coincide.

Although the sub-Farey sequences and the Farey partitions share some properties, they differ from each other for $p \geqslant 4$. Indeed, for $p \geqslant 4$, the cardinality of Farey$_p$ is equal to $2^p + 1$, while $|D_p|$ increases in a polynomial way and Farey$_p \neq D_p$, as we shall clarify in Lemma 20, 6 and 7.

Set $D_p = T_p \cup M_p$, where M_p denotes the set of all mediants of the elements of T_p.

Lemma 20.

1. *For every $p \in \mathbb{N}$, $D_p \subseteq D_{p+1}$;*
2. *all fractions in D_p are in the irreducible form;*
3. *for any two consecutive fractions $\frac{a}{b} < \frac{c}{d}$ in T_p, $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$;*
4. *every irreducible fraction $\frac{r}{s} \in [0,1]$ occurs in some D_p;*
5. *the interval $[\frac{a}{b}, \frac{c}{d}]$ determined by any two consecutive fractions $\frac{a}{b} < \frac{c}{d}$ in D_p has the unimodularity property $cb - ad = 1$;*
6. *for $p \geqslant 4$, $\mathrm{Farey}_p \neq D_p$;*
7. $|D_p| = 2\sum_{i=1}^{p} |\varphi(i)| - 1 = \frac{6p^2}{\pi^2} + 2p^2 \mathscr{O}\left(\frac{\log p}{p}\right) + 1.$

Proof.

1. If $x \in T_p$, then $x \in T_{p+1} \subseteq D_{p+1}$.

Consider $x = \frac{r}{s} \in M_p \setminus T_{p+1}$ and let $\frac{k}{n}$ and $\frac{h}{m}$ be the previous and consecutive elements of x in T_p. Thus $s = n + m \geqslant p + 2$.

Let now $\frac{p}{q}$ be any element such that $\frac{k}{n} \leqslant \frac{p}{q} \leqslant \frac{h}{m}$. From Lemmas 39 and 40, $q \geqslant n + m \geqslant p + 2$, hence $\frac{p}{q} \notin T_{p+1}$.

Thus we can conclude that $\frac{k}{n}$ and $\frac{h}{m}$ are consecutive also in T_{p+1} and that $x = \frac{r}{s} \in M_{p+1} \subseteq D_{p+1}$.

2. It follows from Remark 33, 2 and Remark 19, (i), (ii).
3. It follows from Propositions 18 and 37, and Remark 19, (i), (ii).
4. Trivially $\frac{r}{s} \in T_s \subseteq D_s$.
5. It follows from Remark 19, (i), (ii).

6. We recall that, by Proposition 18, 1, if $\frac{a}{b} \in D_p$, then $b = r + s$ with $r, s \leqslant p$ and $r \neq s$. Hence $b \leqslant 2p - 1$.

For $p = 4$, as we said above, $\frac{3}{8} \notin D_4$, while $\frac{3}{8} \in \mathrm{Farey}_4$.

Assume now $p = 4 + q$, $q \geqslant 1$. The subsequent element of $\frac{3}{8}$ in Farey_p is equal to $\frac{3q+2}{8q+5}$ and $\frac{3q+2}{8q+5} \notin D_p$. Indeed $8q + 5 \leqslant 2p - 1$ implies $q \leqslant 0$, a contradiction.

7. It follows from Lemma 15. □

As we shall see, each sequence D_p allows us to cut out a relative subalgebra of $F(1)$, which is isomorphic to $F_p(1)$. Indeed now we are going to map the set of sub-Farey sequences to a subset of McNaughton functions.

For every $f \in F_p(1)$, let F be the following function:

$$F\colon x \in D_p \to \begin{cases} f(x) & \text{if } x \in \bigcup_{i=1}^{p} S_i, \\ 0 & \text{if } x \notin \bigcup_{i=1}^{p} S_i. \end{cases}$$

For every $x \in [0,1] \setminus D_p$, let x_i be the previous element of x in D_p and x_{i+1} the subsequent element of x in D_p.

Finally set

$$g_p(f)\colon x \in [0,1] \to \begin{cases} \frac{F(x_{i+1}) - F(x_i)}{(x_{i+1} - x_i)}(x - x_i) + F(x_i) & \text{if } x \in [0,1] \setminus D_p, \\ F(x) & \text{otherwise.} \end{cases}$$

Thus $g_p(f)$ is a continuous piecewise linear function, whose graph consists of the segments joining the points $(x_j, F(x_j))$, $x_j \in D_p$.

Let u_p be the unit of $F_p(1)$, $v_p = g_p(u_p)$ and $G_p(1) = \{g_p(f) : f \in F_p(1)\}$.

For every $g = g_p(f) \in G_p(1)$ let $Z(g) = g^{-1}(0)$ be the zeroset of g. Then, with the above notation, by definitions and Remark 19(iii), we get:

Lemma 21. *Let $g \in G_p(1)$, then we get:*

1. *$Z(g) \supset D_p \setminus T_p$;*
2. *if $\frac{a}{b}$ and $\frac{c}{d}$ are two consecutive elements of $D_p \setminus T_p$, then $Z(g) \supset [\frac{a}{b}, \frac{c}{d}]$ iff $f(\frac{a+c}{b+d}) = 0$.*

Theorem 22.

