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CENTRAL ELEMENTS IN PSEUDOEFFECT ALGEBRAS

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ABSTRACT. We introduce the definition of pseudoorthoalgebras and discuss some relationships between orthomodular lattices and pseudoorthoalgebras. Then we study the conditions that a pseudoeffect algebra is isomorphic to an "internal direct product" of ideals generated by orthogonal principal elements. At last, we give some characterizations of central elements in pseudoeffect algebras.

1. Preliminaries

Recently, partial algebraic structures, called pseudoeffect algebras, have been introduced in [2] and [3]. The partial addition pseudoeffect algebras are equipped with is not assumed to be commutative. They can serve as the models of quantum structures as well as noncommutative logic ([8, 17]). In fact, pseudoeffect algebras arise typically from not necessary commutative po-groups, which have been studied in quantum physics for many years. Pseudoeffect algebras are sometimes unit intervals in cones of unital po-groups ([2, 3]). An important po-group used in physics is $B(H)$, the system of all Hermitian operators of a Hilbert space H, and the system of all effect operators $E(H)$, i.e. the system of all Hermitian operators A on H such that $0 \leqslant A \leqslant I$, where 0 and I are the zero and identity operators. Then $E(H)$ is the interval in $B(H)$, and it is one of the most important examples of effect algebras. In [6], the authors have proved that if a lattice ordered pseudoeffect algebra E satisfies the difference

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compatibility property, then any block is a pseudoeffect subalgebra, which is a pseudo MV-algebra, and E is a set-theoretical union of its block.

It is well known that effect algebras which were introduced in [10] by Foulis and Bennett in 1994, can be considered as a generalized form of quantum logic. Since the logical excluding middle law $a \vee a' = 1$ is not satisfied in the effect algebras, they are called "unsharp quantum logic" ([1]). However, the logical excluding middle law holds in orthoalgebras, which are considered as sharp quantum logic and play an important role in quantum logic ([1, 11, 12]). In accordance with all theses facts, we naturally wonder what kind of algebras can be when the logical excluding middle law is satisfied in the pseudoeffect algebras. In this paper, we introduce the definition of pseudoorthoalgebra. We find pseudoorthoalgebra is the sharp part of pseudoeffect algebras. Especially, we also study the weakly commutative pseudoorthoalgebras and give some characterizations of the orthomodular posets (lattices) in the weakly commutative pseudoorthoalgebras.

During the process of studying quantum structures, the principal elements play an important role ([1, 18]). In order to study the structure of pseudoeffect algebras, we introduce the definition of principal elements in pseudoeffect algebras. Then, we study the conditions that guarantee pseudoeffect algebras to be isomorphic to the cartesian product of its principal ideals.

Central elements and the center (the set of all central elements) play an important role in quantum structures — they represent the "classical part" of a given model ([1, 14, 15]). In [15] and [16], J. Tkadlec studied the central elements of effect algebras in details. In [7], A. Dvurečenskij proved that the center of pseudoeffect algebra is a Boolean algebra and proved the Cantor-Bernstein theorem of pseudoeffect algebras. In this paper, we will further study the properties of central elements in pseudoeffect algebra and give some interesting results.

The paper is organized as follows. In Section 2, pseudoorthoalgebras and weakly commutative pseudoorthoalgebras are introduced, and the relationships between weakly commutative pseudoorthoalgebras and orthomodular posets are given too. In Section 3, the central elements in pseudoeffect algebras are discussed in details. Firstly, the definition of principal elements in pseudoeffect algebras is introduced. The conditions are also given that pseudoeffect algebras can be represented as the internal direct product of ideals of generated by orthogonal principal elements. Secondly, we give some equivalent characterizations of central elements of pseudoeffect algebras. Finally, central elements in atomic pseudoeffect algebras are studied, and some conditions ensuring that an atomic pseudoeffect algebra is a Boolean algebra are presented. These results generalize the conclusions in [15] and [16].

Now, we give the basic definitions and facts that we will need in this paper.

Definition 1.1. ([5]) A structure $(P;\oplus,0)$, where $\oplus$ is a partial binary operation and 0 is a constant, is called a *generalized pseudoeffect algebra*, or GPEA for short, if for all $a,b,c\in P$, the following hold.

(GPE1) $a\oplus b$ and $(a\oplus b)\oplus c$ exist if and only if $b\oplus c$ and $a\oplus(b\oplus c)$ exist, and in this case, $(a\oplus b)\oplus c=a\oplus(b\oplus c)$.

(GPE2) If $a\oplus b$ exists, then there are elements $d,e\in P$ such that $a\oplus b=d\oplus a=b\oplus e$.

(GPE3) If $a\oplus b$ and $a\oplus c$ exist and are equal, then $b=c$. If $b\oplus a$ and $c\oplus a$ exist and are equal, then $b=c$.

