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EFFECT-LIKE ALGEBRAS INDUCED BY MEANS OF BASIC ALGEBRAS

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ABSTRACT. Having an MV-algebra, we can restrict its binary operation addition only to the pairs of orthogonal elements. The resulting structure is known as an effect algebra, precisely distributive lattice effect algebra. Basic algebras were introduced as a generalization of MV-algebras. Hence, there is a natural question what an effect-like algebra can be reached by the above mentioned construction if an MV-algebra is replaced by a basic algebra. This is answered in the paper and properties of these effect-like algebras are studied.

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Effect algebras were introduced by D. J. Foulis and M. K. Bennett [12]. An effect algebra is a partial algebra which serves as a generalization of the set of Hilbert-space effects, i.e. self-adjoint operators on a Hilbert space (see e.g. [11] for the motivation in full details). For reader's convenience, we recall the definition of effect algebra.

Definition 1. An *effect algebra* is a partial algebra $\mathscr{E} = (E; +, 0, 1)$ of type $(2, 0, 0)$ satisfying the axioms

(EA1) if $x + y$ is defined then $y + x$ is defined and $x + y = y + x$;

(EA2) if $x + y$ and $(x + y) + z$ are defined then $y + z$ and $x + (y + z)$ are defined and $x + (y + z) = (x + y) + z$;

(EA3) for each $a \in E$ there exists a unique $b \in E$ such that $a + b = 1$; let us denote this b by a';

(EA4) if $1 + a$ is defined then $a = 0$.

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For an overview of results on effect algebras, the reader is recommended to consult e.g. [11]. It is well-known that every effect algebra is an ordered set with respect to the order defined by

$$x \leq y \qquad \text{if and only if} \qquad y = x + z \quad \text{for some } z \in E$$

and $0 \leq x \leq 1$ for each $x \in E$. If the ordered set $(E; \leq)$ is a lattice, $\mathscr{E}$ is called a *lattice effect algebra*.

Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be an MV-algebra (see e.g. [11] for the definition). Denote $1 = \neg 0$ and define $x + y = x \oplus y$ whenever $x \leq \neg y$ in $\mathscr{A}$ (where $\leq$ is the induced order of $\mathscr{A}$, i.e. $x \leq y$ if and only if $\neg x \oplus y = 1$). Then $\mathscr{E}(A) = (A; +, 0, 1)$ is a lattice effect algebra where the induced order of $\mathscr{A}$ coincides with the order of effect algebra mentioned above and $a' = \neg a$.

On the other hand, not every lattice effect algebra is induced by an MV-algebra. It was shown in [8] and [9] that lattice effect algebras are induced by a much more general structures, the so-called basic algebras satisfying certain identities. For effect algebras which do not have a lattice order this is done in [9]. Our aim is to show what an effect-like algebra can be induced by a basic algebra if no additional conditions are assumed.

The concept of a basic algebra was introduced (under a different name) in [4], the contemporary name was firstly used in [5]. The reader is referred to the monograph [7] for the concept and basic results.

Definition 2. By a *basic algebra* is meant an algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities

(B1) $x \oplus 0 = x$;

(B2) $\neg\neg x = x$;

(B3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$;

(B4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$, where $1 = \neg 0$.

Let us note that the definition from [5] or [7] contains one more axiom which is, however, a conclusion of identities (B1)–(B4) as it was recently proved in [10].

It is known that every basic algebra $\mathscr{A}$ induces an order

$$x \leq y \qquad \text{if and only if} \qquad \neg x \oplus y = 1$$

the so-called *induced order* of $\mathscr{A}$. Moreover, $0 \leq x \leq 1$ for each x of $\mathscr{A}$. Further, $(A; \leq)$ is a lattice, where

$$x \vee y = \neg(\neg x \oplus y) \oplus y \qquad \text{and} \qquad x \wedge y = \neg(\neg x \vee \neg y)$$

and $\neg$ is an antitone involution on A (i.e. $\neg\neg x = x$ and $x \leq y$ implies $\neg y \leq \neg x$). Consider a bounded lattice $(L; \vee, \wedge, 0, 1)$. A structure $\mathscr{L} = (L; \vee, \wedge, ({}^a)_{a \in L}, 0, 1)$ is called a *lattice with section antitone involutions* (see e.g. [4], [6], [7]) if for each

$a \in L$ there exists an antitone involution $x \mapsto x^a$ on the *section* (i.e. interval $[a, 1]$); in other words, if for any $x \geq a$ we have $x^{aa} = x$ and, if also $y \geq a$ and $x \leq y$ then $y^a \leq x^a$.