1. *For every $f \in F_p(1)$, $g_p(f)$ is a McNaughton function.*
2. *g_p is an injective map from $F_p(1)$ onto $G_p(1) \subseteq F(1)$.*

Proof.

1. We have to show that the coefficients of $g_p(f)$ are integer numbers.

a) Let $x_i = \frac{k}{n} \in T_p$ and let $\frac{h}{m}$ be its subsequent element in T_p. Then $x_{i+1} \in D_p \setminus T_p$ and $x_{i+1} = \frac{k+h}{n+m}$.

Recalling that $F(x_i) = \frac{k'}{n}$, $k' \in \{0, \ldots, n\}$, $F(x_{i+1}) = 0$ and $(k+h, n+m) \in S^+(k, n)$ (see Section 8 and Remark 19), we have $\frac{F(x_{i+1})-F(x_i)}{(x_{i+1}-x_i)} = \frac{-\frac{k'}{n}}{\frac{1}{n(n+m)}} = -k'(n+m) \in \mathbb{Z}$.

Moreover, using Remark 19, $\frac{F(x_{i+1})-F(x_i)}{(x_{i+1}-x_i)}(-x_i) + F(x_i) = -k'(n+m)\left(-\frac{k}{n}\right) + \frac{k'}{n} = \frac{k'}{n}((n+m)k+1) = \frac{k'}{n}n(h+k) = k'(h+k) \in \mathbb{Z}$.

b) Let $x_{i+1} = \frac{k}{n} \in T_p$ and let $\frac{h}{m}$, $((h, m) \in S^-(k, n))$ (see Section 8) be its previous element in T_p.

Then $x_i \in D_p \setminus T_p$ and $x_i = \frac{k+h}{n+m}$.

Recalling that $F(x_{i+1}) = \frac{k'}{n}$, $k' \in \{0, \ldots, n\}$, $F(x_i) = 0$ and $(k+h, n+m) \in S^-(k, n)$ (see Section 8 and Remark 19), we have

$$\frac{F(x_{i+1}) - F(x_i)}{(x_{i+1} - x_i)} = \frac{\frac{k'}{n}}{\frac{1}{n(n+m)}} k'(n+m) \in \mathbb{Z}.$$

Moreover, using Remark 19, $\frac{F(x_{i+1})-F(x_i)}{(x_{i+1}-x_i)}(-x_i) + F(x_i) = k'(n+m)\left(-\frac{k}{n}\right) + \frac{k'}{n} = -\frac{k'}{n}((n+m)k+1) = -\frac{k'}{n}n(h+k) = -k'(h+k) \in \mathbb{Z}$.

2. Let $f, f' \in F_p(1)$ and $f \neq f'$. Then there is an element $x \in T_p$ such that $f(x) \neq f'(x)$. Since $g_p(f)(x) = f(x)$ and $g_p(f')(x) = f'(x)$, we get $g_p(f) \neq g_p(f')$.

The theorem is completely proved.

□

Remark 23. For any $p \in \mathbb{N}$ and $\frac{k}{n} \in T_p$, set:

1. $\frac{h'}{m'}$ as the previous element of $\frac{k}{n}$ in T_p if $\frac{k}{n} \in T_p \setminus \{0\}$,
2. $\frac{h''}{m''}$ as the subsequent element of $\frac{k}{n}$ in T_p if $\frac{k}{n} \in T_p \setminus \{1\}$,
3. $\alpha_p(\frac{k}{n})(x) = ((m'+n)x-(h'+k))^{\#} \wedge (-(m''+n)x+(h''+k))^{\#}$ if $\frac{k}{n} \in T_p \setminus \{0, 1\}$,
4. $\alpha_p(0)(x) = \alpha_p\left(\frac{0}{1}\right)(x) = (1-(p+1)x)^{\#}$,
5. $\alpha_p(1)(x) = \alpha_f\left(\frac{1}{1}\right)(x) = ((p+1)x-p)^{\#}$.

Then, by Remark 23, each $\alpha_p\left(\frac{k}{n}\right)$, $p \in \mathbb{N}$, $\frac{k}{n} \in T_p$, is, like a Schauder hat (see [2, p. 58]), a function whose graph consists of the four segments joining the points $(0, 0)$, $\left(\frac{k+h'}{n+m'}, 0\right)$, $\left(\frac{k}{n}, \frac{1}{n}\right)$, $\left(\frac{k+h''}{n+m''}, 0\right)$, $(1, 0)$.

Proposition 24. *Let $p \in \mathbb{N}$. Then*

1. *for any two elements* $\frac{k}{n}, \frac{r}{s} \in T_p$, $\frac{k}{n} \neq \frac{r}{s}$,
   (i) $\left(k'\alpha_p\left(\frac{k}{n}\right) \oplus \chi'\alpha_p\left(\frac{k}{n}\right)\right) \wedge s\alpha_p\left(\frac{r}{s}\right) = 0,\ 0 \leqslant k', \chi' \leqslant n;$
   (ii) $\left(\alpha_p\left(\frac{k}{n}\right)\right)^* \left(\rho\alpha_p\left(\frac{r}{s}\right)\right) = \rho\alpha_p\left(\frac{r}{s}\right),\ 0 \leqslant \rho \leqslant s.$
2. *If* $f \in F_p(1)$ *and* $f(\frac{k}{n}) = \frac{k'}{n}$, *then*

$$k'\alpha_p\left(\frac{k}{n}\right) = \begin{cases} g_p(f) & \text{if } x \in \left[\frac{h'+k}{m'+n}, \frac{h''+k}{m''+n}\right], \\ 0 & \text{otherwise.} \end{cases}$$

P r o o f.