(GPE4) If $a\oplus b$ exists and $a\oplus b=0$, then $a=b=0$.

(GPE5) $a\oplus 0$ and $0\oplus a$ exist and are both equal to a.

Remark 1.1. ([5]) We can introduce the binary relation $\leqslant$ in the GPEA P. Let $a,b\in P$. Define $a\leqslant b$ if and only if there is a $c\in P$ such that $a\oplus c=b$. It is easy to prove that $\leqslant$ is a partial order in P.

Remark 1.2. ([18]) We can introduce the partial binary operations $\ominus_l$ and $\ominus_r$ in the GPEA P. For any $a,b,c,d\in P$, $a\ominus_l b=c$ if and only if $b\oplus c$ exists and $a=b\oplus c$; $a\ominus_r b=d$ if and only if $d\oplus b$ exists and $a=d\oplus b$.

Let us recall some basic properties of $\leqslant,\oplus,\ominus_l$ and $\ominus_r$ because these properties are used in calculations.

Proposition 1.1. ([5, 6]) *Let E be a pseudoeffect algebra. For all $a,b,c,d\in E$, the following properties hold.*

(1) *If $a\oplus b$ exists then $c\oplus d$ exists for all $c\leqslant a$ and $d\leqslant b$.*
(2) *If $a\leqslant b$, then $b\ominus_r a, b\ominus_l a\leqslant b$, and $b\ominus_r(b\ominus_l a)=b\ominus_l(b\ominus_r a)=a$.*
(3) $a\ominus_l a=a\ominus_r a=0$.
(4) *If $a,b\leqslant c$, and $c\ominus_l a=c\ominus_l b$ (or $c\ominus_r a=c\ominus_r b$), then $a=b$.*
(5) *If $a\leqslant b\leqslant c$, then $b\ominus_l a\leqslant c\ominus_l a, b\ominus_r a\leqslant c\ominus_r a$ and $(c\ominus_l a)\ominus_l(b\ominus_l a)=c\ominus_l b$, $(c\ominus_r a)\ominus_r(b\ominus_r a)=c\ominus_r b$.*
(6) *$a\oplus b\leqslant c$ if and only if $a\leqslant c\ominus_r b$ if and only if $b\leqslant c\ominus_l a$.*

Definition 1.2. ([20]) A GPEA P is called a *weakly commutative GPEA* if it satisfies the condition:

(C) For any $a,b\in E$, $a\oplus b$ exists if and only if $b\oplus a$ exists.

Definition 1.3. ([1, 2]) A weakly commutative GPEA P is called a *generalized effect algebra* (or GEA for short) if for any $a,b\in P$, if $a\oplus b$ exists, then $a\oplus b=b\oplus a$.

Definition 1.4. ([2]) A structure $(P;\oplus,0,1)$, where $\oplus$ is a partial binary operation and 0 and 1 are constants, is called a *pseudoeffect algebra*, or PEA for short, if for all $a,b,c\in P$, the following hold.

(PE1) $a\oplus b$ and $(a\oplus b)\oplus c$ exist if and only if $b\oplus c$ and $a\oplus(b\oplus c)$ exist, and in this case, $(a\oplus b)\oplus c=a\oplus(b\oplus c)$.

(PE2) There is exactly one $d\in P$ and exactly one $e\in P$ such that $a\oplus d=e\oplus a=1$.

(PE3) If $a\oplus b$ exists, there are elements $d,e\in P$ such that $a\oplus b=d\oplus a=b\oplus e$.

(PE4) If $a\oplus 1$ or $1\oplus a$ exists, then $a=0$.

Remark 1.3. ([5]) Pseudoeffect algebras are equivalent to generalized pseudoeffect algebras with greatest element.

Remark 1.4. ([2]) If $a\oplus b$ exists and $a\oplus b=1$, then we write $b^-=a$, $a^\sim=b$. Thus, $-\colon P\to P$ and $\sim\colon P\to P$ are two unary operations satisfying the following:

(1) if $a\leqslant b$, then $b^-\leqslant a^-$, $b^\sim\leqslant a^\sim$.

(2) For any $a\in P$, $a^{-\sim}=a^{\sim-}=a$.

Definition 1.5. ([20]) A PEA P is called a *weakly commutative PEA* if it satisfies the condition:

(C) For any $a,b\in E$, $a\oplus b$ exists if and only if $b\oplus a$ exists.

Definition 1.6. ([1]) A weakly commutative PEA P is called an *effect algebra* if for any $a,b\in P$, if $a\oplus b$ exists, then $a\oplus b=b\oplus a$.