The following representation is presented in [7]:

Proposition 1.

(a) *Let* $\mathscr{L} = (L; \vee, \wedge, ({}^a)_{a \in L}, 0, 1)$ *be a lattice with section antitone involutions. Define* $\neg x = x^0$ *and* $x \oplus y = (x^0 \vee y)^y$. *Then* $\mathscr{A}(L) = (L; \oplus, \neg, 0)$ *is a basic algebra.*

(b) *Let* $\mathscr{A} = (A; \oplus, \neg, 0)$ *be a basic algebra. Define* $1 = \neg 0$, $x \vee y = \neg(\neg x \oplus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$ *and for every* $a \in A$ *and each* $x \geq a$ *let* $x^a = \neg x \oplus a$. *Then* $\mathscr{L}(A) = (A; \vee, \wedge, ({}^a)_{a \in A}, 0, 1)$ *is a lattice with section antitone involutions.*

(c) *The mappings* $\mathscr{A} \mapsto \mathscr{L}(A)$ *and* $\mathscr{L} \mapsto \mathscr{A}(L)$ *are the mutual one-to-one correspondences, i.e.* $\mathscr{A}(\mathscr{L}(A)) = \mathscr{A}$ *and* $\mathscr{L}(\mathscr{A}(L)) = \mathscr{L}$.

Due to Proposition 1, we will often use the lattice $\mathscr{L}(A)$ when discussing properties of a basic algebra $\mathscr{A}$. Let us note that every basic algebra satisfies the following:

- $x \oplus 1 = 1 = 1 \oplus x$;
- $x \oplus \neg x = 1 = \neg x \oplus x$;
- $x \leq y$ if and only if $\neg y \leq \neg x$;
- $x \leq y$ implies $x \oplus a \leq y \oplus a$.

(see [7] and [10]).

Definition 3. Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a basic algebra. Take $1 = \neg 0$ and define a partial binary operation $+$ on A as follows

$$x + y \quad \text{is defined if and only if} \quad x \leq \neg y \quad \text{and then} \quad x + y = x \oplus y.$$

Further, put x' equal to $\neg x$. Denote by $\mathscr{E}(A) = (A; +, 0, 1)$ the induced partial algebra.

Lemma 1. *Let* $\mathscr{A} = (A; \oplus, \neg, 0)$ *be a basic algebra. Then*

(i) $a + b = (\neg a)^b$ *(in* $\mathscr{L}(A)$*);*

(ii) $a \leq b$ *if and only if* $b = x + a$ *for some* $x \in A$.

Proof.

(i) Since $a + b$ is defined if $a \leq \neg b$, i.e. $b \leq \neg a$, we have $a + b = a \oplus b = (\neg a \vee b)^b = (\neg a)^b$ in the induced lattice $\mathscr{L}(A)$.

(ii) Let $\leq$ be the induced order of $\mathscr{A}$ and assume $b = x + a$ in $\mathscr{E}(A)$. Then $b = (\neg x)^a$ and hence $a \leq b$.

Conversely, if $a \leq b$ then $\neg b \leq \neg a$ and hence $b' + a$ is defined. Denote $b' + a = y$. Then

$$b^a = (b \vee a)^a = \neg b \oplus a = b' + a = y$$

thus also $a \leq y$ and

$$b = b^{aa} = y^a = (y \vee a)^a = \neg y \oplus a = y' + a = x + a$$

for $x = y'$. □

Lemma 2. *Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a basic algebra. Then $\mathscr{E}(A)$ satisfies the following*

(a) *if $a + b = c$ then $a' = c' + b$;*

(b) *$a = x + a$ implies $x = 0$ and $a = x + (y + a)$ implies $x = y = 0$;*

(c) *$a + b = 0$ implies $a = 0 = b$.*

Proof.