1. Assume $\frac{k}{n} < \frac{r}{s}$. Let $\frac{u'}{v'}$ and $\frac{u''}{v''}$ be the previous and the subsequent element of $\frac{r}{s}$ in T_p, respectively.

To show (i), we consider three different cases.

a) Let $0 \leqslant x \leqslant \frac{u'+r}{v'+s}$.
Then $\left(k'\alpha_p\left(\frac{k}{n}\right) \oplus \chi'\alpha_p\left(\frac{k}{n}\right)\right) \wedge s\alpha_p(\frac{r}{s})(x) \leqslant s\alpha_p(\frac{r}{s})(x) \leqslant (s(v'+s)x-(u'+r))^{\#}$ $= 0$. Indeed $(v'+s)x-(u'+r) \leqslant 0$, for $x \leqslant \frac{u'+r}{v'+s}$.

b) Let $\frac{u''+r}{v''+s} \leqslant x \leqslant 1$.
Then $\left(k'\alpha_p\left(\frac{k}{n}\right) \oplus \chi'\alpha_p\left(\frac{k}{n}\right)\right) \wedge s\alpha_p(\frac{r}{s})(x) \leqslant s\alpha_p(\frac{r}{s})(x) \leqslant (-s[(v''+s)x - (u''+r)])^{\#} = 0$. Indeed $-[(v''+s)x-(u''+r)] \leqslant 0$, for $x \geqslant \frac{u''+r}{v''+s}$.

c) Let $\frac{u'+r}{v'+s} \leqslant x \leqslant \frac{u''+r}{v''+s}$; thus $x \geqslant \frac{k+h''}{n+m''}$.
Then $\left(k'\alpha_p\left(\frac{k}{n}\right) \oplus \chi'\alpha_p\left(\frac{k}{n}\right)\right) \wedge s\alpha_p\left(\frac{r}{s}\right)(x) \leqslant \left(k'\alpha_p\left(\frac{k}{n}\right) \oplus \chi'\alpha_p\left(\frac{k}{n}\right)\right) \leqslant (-n[(m''+n)x-(h''+k)])^{\#} = 0$. Indeed $-[(m''+n)x-(h''+k)] \leqslant 0$, for $x \geqslant \frac{k+h''}{n+m''}$.

To show (ii), observe that, for $x \in [0, \frac{u'+r}{v'+s}]$, $\rho\alpha_p(\frac{r}{s}) = 0$, while, for $x \in [\frac{u'+r}{v'+s}, 1]$, $\left(\alpha_p\left(\frac{k}{n}\right)\right)^* = 1$.

2. Proving 1, it remains to show that $k'\alpha_p\left(\frac{k}{n}\right) = g_p(f)$ on $\left[\frac{h'+k}{m'+n}, \frac{h''+k}{m''+n}\right]$. It is enough to consider that both coincide with the line for $\left(\frac{h'+k}{m'+n}, 0\right)$ and $\left(\frac{k}{n}, \frac{k'}{n}\right)$ on $\left[\frac{h'+k}{m'+n}, \frac{k}{n}\right]$ and the line for $\left(\frac{k}{n}, \frac{k'}{n}\right)$ and $\left(\frac{h''+k}{m''+n}, 0\right)$ on $\left[\frac{k}{n}, \frac{h''+k}{m''+n}\right]$. □

Corollary 25. *For every* $f \in F_p(1)$,

$$g_p(f) = \bigvee_{\frac{k}{n} \in T_p} k'\alpha_p\left(\frac{k}{n}\right)$$

where

1. $f\left(\frac{k}{n}\right) = \frac{k'}{n}$,
2. $\frac{h'}{m'}$ *is the previous element of* $\frac{k}{n}$ *in* T_p *if* $\frac{k}{n} \in T_p \setminus \{0\}$,
3. $\frac{h''}{m''}$ *is the subsequent element of* $\frac{k}{n}$ *in* T_p *if* $\frac{k}{n} \in T_p \setminus \{1\}$,
4. $\alpha_p\left(\frac{k}{n}\right)(x)$ *is defined like in* 3,4,5 of Remark 23.

With notations of Section 1, Proposition 24 and Corollary 25 we get:

Theorem 26. *For every* $p \in \mathbb{N}$,

1. $(G_p(1), \uplus, ^-, \underline{0})$ *is a relative MV-subalgebra of* $F(1)$, *where* $\overline{0} = g_p(u_p) = v_p$;
2. g_p *is an MV-isomorphism between* $F_p(1)$ *and MV-algebra* $(G_p(1), \uplus, ^-, \underline{0})$.