Proposition 1.2. ([20]) *If E is a weakly commutative PEA, then for any $a\in E$, $a^-=a^\sim$.*

Remark 1.5. By the above proposition, in weakly commutative PEA, if $a\oplus b$ $=1$, then we define $a'=b$, and a' is called the orthosupplement of a.

Remark 1.6. In weakly commutative PEA, if $a\oplus b$ exists, then $b\oplus a$ exists. Hence, we write $a\perp b$ whenever $a\oplus b$ exists.

In the following part of this section, we will show that every weakly commutative GPEA is an order ideal of weakly commutative PEA and how to construct weakly commutative PEA from weakly commutative GPEA.

Let $(P;\oplus,0)$ be a weakly commutative GPEA. Let $P^\sharp$ be a set disjoint from P with the same cardinality. Consider a bijection $a\mapsto a^\sharp$ from P onto $P^\sharp$ and let us denote $P\cup P^\sharp=\hat{P}$. Define a partial operation $\oplus^*$ on $\hat{P}$ by the following rules.

For $a, b \in P$,

(1) $a \oplus^* b$ is defined if and only if $a \oplus b$ is defined, and $a \oplus^* b = a \oplus b$.

(2) $a \oplus^* b^\sharp$ is defined if and only if $b \ominus_r a$ is defined, and then $a \oplus^* b^\sharp = (b \ominus_r a)^\sharp$.

(3) $b^\sharp \oplus^* a$ is defined if and only if $b \ominus_l a$ is defined, and then $b^\sharp \oplus^* a = (b \ominus_l a)^\sharp$.

Proposition 1.3. ([20]) *If $(P; \oplus, 0)$ is a weakly commutative GPEA, then the structure $(\hat{P}; \oplus^*, 0, 0^\sharp)$ is a weakly commutative PEA. Moreover, P is an order ideal in $\hat{P}$ closed under $\oplus$, and the partial order induced by $\oplus^*$ when restricted to P, coincides with the partial order induced by $\oplus$.*

Proposition 1.4. ([20]) *If $(P; \oplus, 0)$ is a GPEA and the structure $(\hat{P}; \oplus^*, 0, 0^\sharp)$ is a PEA, then $(P; \oplus, 0)$ is a weakly commutative GPEA and $(\hat{P}; \oplus^*, 0, 0^\sharp)$ is a weakly commutative PEA.*

2. Weakly commutative generalized pseudoorthoalgebras and orthomodular posets

In this section, we introduce the definitions of (weakly commutative) generalized pseudoorthoalgebras and (weakly commutative) pseudoorthoalgebras, then discuss their relationships between orthomodular posets. Further, some results about effect algebras are generalized to weakly commutative pseudoeffect algebras.

Definition 2.1. A structure $(E; \oplus, 0)$, where $\oplus$ is a partial binary operation and 0 is constant, is called a *generalized pseudoorthoalgebra*, or GPOA for short, if for all $a, b, c \in E$, the following hold.

(GPOA1) $a \oplus b$ and $(a \oplus b) \oplus c$ exist if and only if $b \oplus c$ and $a \oplus (b \oplus c)$ exist, and in this case, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(GPOA2) If $a \oplus b$ and $a \oplus c$ exist and are equal, then $b = c$. If $b \oplus a$ and $c \oplus a$ exist and are equal, then $b = c$.

(GPOA3) If $a \oplus b$ exists, then there are elements $d, e \in E$ such that $a \oplus b = d \oplus a = b \oplus e$.

(GPOA4) If $a \oplus a$ exists, then $a = 0$.

(GPOA5) $a \oplus 0$ and $0 \oplus a$ exist and are both equal to a.

Definition 2.2. A generalized pseudoorthoalgebra $(E; \oplus, 0)$ is called *weakly commutative* if it satisfies the following condition:

(C) For any $a, b \in E$, $a \oplus b$ exists if and only if $b \oplus a$ exits.

Definition 2.3. A structure $(E; \oplus, 0, 1)$, where $\oplus$ is a partial binary operation and 0 and 1 are constants, is called a *pseudoorthoalgebra*, or POA for short, if for all $a, b, c \in E$, the following hold.

(POA1) $a \oplus b$ and $(a \oplus b) \oplus c$ exist if and only if $b \oplus c$ and $a \oplus (b \oplus c)$ exist, and in this case, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(POA2) There is exactly one $d \in E$ and exactly one $e \in E$ such that $a \oplus d = e \oplus a = 1$.

(POA3) If $a \oplus b$ exists, then there are elements $d, e \in E$ such that $a \oplus b = d \oplus a = b \oplus e$.

(POA4) If $a \oplus a$ exists, then $a = 0$.

Remark 2.1. POA's are equivalent to GPOA's with greatest element.

From the following proposition, we can get the relationships between PEA and POA.

Proposition 2.1.