(a) If $a+b = c$ then $c = (\neg a)^b$, i.e. $b \leq c$ and $c^b = \neg a$, i.e. $a' = c^b = \neg c \oplus b = c' + b$ since $\neg c \leq \neg b$ thus $c' + b$ is defined.

(b) Assume $a = x + a$. Then $a \leq \neg x$ and $a = (\neg x)^a$ which is possible if and only if $\neg x = 1$, i.e. $x = 0$.

Assume $a = x + (y + a)$. Then $x \leq \neg(y + a)$, i.e. $y + a \leq \neg x$ and $a = (\neg x \vee (y+a))^{(y+a)}$. Since also $y+a$ is defined, we have $a \leq \neg y$ and $y+a = (\neg y)^a$. Thus $a = (\neg x)^{(\neg y)^a}$. This yields $a \geq (\neg y)^a$ and hence $1 = a^a \leq (\neg y)^{aa} = \neg y$ whence $y = 0$. Due to this, $a = (\neg x)^{1^a} = (\neg x)^a$ which yields $x = 0$ as shown above.

(c) If $a + b = 0$ then $b \leq \neg a$ and $0 = a + b = (\neg a)^b$. This is possible if and only if $b = 0$ and $\neg a = 1$, i.e. $a = 0$. □

Lemma 3. *Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a basic algebra. Then $\mathscr{E}(A)$ satisfies the following*

(a) *if $b = x + a$ and $c = y + b$ then there exists $z \in A$ such that $c = z + a$;*

(b) *if $b = x + a$ then there exists $y \in A$ such that $a' = y + b'$;*

(c) *if $b = x + a$ and $z \in A$ then there exists $v \in A$ such that $b + z = v + (a + z)$ whenever $b + z$, $a + z$ are defined.*

Proof.

(a) follows directly from Lemma 1 and the fact that the order is transitive.

To prove (b), $b = x + a$ yields $a \leq b$ thus $\neg b \leq \neg a$ and, by Lemma 1, there exists $y \in A$ with $a' = \neg a = y + \neg b = y + b'$.

We prove (c). If $b = x + a$ then $a \leq b$, i.e. $\neg b \leq \neg a$ and hence $\neg b \vee z \leq \neg a \vee z$, thus $b \oplus z = (\neg b \vee z)^z \geq (\neg a \vee z)^z = a \oplus z$. If $a + z$, $b + z$ are defined then $a + z = a \oplus z \leq b \oplus z = b + z$. By Lemma 1 we obtain the statement. □

Taking in account the just proved properties, we can define our crucial concept.

Definition 4. A partial algebra $\mathscr{E} = (E; +, 0, 1)$ of type $(2, 0, 0)$ is called a *weak effect algebra* if it satisfies the following conditions:

(W1) for each $a \in E$ there exists a unique $b \in E$ such that $a + b = 1 = b + a$; denote this b as a';

(W2) if $a + 1$ or $1 + a$ is defined then $a = 0$;

(W3) $a = x + a$ implies $x = 0$, $a = x + (y + a)$ implies $x = 0 = y$;

(W4) if $b = x + a$ and $c = y + b$ then there exists $z \in E$ with $c = z + a$;

(W5) if $b = x + a$ then there exists $y \in E$ with $a' = y + b'$;

(W6) if $b = x + a$ and $a + z$, $b + z$ are defined then there exists $v \in E$ such that $b + z = v + (a + z)$;

(W7) if $a + b = c$ then $a' = c' + b$.

It is easy to check that every effect algebra is a weak effect algebra (but not vice versa).