Proof. Let $h = g_p(f) \in G_p(1)$. Then, by Corollary 25,

$$g_p(f) = \bigvee_{\frac{k}{n} \in T_p} k'\alpha_p\left(\frac{k}{n}\right) \leqslant \bigvee_{\frac{k}{n} \in T_p} n\alpha_p\left(\frac{k}{n}\right) = g_p(u_p) = v_p.$$

Thus $\underline{0} \leqslant h \leqslant v_p$, for every $h \in G_p(1)$ and $\overline{0} = \underline{0} \oplus \underline{0}^* v_p = v_p$.

Now we shall prove that $g_p(f \oplus g) = (g_p(f) \oplus g_p(g)) \wedge v_p$ and that $g_p(f^*) = (g_p(f))^* v_p$, for every for every $f, g \in F_p(1)$.

Claim 1. $g_p(f \oplus g) = (g_p(f) \oplus g_p(g)) \wedge v_p$.

Set $f\left(\frac{k}{n}\right) = \frac{k'}{n}$, $g\left(\frac{k}{n}\right) = \frac{\chi'}{n}$ and $k' \oplus \chi' = \min(k' + \chi', n)$.
With these notations $(f \oplus g)(\frac{k}{n}) = \frac{k' \oplus \chi'}{n}$.
By Corollary 25

$$g_p(f \oplus g) = \bigvee_{\frac{k}{n} \in T_p} (k' \oplus \chi')\alpha_p\left(\frac{k}{n}\right).$$

Besides, applying Corollary 25 and Proposition 24,1, for $g(\frac{r}{s}) = \frac{\rho'}{s}$, we have:

$$
\begin{aligned}
&(g_p(f) \oplus g_p(g)) \wedge v_p \\
&= \left(\bigvee_{\frac{k}{n} \in T_p} k' \alpha_p \left(\frac{k}{n}\right) \oplus \bigvee_{\frac{r}{s} \in T_p} \rho' \alpha_p \left(\frac{r}{s}\right) \right) \wedge \left(\bigvee_{\frac{u}{v} \in T_p} v \alpha_p \left(\frac{u}{v}\right) \right) \\
&= \left(\bigvee_{\frac{k}{n} \in T_p} \left(k' \alpha_p \left(\frac{k}{n}\right) \oplus \chi' \alpha_p \left(\frac{k}{n}\right) \right) \vee \bigvee_{\frac{k}{n} \neq \frac{r}{s} \in T_p} \left(k' \alpha_p \left(\frac{k}{n}\right) \vee \rho' \alpha_p \left(\frac{r}{s}\right) \right) \right) \\
&\qquad \wedge \left(\bigvee_{\frac{u}{v} \in T_p} v \alpha_p \left(\frac{u}{v}\right) \right) \\
&= \left(\bigvee_{\frac{k}{n} \in T_p} \left(k' \alpha_p \left(\frac{k}{n}\right) \oplus \chi' \alpha_p \left(\frac{k}{n}\right) \right) \right) \wedge \left(\bigvee_{\frac{u}{v} \in T_p} v \alpha_p \left(\frac{u}{v}\right) \right) \\
&= \bigvee_{\frac{k}{n} \in T_p} \left(k' \alpha_p \left(\frac{k}{n}\right) \oplus \chi' \alpha_p \left(\frac{k}{n}\right) \right) \wedge n \alpha_p \left(\frac{k}{n}\right).
\end{aligned}
$$

In the last expression

if $k' + \chi' \geqslant n$, then $k'\alpha_p\left(\frac{k}{n}\right) \oplus \chi'\alpha_p\left(\frac{k}{n}\right) \geqslant n\alpha_p\left(\frac{k}{n}\right)$,
so $\left(k'\alpha_p\left(\frac{k}{n}\right) \oplus \chi'\alpha_p\left(\frac{k}{n}\right)\right) \wedge n\alpha_p\left(\frac{k}{n}\right) = n\alpha_p\left(\frac{k}{n}\right)$,
if $k' + \chi' < n$, then $k'\alpha_p\left(\frac{k}{n}\right) \oplus \chi'\alpha_p\left(\frac{k}{n}\right) < n\alpha_p\left(\frac{k}{n}\right)$ and
$\left(k'\alpha_p\left(\frac{k}{n}\right) \oplus \chi'\alpha_p\left(\frac{k}{n}\right)\right) \wedge n\alpha_p\left(\frac{k}{n}\right) = (k' + \chi')\alpha_p\left(\frac{k}{n}\right)$.

Therefore

$$(g_p(f) \oplus g_p(g)) \wedge v_p = \bigvee_{\frac{k}{n} \in T_p} (k' \oplus \chi')\, \alpha_p \left(\frac{k}{n}\right) = g_p(f \oplus g).$$

Claim 2. $g_p(f^*) = (g_p(f))^* v_p$.