(1) *Every GPOA is a GPEA, and a GPEA is a GPOA if and only if it satisfies the condition* (GPOA4) *from* Definition 2.1.

(2) *Every POA is a PEA, and a PEA is a POA if and only if it satisfies the condition* (POA4) *from* Definition 2.3.

Remark 2.2. We will see in Proposition 2.3 that in any pseudoorthoalgebra $(E; \oplus, 0, 1)$, for any $a \in E$, we have that the logical excluding middle law $a \vee a' = 1$ and $a \wedge a' = 0$ hold, where $a' = a^{\sim}$ or $a' = a^{-}$. Hence, we can say that pseudoorthoalgebras represent sharp logic.

Definition 2.4. A pseudoorthoalgebra $(E; \oplus, 0, 1)$ is called *weakly commutative*, or WPOA for short, if it satisfies the condition (C).

Remark 2.3. Since every POA is a PEA, hence weakly commutativity yields $a^{-} = a^{\sim}$, and then we put $a' := a^{-} = a^{\sim}$ in WPOA.

Definition 2.5. Let $(E; \oplus, 0, 1)$ be a weakly commutative pseudoeffect algebra. For any $a \in E$, the element a is called a *sharp* element if and only if $a \wedge a' = 0$.

Remark 2.4. Let $(E; \oplus, 0, 1)$ be a weakly commutative pseudoorthoalgebra. Then for any $a \in E$, a is a sharp element of E.

Example 2.1. Let P be the set $\{0, a, b, c, d, e\}$. Define the partial binary operation $\oplus$ on P as the following:

(1) $a \oplus b = b \oplus c = c \oplus a = e$, $b \oplus a = c \oplus b = a \oplus c = d$;

(2) For any $x \in P$, $x \oplus 0 = 0 \oplus x = x$.

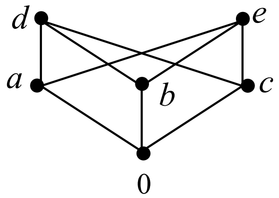

FIGURE 1

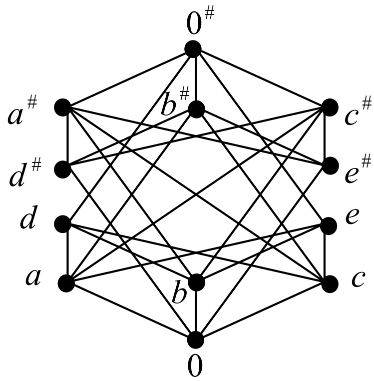

FIGURE 2

It is routine to verify that P is a weakly commutative GPOA. The Hasse diagram is in Fig. 1.

From Proposition 2.1, we can get the following classical result.

Corollary 2.1. ([1]) *An effect algebra* $(E;\oplus,0,1)$ *is an orthoalgebra if and only if for all* $a\in E$*, the existence of* $a\oplus a$ *implies* $a=0$*.*

It is easy to get the following result by Proposition 1.3 and Proposition 2.1.

Proposition 2.2. *Let* P *be a WGPOA. Then* $\hat{P}$ *is a WPOA.*

Example 2.2. Let P be same as in Example 2.1 and $P^\sharp=\{0^\sharp,a^\sharp,b^\sharp,c^\sharp,d^\sharp,e^\sharp\}$ and $P^\sharp\cap P=\emptyset$. $\hat{P}=P\cup P^\sharp$. Then by Proposition 2.2, $(\hat{P};\oplus,0,0^\sharp)$ is a WPOA. The Hasse diagram is in Fig. 2.

The following proposition gives some characterizations of POAs among pseudoeffect algebras.

Proposition 2.3. *For a pseudoeffect algebra* E*, the following conditions are mutually equivalent:*

(1) E *is a POA.*
(2) *For* $p,q\in E$ *such that* $p\oplus q$ *exists, we have that* $p\oplus q$ *is a minimal upper bound for* p *and* q*.*
(3) *For any* $p\in E$, $p\wedge p^-=p\wedge p^\sim=0$*.*
(4) *For any* $p\in E$, $p\vee p^-=p\vee p^\sim=1$*.*

Proof.

(1) $\Longrightarrow$ (2). Clearly, $p, q \leqslant p \oplus q$. Suppose $r \in E$, $p, q \leqslant r, r \leqslant p \oplus q$. Then there are p_1, q_1, r_1 such that $r = p \oplus p_1 = q \oplus q_1, p \oplus q = r \oplus r_1$. By cancellativity, $q = p_1 \oplus r_1$. So $r \oplus r_1 = q \oplus q_1 \oplus r_1 = p_1 \oplus r_1 \oplus q_1 \oplus r_1$, then $r_1 \oplus r_1$ exists, hence $r_1 = 0$ by (POA4).