Theorem 1. *Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a basic algebra, let $1 = \neg 0$. Define $a + b$ if $a \leq \neg b$ and then $a + b = a \oplus b$. Then $\mathscr{E}(A) = (A; +, 0, 1)$ is a weak effect algebra where $a' = \neg a$.*

Proof. By Lemma 2, $\mathscr{E}(A)$ satisfies (W3) and (W7). By Lemma 3, $\mathscr{E}(A)$ satisfies (W4), (W5) and (W6). Hence, we need only to prove (W1) and (W2).

Since $a' = \neg a$ and $\neg\neg a = a$, we have $a \leq a = \neg\neg a = \neg a'$ thus both $a' + a$ and $a + a'$ are defined. As mentioned after Proposition 1, $\neg a \oplus a = 1 = a \oplus \neg a$ thus also $a' + a = 1$ and $a + a' = 1$. Conversely, assume $a + b = 1$. Then $b \leq \neg a$ and $a + b = (\neg a)^b$ in $\mathscr{L}(A)$. Thus $(\neg a)^b = 1$ which is possible if and only if $\neg a = b$, i.e. $b = a'$. For $b + a = 1$ it can be shown analogously $b = a'$ thus $\mathscr{E}(A)$ satisfies (W1).

Prove (W2). If $a + 1$ is defined then $a \leq \neg 1 = 0$ whence $a = 0$. If $1 + a$ is defined then $1 \leq \neg a$, i.e. $\neg a = 1$ giving $a = 0$. □

In what follows, we are going to set up a converse construction. It will be shown that it is possible if the induced order of a weak effect algebra is a lattice order.

Lemma 4. *Let $\mathscr{E} = (E; +, 0, 1)$ be a weak effect algebra. Then:*

(a) $1' = 0$ *and* $a'' = a$*;*

(b) $0 + a = a$*;*

(c) *for any* $a \in E$ *there exists* $y \in E$ *such that* $a = y + 0$*.*

Proof.

(a) By (W1), $1' + 1 = 1 = 1 + 1'$ which, together with (W2), yields $1' = 0$. Moreover, (W1) gets $a'' = a$ for each $a \in E$.

(b) By (W1) we have $a + a' = 1$ which, together with (W7) and (a), implies $a = 0 + a$. Since $a'' = a$, we obtain also $a + 0 = a$.

(c) Since $1 = a + a'$ by (W1), we apply (W7) to get the assertion. $\square$

Theorem 2. *Let* $\mathscr{E} = (E; +, 0, 1)$ *be a weak effect algebra. Define*

$$a \le b \qquad \text{if and only if} \qquad b = x + a \quad \text{for some } x \in E.$$

Then $\le$ *is an order on* E *and* $0 \le a \le 1$ *for each* $a \in E$*.*

Proof. By (b) of Lemma 4, $\le$ is reflexive. Assume $a \le b$ and $b \le a$. Then $a = x + b$ and $b = y + a$ for some $x, y \in E$, i.e. $a = x + (y + a)$. By (W3) we have $x = 0 = y$ thus $a = b$.

If $a \le b$ and $b \le c$ then $b = x + a$ and $c = y + b$ for some $x, y \in E$. By (W4) we conclude $a \le c$ thus $\le$ is an order on E. Due to Lemma 4, $0 \le a$ and, by (W1), $a \le 1$ for any $a \in E$. $\square$

Definition 5. For a weak effect algebra $\mathscr{E} = (E; +, 0, 1)$, the order $\le$ introduced in Theorem 2 will be called the *induced order of* $\mathscr{E}$. If, moreover, $(E, \le)$ is a lattice, $\mathscr{E}$ is called a *lattice weak effect algebra.*

Remark 1. Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a basic algebra. By Lemma 1 and Theorem 2, the induced order of $\mathscr{E}(A)$ coincides with that of $\mathscr{A}$. Due to Proposition 1, $\mathscr{E}(A)$ is a lattice weak effect algebra.

Lemma 5. *Let* $\mathscr{E} = (E; +, 0, 1)$ *be a weak effect algebra and* $\le$ *its induced order. Then*

(a) $x \le y$ *if and only if* $y' \le x'$*;*

(b) $a + b$ *is defined if and only if* $a \le b'$ *(which is equivalent to* $b \le a'$*);*

(c) *if* $a \le x$ *then* $(x' + a)' + a = x$*.*

Proof.