By Corollary 25

$$g_p(f^*) = \bigvee_{\frac{k}{n} \in T_p} (n - k')\, \alpha_p \left(\frac{k}{n}\right).$$

Moreover, applying Proposition 24,1 and the distributive property of the product with respect to $\wedge$ and $\vee$,

$$
\begin{aligned}
(g_p(f))^* v_p &= \left(\bigvee_{\frac{k}{n}\in T_p} k'\alpha_p\left(\frac{k}{n}\right)\right)^* \left(\bigvee_{\frac{r}{s}\in T_p} s\alpha_p\left(\frac{r}{s}\right)\right) \\
&= \left(\bigwedge_{\frac{k}{n}\in T_p} \left(k'\alpha_p\left(\frac{k}{n}\right)\right)^*\right) \left(\bigvee_{\frac{r}{s}\in T_p} s\alpha_p\left(\frac{r}{s}\right)\right) \\
&= \bigvee_{\frac{r}{s}\in T_p} \left(\bigwedge_{\frac{k}{n}\in T_p} \left(k'\alpha_p\left(\frac{k}{n}\right)\right)^* s\alpha_p\left(\frac{r}{s}\right)\right) \\
&= \bigvee_{\frac{r}{s}\in T_p} \left(\rho'\alpha_p\left(\frac{r}{s}\right)\right)^* s\alpha_p\left(\frac{r}{s}\right),
\end{aligned}
$$

where $f(\frac{r}{s}) = \frac{\rho'}{s}$, $0 \leqslant \rho' \leqslant s$.

In conclusion

$$
g_p(f)^* v_p = \bigvee_{\frac{r}{s}\in T_p} (s-\rho')\alpha_p\left(\frac{r}{s}\right) = g_p(f^*).
$$

The above statements show that $G_p(1)$ is closed under $\uplus$ and $^-$. Thus $G_p(1)$ is a relative subalgebra of $F(1)$ and g_p respects the operations.

By Theorem 22,2 g_p is an MV-isomorphism between $F_p(1)$ and MV-algebra $(G_p(1), \uplus, ^-, \underline{0})$. □

With the aim to give a definition of *relative directed family* of MV-algebras, we introduce some notations.

For $p \leqslant q$, set

$$
F_{p,q}(1) = \left\{ f \in F_q(1) : \ f(x) = 0 \text{ for every } x \in \bigcup_{i\in p+1}^{q} S_i \right\}.
$$

Let $u_{p,q} \in F_{p,q}(1)$ be the function defined by:

$$
u_{p,q}(x) = \begin{cases} 1 & \text{if } x \in \bigcup\limits_{i=1}^{p} S_i, \\ 0 & \text{if } x \in \bigcup\limits_{i=p+1}^{q} S_i. \end{cases}
$$

Then with the above notations we have:

Proposition 27. *For $p, q \in \mathbb{N}$ and $p \leqslant q$, $F_{p,q}(1) = P(\underline{0}, u_{p,q})$ is a relative subalgebra of $F_q(1)$.*

P r o o f. It is easy to check that $\underline{0}$ and $u_{p,q}$ are the smallest and the greatest element in $F_{p,q}(1)$, respectively, and that $F_{p,q}(1)$ is closed with respect to the operations $\uplus$ and $^-$, as defined in Section 1. □

For $p, q \in \mathbb{N}$ and $p \leqslant q$, we define the mapping

$$e_{p,q} \colon F_p(1) \to F_{p,q}(1),$$

as follows:

$$e_{p,q}(f)(x) = \begin{cases} f(x) & \text{if } x \in \bigcup_{i=1}^{p} S_i, \\ 0 & \text{if } x \in \bigcup_{i=p+1}^{q} S_i. \end{cases}$$

Proposition 28. *For $p, q \in \mathbb{N}$ and $p \leqslant q$, $e_{p,q}$ is a relative isomorphism from $F_p(1)$ to $F_q(1)$.*

P r o o f. Trivial. □

Finally we set

$$\varphi_{p,q} \colon h \in G_p(1) \to k \in G_q(1),$$

where k is defined by

$$k = g_q(e_{p,q}(g_p^{-1}(h))).$$

Proposition 29. *For $p \leqslant q$, $\varphi_{p,q} = g_q \circ e_{p,q} \circ g_p^{-1}$ is a relative isomorphism from $G_p(1)$ to $G_q(1)$.*

P r o o f. It follows by Theorems 8 and 22 and Proposition 27. □

Definition 30. A *relative directed family of MV-algebras* is defined to be a triplet of the following objects:

(i) A directed partially ordered set $(I, \leqslant)$;

(ii) a family of MV-algebras $(A_i)_{i\in I}$;

(iii) a family of relative homomorphisms $\varphi_{i,j}$ from A_i to A_j, for all $i \leqslant j$ such that

$$\varphi_{i,j}\varphi_{j,k} = \varphi_{i,k} \qquad \text{if} \quad i \leqslant j \leqslant k$$

and $\varphi_{i,i}$ is the identity map for all $i \in I$.

Proposition 31. $((G_p(1))_{p\in\mathbb{N}}, \varphi_{p,q})$ *is a relative directed family of MV-algebras.*

P r o o f. From Propositions 27 and 28 and Theorem 8. □

5. A retraction

Let p be a positive integer. We define the following binary relation on $F(1)$: $s, t \in F(1)$,

$$s \equiv_p t$$

iff

$$\forall x \in T_p \subseteq [0,1] \quad s(x) = t(x).$$

The relation $\equiv_p$ is a congruence of $F(1)$ and $(F(1)/\equiv_p) \cong F_p(1)$. Thus, by $\equiv_p$, we can define an MV-homomorphism h_p from $F(1)$ to $F_p(1)$. In symbols

$$h_p \colon f \in F(1) \to f_{T_p}.$$

Then the map $g_p \circ h_p$, which we shall denote by δ_p, is a relative-homomorphism from $F(1)$ to $G_p(1)$. Since $G_p(1)$ is a relative-subalgebra of $F(1)$, the identity map i_p provides a relative-homomorphism from $G_p(1)$ to $F(1)$, too.