(2) $\Longrightarrow$ (3). Assume (2) and suppose $r \in E$ with $r \leqslant p, p^-$. Then $p, p^\sim \leqslant r^\sim \leqslant 1 = p \oplus p^\sim$, so $r^\sim = 1$ and $r = 0$. Hence, $p \wedge p^- = 0$. Similarly, we have $p \wedge p^\sim = 0$.

(3) $\Longrightarrow$ (4). It is easy to see that the mappings $p \mapsto p^-$ and $p \mapsto p^\sim$ are order-inverting and $p^{-\sim} = p^{\sim -} = p$ on the bounded poset E, so that the de Morgan law is valid. Therefore, since $p \wedge p^- = p \wedge p^\sim = 0$ for all $p \in E$, it follows that $p \vee p^- = p \vee p^\sim = 1$, for all $p \in E$.

(4) $\Longrightarrow$ (1). Suppose $p \oplus p$ exists. Then $p \leqslant p^-$, $p = p \wedge p^- = 0$. $\square$

In the following, we will give some characterizations of OMPs among weakly commutative pseudoeffect algebras and pseudoorthoalgebras.

Let us recall the definition of an OMP.

Definition 2.6. ([1]) An *orthomodular poset* (OMP, in short) is a bounded poset $(L; \leqslant, ', 0, 1)$ with an unary operation $' \colon L \to L$ (an orthocomplementation) such that the following conditions are satisfied for all $a, b, c \in L$:

(OMP1) $a \leqslant b \implies b' \leqslant a'$.

(OMP2) $(a')' = a$.

(OMP3) $a \vee a' = 1$.

(OMP4) If $a \leqslant b'$ then $a \vee b$ exists in L.

(OMP5) If $a \leqslant b$, then there is a $c \in L$ such that $c \leqslant a'$ and $a \vee c = b$.

We will say that a weakly commutative pseudoeffect algebra E satisfies the *coherence law* if for all $p, q, r \in E$ with $p \perp q$, $q \perp r$, $r \perp p$, the sum $p \oplus q \oplus r$ exists.

Proposition 2.4. *A weakly commutative PEA $(E; \oplus, 0, 1)$ can be organized into an OMP such that $a \oplus b = a \vee b$ provided $a \perp b$ if and only if it satisfies the coherence law.*

Proof. Assume that E is a weakly commutative PEA satisfying the coherence law. We will prove that $a \oplus b = a \vee b$ whenever $a \perp b$. Clearly, $a, b \leqslant a \oplus b$. Suppose $a, b \leqslant r$. It means that $a, b \perp r'$, so that a, b, r' are pairwise orthogonal. By coherence law, $a \oplus b \oplus r'$ exists, so that $a \oplus b \perp r'$. In other words, $a, b \leqslant r$ implies $a \oplus b \leqslant r$, hence $a \oplus b = a \vee b$. Then $a \oplus b = b \oplus a$ for all $a, b \in E$

whenever $a \perp b$, thus E is an effect algebra and an OMP. The other direction is obvious. □

Especially, we can get the following result.

Corollary 2.2. ([1]) *An effect algebra $(E; \oplus, 0, 1)$ can be organized into an OMP such that $a \oplus b = a \vee b$ provided $a \perp b$ if and only if it satisfies the coherence law.*

Proposition 2.5. *The following conditions are equivalent for a WPOA E:*

(1) *E is an OMP.*

(2) *If $a \oplus b$ is defined, then $a \vee b$ exists.*

(3) *If $a \oplus b$ is defined, then $a \vee b$ exists, and $a \oplus b = a \vee b$.*

(4) *If $a \oplus b, a \oplus c$ and $b \oplus c$ exist, then $(a \oplus b) \oplus c$ exists.*

Proof. This easily follows from Proposition 2.3 and 2.4. □

Proposition 2.6. *A lattice ordered WPOA is an OML.*

Proof. This easily follows from Proposition 2.5. □

Corollary 2.3. ([1]) *A lattice ordered OA is an OML.*

The following example shows that above two propositions fail in POA.

Example 2.3. Let A be the set $\{0, a, b, c, 1\}$. Define the partial binary operation $\oplus$ on A as the following:

(1) $a \oplus b = b \oplus c = c \oplus a = 1$,

(2) For any $x \in A, x \oplus 0 = 0 \oplus x = x$.

It is routine to verify that A is a pseudoorthoalgebra. It is not a weakly commutative POA. It is a lattice and $x \oplus y = x \vee y$ for any $x, y \in A$ and $x \oplus y \in A$. However, it is not an OML. The Hasse diagram is in Fig. 3.

Remark 2.5. An effect algebra is an orthomodular poset if and only if every its element is principal [1]. However, this result fails in pseudoeffect algebra from Example 2.3.