(a) Assume $x \le y$. By Theorem 2, $y = c + x$ for some $c \in E$. Due to (W5), there exists $d \in E$ with $x' = d + y'$ whence $y' \le x'$. By (a) of Lemma 4 we can show the converse.

(b) If $a + b$ is defined, say $a + b = c$ then, by (W7), $c' + b$ is defined and $a' = c' + b$. Hence $b \leq a'$ and, by (a), also $a \leq b'$. The converse follows by (W7) immediately.

(c) Assume $a \leq x$. Then $x' + a$ is defined and $a \leq x' + a$. Thus also $(x' + a)' + a$ is defined and, due to (W7) and (a) of Lemma 4, for $y = x' + a$ we obtain

$$x = x'' = y' + a = (x' + a)' + a.$$

□

Remark 2. By Lemma 5(b), $a + b$ is defined in a weak effect algebra if and only if $b + a$ is defined.

Now, we can prove our main result.

Theorem 3. *Let $\mathscr{E} = (E; +, 0, 1)$ be a lattice weak effect algebra. Define $\neg x = x'$ and $x \oplus y = (x \wedge y') + y$. Then $\mathscr{A}(E) = (E; \oplus, \neg, 0)$ is a basic algebra.*

Proof. Denote by $\vee$ and $\wedge$ the lattice operations of $(E; \leq)$. Then $(E; \vee, \wedge, ', 0, 1)$ is a bounded lattice with an antitone involution (and hence it satisfies the DeMorgan laws).

Now, let $a \in E$. Suppose $a \leq x$ and denote by $x^a = x' + a$. Due to Lemma 5, x^a is defined for each $x \in [a, 1]$ and $x^{aa} = (x' + a)' + a = x$. Assume $x, y \in [a, 1]$ with $x \leq y$. By Lemma 5, $y' \leq x'$ and, due to (W6) and Theorem 2,

$$y^a = y' + a \leq x' + a = x^a$$

thus $x \mapsto x^a$ is an antitone involution on the section $[a, 1]$ for every $a \in E$. Hence, $(E; \vee, \wedge, (^a)_{a \in E}, 0, 1)$ is a lattice with section antitone involutions and, by Proposition 1, $\mathscr{A}(E) = (E; \oplus, \neg, 0)$ is a basic algebra for $\neg x = x'$ and $x \oplus y = (x' \vee y)^y = (x' \vee y)' + y = (x \wedge y') + y$. □

Remark 3. One can easily check that the mappings $\mathscr{A} \mapsto \mathscr{E}(A)$ and $\mathscr{E} \mapsto \mathscr{A}(E)$ between basic algebras and lattice weak effect algebras are mutually inverse correspondences, i.e. $\mathscr{A}(\mathscr{E}(A)) = \mathscr{A}$ and $\mathscr{E}(\mathscr{A}(E)) = \mathscr{E}$.

Lattice weak effect algebras have some more nice properties which are common with effect algebras, see e.g. the following lemma:

Lemma 6. *Let $\mathscr{E} = (E; +, 0, 1)$ be a lattice weak effect algebra, $a, b, c \in E$. If $b + a$ and $c + a$ are defined then*

$$b + a = c + a \quad \textit{implies} \quad b = c.$$

Proof. Let $b + a, c + a$ be defined and $b + a = c + a$. By Theorem 3, we can compute in the assigned basic algebra and its induced lattice as follows

$$(\neg b)^a = b \oplus a = b + a = c + a = c \oplus a = (\neg c)^a$$

whence $\neg b = (\neg b)^{aa} = (\neg c)^{aa} = \neg c$, i.e. $b = c$. $\square$

We say that a weak effect algebra $\mathscr{E} = (E; +, 0, 1)$ is *commutative* if $x + y = y + x$ whenever $x + y$ is defined. In the remaining part of the paper, we will study commutative weak effect algebras. First of all we show that there exist lattice weak effect algebras which are not commutative although they can be associative.