Summarizing, we get the following relative-homomorphisms:

$$\delta_p \colon F(1) \to G_p(1) \qquad i_p \colon G_p(1) \to F(1).$$

A direct inspection proves that the following relation holds:

$$\delta_p \circ i_p = ID_{G_p(1)},$$

being $ID_{G_p(1)}$ the identity map of $G_p(1)$. By the above relation we get:

Proposition 32. *For every $p \in \mathbb{N}$, $G_p(1)$ is a retract of $F(1)$ in the category $r\mathscr{MV}$.*

6. Appendix

Let $\mathbb{Z}$ be the set of all the integers, and $\mathbb{N} = \mathbb{Z}^+$ be the set of all the positive integers and $\mathbb{Z}^-$ the set of all the negative integers.

Let $n \in \mathbb{N} \setminus \{1\}$ and $k \in \varphi(n)$. It is well known that the set

$$S(k,n) = \{(h,m) \in \mathbb{Z} \times \mathbb{Z} : \ hn - mk = 1\} \neq \emptyset$$

and that

$$S(k,n) \subseteq (\mathbb{Z}^- \cup \{0\} \times \mathbb{Z}^-) \cup (\mathbb{Z}^+ \times \mathbb{Z}^+).$$

To make easier the notations we set:

$$S^+(k,n) = S(k,n) \cap (\mathbb{Z}^+ \times \mathbb{Z}^+),$$

$$S^-(k,n) = \left\{(h,m) \in \mathbb{Z} \times \mathbb{Z} : \ (-h,-m) \in S(k,n) \cap (\mathbb{Z}^- \cup \{0\} \times \mathbb{Z}^-)\right\},$$

$$-S^-(k,n) = S(k,n) \cap (\mathbb{Z}^- \cup \{0\} \times \mathbb{Z}^-).$$

So

$$|S(k,n)| = S^+(k,n) \cup S^-(k,n) \subseteq \mathbb{Z}^+ \cup \{0\} \times \mathbb{Z}^+,$$

and

$$S(k,n) = S^+(k,n) \cup -S^-(k,n).$$

Remark 33.

1. $(h,m) \in S^-(k,n)$ if and only if $(h,m) > (0,0)$ and $hn - mk = -1$.
2. for any $(h,m) \in |S(k,n)|$, and g.c.d. of h, m is 1.

With the above notations we have:

Lemma 34. *Let $k \in \varphi(n)$ and $(h,m) \in |S(k,n)|$. Then the following statements hold:*

1. *If $(h,m) \in S^+(k,n)$, then $h \leqslant m$, and $h = m$ if and only if $h = m = 1$ and $k = n-1$;*
2. *if $(h,m) \in S^-(k,n)$, then $h < m$;*
3. *$h \in \varphi(m)$.*

P r o o f.

1. If $h > m$, then $mk + 1 = hn > mn = m(n-1) + m \geqslant m(n-1) + 1$. From that $k > n-1$, absurd.

To show the second part of 1, it is enough to observe that $h = m$ is equivalent to $h(n-k) = m(n-k)= 1$.

2. If $h \geqslant m$, then we get $-1 = hn - mk \geqslant m(n-k) > 0$, absurd.

3. It is trivial. □

From Lemma 34 for every $(h,m) \in |S(k,n)|$, $\frac{h}{m} \in [0,1]$ where $\frac{h}{m}$ is in the irreducible form.

Lemma 35. *Let $k \in \varphi(n)$. Then there is a pair of integer numbers $(h,m) \in S(k,n)$ satisfying the following properties:*

1. *$(h,m) \in S^+(k,n)$ and $m < n$,*
2. *$h \leqslant k$,*
3. *$h = k$ if and only if $h = k = 1$ and $m = n-1$.*

P r o o f.

1. Since $k \in \varphi(n)$, the congruencial equation modulo n, $xk \cong -1(n)$, has solutions, which constitute a whole class in the set of the classes modulo n. Thus there is an integer m such that $0 < m < n$ and $mk \cong -1(n)$, that is $mk + 1 = hn$, for some $h > 0$.

2. If $h > k$, then $1 > k(n-m) \geqslant k$, absurd.

3. It is enough to observe that $h = k$ is equivalent to $k(n-m) = h(n-m) = 1$. □

Proposition 36. *Let $k \in \varphi(n)$. Then there is just a pair $(h,m) \in S(k,n)$ such that*

1. *$(h,m) \in S^+(k,n)$ and $m < n$,*
2. *$h \leqslant k$,*
3. *$h < k$ or $h = k = 1$ and $m = n-1$.*

Proof. By Lemmas 34 and 35 such a pair exists. Assume now there are two elements in $S(k,n)$ with the above properties. Let them be (h,m) and (h',m'). Being $hn - mk = h'n - mk'$, we have $(h-h')n = (k'-k)m$. Thus m divides $h-h'$ ($m \in \varphi(n)$), which is absurd, since $|h-h'| < m$. □

In the sequel we shall denote by (k_0,n_0) the unique element of $S^+(k,n)$ with the properties of Proposition 36.