3. Central elements of pseudoeffect algebras

In this section, we will study the central elements of pseudoeffect algebras. Firstly, we introduce the definition of principal elements of pseudoeffect algebras and prove that a pseudoeffect algebra can be represented as an internal direct

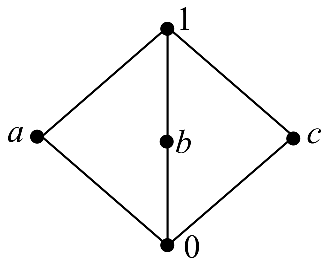

FIGURE 3

product of ideals generated by orthogonal principal elements. Secondly, we study the central elements of pseudoeffect algebras in detail, and give some equivalent characterizations of central elements in pseudoeffect algebras. Finally, we further discuss the central elements in atomic pseudoeffect algebras, and get some interesting results.

3.1. Principal elements of pseudoeffect algebras

In order to study the structure of pseudoeffect algebras, we introduce the definition of principal element in pseudoeffect algebras. Then we give the conditions which guarantee that pseudoeffect algebras are isomorphic to the direct product of its principal ideals.

DEFINITION 3.1. ([4]) A nonempty subset I of a pseudoeffect algebra E is called an *ideal* in E if, for $p, q \in E$ such that $p \oplus q$ exists, $p \oplus q \in I$ if and only if $p, q \in I$. The ideal I generated by a single element $e \in E$ is called a *principal ideal.*

Let E be a pseudoeffect algebra. Then the interval $E[0, e] = \{x \in E : \ x \leqslant e\}$ is a principal ideal if and only if e is principal in the sense of the following definition.

DEFINITION 3.2. An element e of a pseudoeffect algebra E is said to be *principal* if, for $p, q \in E$ such that $p, q \leqslant e$, whenever $p \oplus q$ exists, then $p \oplus q \leqslant e$.

PROPOSITION 3.1. *If q is principal, and $q \leqslant u$, $u \in E$, then $q \wedge (u \ominus_r q) = q \wedge (u \ominus_l q) = 0$. In particular, if E is a weakly commutative pseudoeffect algebra, then $q \wedge q' = 0$.*

P r o o f. If q is principal and $p \leqslant q, u \ominus_r q$, then $p \oplus q$ exists and $p, q \leqslant q$, hence $p \oplus q \leqslant q = 0 \oplus q$, hence $p = 0$ by the cancellation law. Thus, $q \wedge (u \ominus_r q) = 0$. Similarly, we can prove $q \wedge (u \ominus_l q) = 0$. □

In general, we do not necessarily assume that e is principal. Thus the interval $E[0, e]$ is not necessarily an ideal in E. For instance, in Example 2.2, $\hat{P}[0, e] =$

$\{0, a, b, c, e\}$ is not an ideal of $\hat{P}$, since $b \oplus a$ exists in $\hat{P}$ but $b \oplus a \notin \hat{P}[0, e]$. In fact, the interval $E[0, e]$ is a de Morgan poset in the sense of the following proposition.

Proposition 3.2. *Let E be a pseudoeffect algebra, and $0 \neq e \in E$. Define $\tau, \bar{\tau}\colon E[0, e] \to E[0, e]$ by $\tau(p) = e \ominus_r p$ and $\bar{\tau}(p) = e \ominus_l p$ for all $p \in E[0, e]$. Then $\tau(0) = \bar{\tau}(0) = e, \tau(e) = \bar{\tau}(e) = 0$, and the following de Morgan-type laws hold:*

1. *If $p, q \in E[0, e]$ and $p \vee q$ exists in E, then $\tau(p) \wedge \tau(q) = \tau(p \vee q)$ and $\bar{\tau}(p) \wedge \bar{\tau}(q) = \bar{\tau}(p \vee q)$.*
2. *If $p, q \in E[0, e]$ and $p \wedge q$ exists in E, then $\tau(p \wedge q) = \tau(p) \vee \tau(q)$ and $\bar{\tau}(p \wedge q) = \bar{\tau}(p) \vee \bar{\tau}(q)$.*

Proof. We will prove (1), and (2) is similar to prove. Obviously, $\tau(p \vee q) \leqslant \tau(p), \tau(q)$. If $x \in E$ and $x \leqslant \tau(p), \tau(q)$, then $x \oplus p, x \oplus q \leqslant e$ and $p, q \leqslant e \ominus_l x$, hence $p \vee q \leqslant e \ominus_l x$ and $x \oplus (p \vee q) \leqslant e$, and so $x \leqslant e \ominus_r (p \vee q) = \tau(p \vee q)$. Then $\tau(p) \wedge \tau(q) = \tau(p \vee q)$. □

Remark 3.1. Let E be a pseudoeffect algebra and e be principal. If we restrict the partial addition operation on the interval $E[0, e]$, then $E[0, e]$ is a pseudoeffect algebra.