Example 1. Let $E = \{0, a, b, c, a^0, b^0, c^0, p, q, 1\}$ and the partial operation $+$ is given by the table

$+$	0	a	b	c	p	a^0	b^0	c^0	q	1
0	0	a	b	c	p	a^0	b^0	c^0	q	1
a	a	$-$	p	$-$	$-$	1	$-$	$-$	b^0	$-$
b	b	p	$-$	$-$	$-$	$-$	1	$-$	a^0	$-$
c	c	$-$	$-$	c^0	$-$	$-$	$-$	1	c^0	$-$
p	p	$-$	$-$	$-$	$-$	$-$	$-$	$-$	1	$-$
a^0	a^0	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
b^0	b^0	$-$	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$
c^0	c^0	$-$	$-$	1	$-$	$-$	$-$	$-$	$-$	$-$
q	q	b^0	a^0	p	1	$-$	$-$	$-$	$-$	$-$
1	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$

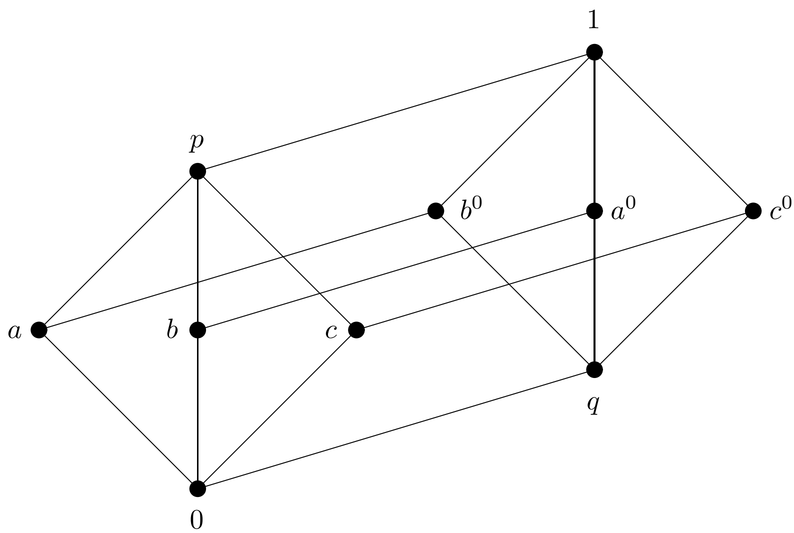

FIGURE 1

It can be easily checked that $\mathscr{E} = (E; +, 0, 1)$ is a lattice weak effect algebra which is not commutative (e.g. $q + c = p \neq c^0 = c + q$). Its order is depicted in Fig. 1.

If we define for $x \in [z, 1]$, $x^z = x' + z$ (where $a' = a^0$, $b' = b^0$, $c' = c^0$, $p' = q$, $0' = 1$) then we have the following section antitone involutions:

[0,1]: $a \mapsto a^0$, $b \mapsto b^0$, $c \mapsto c^0$, $p \mapsto q$, $0 \mapsto 1$;
[c,1]: $c \mapsto 1$, $p \mapsto p$, $c^0 \mapsto c^0$;
[a,1]: $a \mapsto 1$, $b^0 \mapsto p$;
[b,1]: $b \mapsto 1$, $a^0 \mapsto p$;
[q,1]: $q \mapsto 1$, $a^0 \mapsto b^0$, $c^0 \mapsto c^0$

and uniquely in the sections with at most two elements. Then the lattice in Fig. 1 is a lattice with section antitone involutions and hence it induces the basic algebra $\mathscr{A}(E) = (E; \oplus, \neg, 0)$.

A basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ is *commutative* if $x \oplus y = y \oplus x$ holds for each $x, y \in A$. Commutative basic algebras were treated in [1, 2]. It was shown by M. Botur [1] that every finite commutative basic algebra is in fact an MV-algebra and that there exists an infinite commutative basic algebra which is not an MV-algebra.