Proposition 37. *Let $k \in \varphi(n)$. Then we get:*

1. $S(k,n) = \{(h,m) \in \{((\mathbb{Z}^- \cup \{0\}) \times \mathbb{Z}^-) \cup (\mathbb{Z}^+ \times \mathbb{Z}^+) :$
$$(\exists t \in \mathbb{Z})((h,m) = (k_0,n_0) + t(k,n))\},$$
2. *$(h,m) \in S^+(k,n)$ if and only if $t \in \mathbb{N} \cup \{0\}$,*
3. $S^-(k,n) = \{-(k_0,n_0) + t(k,n) : \ t \in \mathbb{N}\}$,
4. $\left(\frac{k_0+tk}{n_0+tn}\right)_{t \in \mathbb{N}\cup\{0\}}$ *is a strictly decreasing sequence and* $\lim_t \frac{k_0+tk}{n_0+tn} = \frac{k}{n}$,
5. $\left(\frac{-k_0+tk}{-n_0+tn}\right)_{t \in \mathbb{N}}$ *is a strictly increasing sequence and* $\lim_t \frac{-k_0+tk}{-n_0+tn} = \frac{k}{n}$,
6. $(k_0,n_0) = \min S^+(k,n)$,
7. *for every $(h,m) \in S(k,n)$, $\frac{h}{m} > \frac{k}{n}$ if and only if $t \in \mathbb{N} \cup \{0\}$.*

Proof.

1. By an easy calculation we can prove that $(k_0,n_0) + t(k,n) \in S(k,n)$, for every $t \in \mathbb{Z}$. Assume now $(h,m) \in S(k,n)$. Then $hn - mk = 1$ and $(h-k_0)n = (m-n_0)k$. Since $k \in \varphi(n)$, k divides $(h-k_0)$ and $h = k_0 + tk$, for some $t \in \mathbb{Z}$. Analogously n divides $(m-n_0)$ and $m = n_0 + sk$, for some $s \in \mathbb{Z}$. Substituting h and m in $hn - mk = 1$, we get $t = s$. Thus 1 is proved.

2. It is trivial that $t \in \mathbb{N} \cup \{0\}$ implies $(h,m) \in S^+(k,n)$. Assume $(k_0,n_0) + t(k,n)) \in S^+(k,n)$. Then $k_0 + tk > 0$ and by Proposition 35, 2, $t > -\frac{k_0}{k} \geqslant -1$, that is $t \in \mathbb{N} \cup \{0\}$.

3. From 1 and 2.

4. Reminding that $nk_0 - kn_0 = 1$, by an easy calculation we get

$$\frac{k_0+(t+1)k}{n_0+(t+1)n} - \frac{k_0+tk}{n_0+tn} = -\frac{1}{(n_0+(t+1)n))(n_0+tn)} < 0.$$

Thus

$$\frac{k_0+(t+1)k}{n_0+(t+1)n} < \frac{k_0+tk}{n_0+tn} \qquad \text{and} \qquad \lim_t \frac{k_0+tk}{n_0+tn} = \frac{k}{n}.$$

5. As in 3, since $nk_0 - kn_0 = 1$,

$$\frac{-k_0+(t+1)k}{-n_0+(t+1)n} - \frac{-k_0-+k}{-n_0+tn} = \frac{1}{(-n_0+(t+1)n))(-n_0+tn)} > 0.$$

Thus

$$\frac{-k_0+(t+1)k}{-n_0+(t+1)n} > \frac{-k_0+tk}{-n_0+tn} \qquad \text{and} \qquad \lim_t \frac{k_0-tk}{n_0-tn} = \frac{k}{n}.$$

6. It follows immediately from 2.

7. Let $h = k_0 + tk$ and $m = n_0 + tn$, then

$$\frac{h}{m} > \frac{k}{n} \iff \frac{h}{m} - \frac{k}{n} = \frac{1}{n(n_0+tn)} > 0,$$

that is if and only if $n_0 + tn > 0$. By Proposition 36, 1, $n_0 + tn > 0$ if and only if $t \geqslant 0$. □

Lemma 38. *$(h, m) \in S^+(k, n) \setminus \{(1, 1)\}$ if and only if $(k, n) \in S^-(h, m)$.*

Moreover, if

$$h = k_0 + tk \qquad \textit{and} \tag{1}$$

$$m = n_0 + tn, \tag{2}$$

then we get:

for $m > n$,

$$h_0 = k_0 + (t-1)k \qquad \textit{and} \tag{3}$$

$$m_0 = n_0 + (t-1)n; \tag{4}$$

for $h = k_0$ and $m = n_0$,

$$h_0 = t'k_0 - k \qquad \textit{and} \tag{5}$$

$$m_0 = t'n_0 - n, \qquad \textit{where} \quad t' = \left[\frac{n}{n_0}\right] + 1. \tag{6}$$

P r o o f. $1 = hn - mk$ if and only if $km - nh = -1$. Hence the first statement is proved.

Let $m > n$. We claim that $(k_0 + (t-1)k, n_0 + (t-1)n) \in S(h, m)$.

Indeed, by (1) and (2), $(k_0 + (t-1)k)m - (n_0 + (t-1)n)h = k_0 m - n_0 h + (t-1)(km - nh) = t - (t-1) = 1$.