Definition 3.3. ([9]) A mapping $\phi\colon E \to F$, where E and F are pseudoeffect algebras, is called a *homomorphism* if

1. $\phi(0) = 0$ and $\phi(1) = 1$;
2. $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$, whenever $a \oplus b$ exists in E. A homomorphism $\phi\colon E \to F$ is called a *monomorphism* if for any $a, b \in E$, $\phi(a) \leqslant \phi(b)$ implies $a \leqslant b$. A surjective monomorphism is called an *isomorphism*.

Remark 3.2. If $\{E_\alpha\}_{\alpha \in J}$ is a family of pseudoeffect algebras, then the cartesian product $E := \prod_{\alpha \in J} E_\alpha$ is organized into a pseudoeffect algebra in the obvious way using coordinate operations and relations. The projection mappings $\pi_\alpha\colon E \to E_\alpha$ are surjective pseudoeffect algebra morphisms for all $\alpha \in J$.

The proof of the following theorem is a routine verification.

Theorem 3.1. *For $j = 1, 2, \ldots, n$ ($n \geqslant 2$), let E_j be a pseudoeffect algebra with the unit 1_j, let $E = E_1 \times E_2 \times \cdots \times E_n$, and define $e_j := (0, \ldots, 0, 1_j, 0, \ldots, 0)$ with 1_j in the jth coordinate position. Then:*

1. *Each e_j is principal in E.*
2. *The principal ideal $E[0, e_j]$ in E is isomorphic to E_j under the restriction to $E[0, e_j]$ of the projection homomorphism $\pi_j\colon E \to E_j$.*

(3) *$e_1, e_2, \dots, e_n$ is an orthogonal sequence in E and $e_1 \oplus \cdots \oplus e_n = 1$ is the unit element in E.*

(4) *Every element $p \in E$ can be represented uniquely in the form $p = p_1 \oplus \cdots \oplus p_n$ with $p_j \in E[0, e_j]$; indeed, $p_j = p \wedge e_j$ for $j = 1, 2, \dots, n$.*

By Remark 3.1 and Remark 3.2, if $e_1, e_2, \dots, e_n$ $(n \geqslant 2)$ are principal elements of pseudoeffect algebra E, then $\prod_j E[0, e_j]$ is a pseudoeffect algebra. In the following lemma, we will give the condition under which $\prod_j E[0, e_j]$, the direct product of intervals $E[0, e_j]$, is isomorphic to the pseudoeffect algebra E.

Lemma 3.1. *Let $e_1, e_2, \dots, e_n$ $(n \geqslant 2)$ be a finite orthogonal sequence of nonzero, principal elements of pseudoeffect algebra E and $e_1 \oplus e_2 \oplus \cdots \oplus e_n = 1$. Define $\phi\colon \prod_j E[0, e_j] \to E$ by $\phi(p_1, p_2, \dots, p_n) := p_1 \oplus p_2 \oplus \cdots \oplus p_n$ for all $(p_1, p_2, \dots, p_n) \in \prod_j E[0, e_j]$. Then ϕ is a homomorphism if and only if the following condition* (C1) *is satisfied:*

(C1) *if $x \leqslant e_i, y \leqslant e_j$ for any $i \neq j$, $i, j = 1, 2, \dots, n$, then $x \oplus y = y \oplus x$.*

P r o o f. Suppose $\phi\colon \prod_j E[0, e_j] \to E$ is a homomorphism between the two pseudoeffect algebras. Then we have $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$, for $a, b \in \prod_j E[0, e_j]$, with $a \oplus b$ existing. We can assume $i = 1$ and $j = 2$. Let $x \leqslant e_1, y \leqslant e_2$. Since $(x, y, 0, \dots, 0) = (x, 0, \dots, 0) \oplus (0, y, \dots, 0)$, we have $x \oplus y = \phi((x, y, 0, \dots, 0)) = \phi((x, 0, \dots, 0)) \oplus \phi((0, y, 0, \dots, 0)) = \phi((0, y, 0, \dots, 0)) \oplus \phi((0, x, 0, \dots, 0)) = y \oplus x$.