In what follows, we present the statement proved in [3] characterizing commutative basic algebras by two (independent) conditions. The first condition will be used for a characterization of commutative weak effect algebras.

Proposition 2. *A basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ is commutative if and only if it satisfies the following two (independent) conditions:*

(i) *if $y \leq x$ then $x^y = (\neg y)^{(\neg x)}$ in $\mathscr{L}(A)$;*
(ii) *$x \oplus (y \wedge \neg x) = x \oplus y$.*

For our reasons, call a basic algebra $\mathscr{A}$ to be *quasi-commutative* if it satisfies (i) of Proposition 2. Let us note that the basic algebra from Example 1 is not quasi-commutative since $c \leq p$ but

$$p^c = p \neq c^0 = (c^0)^q = (c^0)^{(p^0)} = (\neg c)^{(\neg p)}.$$

Example 2. Let $\mathscr{L}$ be the lattice whose diagram is visualized in Fig. 2

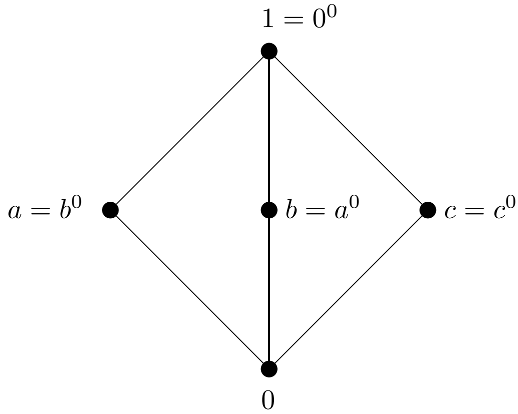

FIGURE 2

and the antitone involutions in at most two-element sections are determined uniquely. Then the assigned basic algebra $\mathscr{A}(L)$ is as follows: $\neg x = x^0$ and $\oplus$ is determined by the table

$\oplus$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	1	c	1
b	b	1	b	c	1
c	c	a	b	1	1
1	1	1	1	1	1

Evidently, $\mathscr{A}(L)$ is not commutative since

$$a \oplus c = c \neq a = c \oplus a$$

but it is quasi-commutative (it can be checked easily). The lattice weak effect algebra induced by $\mathscr{A}(L)$ has the following table

$+$	0	a	b	c	1
0	0	a	b	c	1
a	a	–	1	–	–
b	b	1	–	–	–
c	c	–	–	1	–
1	1	–	–	–	–

It is an immediate reflexion that it is commutative.

THEOREM 4. *Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a basic algebra. The induced lattice weak effect algebra $\mathscr{E}(A) = (A; +, 0, 1)$ is commutative if and only if $\mathscr{A}$ is quasi-commutative.*

P r o o f. Assume that $a + b$ is defined. Then $a \leq b'$, i.e. $a \leq \neg b$ and $b \leq \neg a$ in $\mathscr{A}$ and, due to quasi-commutativity of $\mathscr{A}$ we have

$$a + b = (\neg a)^b = (\neg b)^a = b + a.$$

Conversely, if $\mathscr{E}(A)$ is commutative, $x, y \in A$ and $y \leq x$ then $\neg x \leq \neg y$ thus $y + x'$, $x' + y$ are defined and $(\neg y)^{(\neg x)} = (\neg y \vee \neg x)^{(\neg x)} = y \oplus \neg x = y + x' = x' + y = \neg x \oplus y = x^y$ thus $\mathscr{A}$ is quasi-commutative. □

Due to the mutual one-to-one correspondence mentioned in Remark 3, we can conclude:

COROLLARY 1. *A lattice weak effect algebra $\mathscr{E}$ is commutative if and only if the induced basic algebra $\mathscr{A}(E)$ is quasi-commutative.*

If a weak effect algebra is commutative then clearly (W7) implies (W5) thus the axiom system can be simplified.

COROLLARY 2. *A partial algebra $\mathscr{E} = (E; +, 0, 1)$ is a commutative weak effect algebra if and only if it satisfies the axioms* (W1)–(W4), (W6), (W7) *and*

$$x + y = y + x.$$

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