Moreover $0 < n_0 + (t-1)n < n_0 + tn = m$, thus (3) and (4) follow by Proposition 37.

Let now $h = k_0$, $m = n_0$. Dividing n by n_0, we get $n = (t'-1)n_0 + r,\ r < n_0$.

From that $0 < t'n_0 - n = n_0 + (t'-1)n_0 - n = n_0 - r < n_0 = m$. Since $(t'k_0 - k, t'n_0 - n) \in S(k_0, n_0)$, (5) and (6) follow by Proposition 37. □

Lemma 39. *Let* $\frac{k}{n}, \frac{h}{m} \in]0,1[$, $(h,m) \in S^+(k,n)$ *and* $\frac{k}{n} < \frac{r}{s} < \frac{h}{m}$. *Then* $s \geqslant n + m$.

Proof. If $h = m = 1$, then $k = n - 1$ and the thesis is trivial.

By hypothesis $(h,m) \in S^+(k,n) \setminus \{(1,1)\}$ we get:

$$hn = km + 1, \tag{7}$$

$$m = t_0 n + n_0 \quad \text{for some} \quad t_0 \in \mathbb{N} \cup \{0\} \quad \text{(Proposition 37, 2)}, \tag{8}$$

$$(k,n) \in S^-(h,m) \quad \text{(Lemma 38)}, \tag{9}$$

$$n = t_1 m - m_0 \quad \text{for some} \quad t_1 \in \mathbb{N} \quad \text{(Proposition 37, 3)}. \tag{10}$$

By hypothesis $\frac{k}{n} < \frac{r}{s} < \frac{h}{m}$,

$$kms < rnm < hns. \tag{11}$$

Using (7) and dividing by m the three terms of the inequalities (11), we get

$$ks < rn < ks + \frac{s}{m}.$$

If $rn = ks + 1$, then $(r,s) \in S^+(k,n)$, so, by Proposition 37, 2 and 4, $s = t_2 n + n_0$, where $t_2 > t_0$. Then, by (8) $s = t_0 n + n_0 + (t_2 - t_0)n \geqslant m + n$.

Otherwise it has to be $\frac{s}{m} > 2$ and $s > 2m$.

On the other hand, in a similar way, setting $km = hn - 1$ in (11), it results

$$hs - \frac{s}{n} < rm < hs.$$

Thus, if $rm = hs - 1$, then $(r,s) \in S^-(h,m)$. By (9), (10) and Proposition 37, 5, $s = t_3 m - m_0$, $t_3 > t_1$. Then $s = t_1 m - m_0 + (t_3 - t_1)m \geqslant n + m$.

Otherwise it must be $\frac{s}{n} > 2$ and $s > 2n$. Then we can infer that either $s \geqslant n + m$ or $s > 2n$ and $s > 2m$. Set $i = \max\{n, m\}$, it is $s > 2i > m + n$. □

Lemma 40. *Let* $\frac{k}{n}, \frac{h}{m} \in]0,1[$, $(h,m) \in S^-(k,n)$ *and* $\frac{h}{m} < \frac{r}{s} < \frac{k}{n}$. *Then* $s \geqslant n + m$.

Proof. Follows from Lemmas 38 and 39. □

REFERENCES

[1] BELLUCE, L. P.—DI NOLA, A.: *Yosida-type representation for perfect MV-algebras*, MLQ Math. Log. Q. **42** (1996), 551–563.
[2] CIGNOLI, R. L. O.—D'OTTAVIANO, I. M. L.—MUNDICI, D.: *Algebraic Foundations of Many-valued Reasoning*. Trends Log. Stud. Log. Libr., Kluwer, Dordrecht, 2000.
[3] DI NOLA, A.—GRIGOLIA, R.—PANTI, G.: *Finitely generated free MV-algebras and their automorphism groups*, Studia Logica **61** (1998), 65–78.
[4] DE SIMONE, A.—MUNDICI, D.—NAVARA, M.: *A Cantor-Bernstein theorem for σ-complete MV-algebras*, Czechoslovak Math. J. **53(128)** (2003), 437–447.
[5] DI NOLA, A.—NAVARA, M.: *Cantor-Bernstein property for MV-algebras*. In: Algebraic and Proof-theoretic Aspects of Non-classical Logics, Springer, Berlin-Heidelberg, 2006, pp. 107–118.
[6] JAKUBÍK, J.: *Cantor-Bernstein theorem for MV-algebras*, Czechoslovak Math. J. **49(124)** (1999), 517–526.
[7] LACAVA, F.: *Sulla struttura delle Ł-algebre*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **8**, (1979), 275–281.
[8] LACAVA, F.: *Una caratterizzazione delle Ł-algebre complete, prive di atomi*, Boll. Unione Mat. Ital. Sez. A (7) **9** (1995), 609–618.
[9] MONK, J. D.: *Handbook of Boolean Algebras*, North Holland, Basel, 1989.
[10] SIKORSKI, R.: *A generalization of a theorem of Banach and Cantor-Bernstein*, Colloq. Math. **1** (1948), 140–144.
[11] TARSKI, A.: *Cardinal Algebras*, Oxford University Press, New York, 1949.

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