Conversely, we have to prove that $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$, for $a, b \in \prod_j E[0, e_j]$. If $a = (a_1, a_2, \dots, a_n), b = (b_1, b_2, \dots, b_n) \in \prod_j E[0, e_j]$ and $a \oplus b$ exists, then $a_i \oplus b_i$ exist for $i = 1, 2, \dots, n$. By the assumption, thus we have $\phi(a \oplus b) = (a_1 \oplus b_1) \oplus (a_2 \oplus b_2) \oplus \cdots \oplus (a_n \oplus b_n) = (a_1 \oplus a_2 \cdots \oplus a_n) \oplus (b_1 \oplus b_2 \oplus \cdots \oplus b_n) = \phi(a) \oplus \phi(b)$. And $\phi(e_1, e_2, \dots, e_n) = e_1 \oplus e_2 \oplus \cdots \oplus e_n = 1$, so we have that ϕ is a homomorphism. □

The following theorem deals with factoring a pseudoeffect algebra as an internal direct product of ideals of generated by orthogonal principal elements.

Theorem 3.2. *Let $e_1, e_2, \dots, e_n$ $(n \geqslant 2)$ be a finite orthogonal sequence of nonzero, principal elements of pseudoeffect algebra E and $e_1 \oplus e_2 \oplus \cdots \oplus e_n = 1$. If ϕ is a homomorphism between $\prod_j E[0, e_j]$ and E just as in* Lemma 3.1*, then ϕ is an isomorphism if and only if it is surjective. Furthermore, if ϕ is an* *isomorphism, then, for* $p \in E$*, the infima* $p \wedge e_1, p \wedge e_2, \dots, p \wedge e_n$ *exist in* E *and* $p = (p \wedge e_1) \oplus (p \wedge e_2) \oplus \dots \oplus (p \wedge e_n)$*.*

P r o o f. If ϕ is an isomorphism, then it is surjective by definition.

Conversely, it suffices to show that ϕ is a monomorphism. Suppose $(p_1, \dots, p_n)$, $(q_1, \dots, q_n) \in \prod_j E[0, e_j]$ with $p := \phi(p_1, p_2, \dots, p_n) \leqslant q := \phi(q_1, q_2, \dots, q_n)$ in E. Let $(q_1, q_2, \dots, q_n) \oplus (r_1, r_2, \dots, r_n) = 1$, so that $r := \phi(r_1, r_2, \dots, r_n) = q^{\sim}$. Then $r_1 \leqslant r_1 \oplus r_2 \oplus \dots \oplus r_n = r = q^{\sim}$, so $q \leqslant r_1^{-}$ and, therefore, $p_1 \leqslant p_1 \oplus p_2 \oplus \dots \oplus p_n = p \leqslant q \leqslant r_1^{-}$. Since e_1 is principal and $p_1, r_1 \leqslant e_1$, it follows that $p_1 \oplus r_1 \leqslant e_1 = q_1 \oplus r_1$; hence, $p_1 \leqslant q_1$ by the cancellation law. By symmetry, $p_j \leqslant q_j$ for all j, so ϕ is an isomorphism.

Again suppose that $(p_1, p_2, \dots, p_n) \in \prod_j E[0, e_j]$ with $p := p_1 \oplus p_2 \oplus \dots \oplus p_n$. We note that $p_1 \leqslant p, e_1$ and claim that, in fact, $p_1 = p \wedge e_1$. To prove this, suppose that $x \leqslant p, e_1$. We have to prove that $x \leqslant p_1$. Then $(x, 0, \dots, 0) \in \prod_j E[0, e_j]$ with $\phi(x, 0, \dots, 0) = x \leqslant p = \phi(p_1, p_2, \dots, p_n)$. Because ϕ is an isomorphism, we conclude that $x \leqslant p_1$. Similarly, $p_i = p \wedge e_i$, $i = 2, \dots, n$. □

Corollary 3.1. *Let* $e_1, e_2, \dots, e_n$ *be a finite orthogonal sequence of nonzero, principal elements of a pseudoeffect algebra* E *satisfying the condition* (C1)*, such that for all* $p \in E$*, there are elements* $p_j \leqslant e_j$ *for* $j = 1, 2, \dots, n$ *satisfying the condition* $p = p_1 \oplus p_2 \oplus \dots \oplus p_n$*. Then all orthogonal sums of subsequences of* $e_1, e_2, \dots, e_n$ *are principal elements of* E*. In particular,* $e_1 \oplus e_2 \oplus \dots \oplus e_n = 1$*.*

3.2. Central elements of pseudoeffect algebras

In [7], A. Dvurečenskij introduced the definition of central elements of pseudoeffect algebras and proved that the center is a Boolean algebra of a pseudoeffect algebra. Central elements are important to study the structure of pseudoeffect algebras. For example, A. Dvurečenskij proved the Cantor-Bernstein theorem using central elements ([7]). In this subsection, we give some conditions that guarantee that an element in a pseudoeffect algebra is central.

Definition 3.4. ([7]) An element a of a pseudoeffect algebra E is said to be *central* if there exists an isomorphism $f_a \colon E \to [0, a] \times [0, a^{\sim}]$ such that $f_a(a) = (a, 0)$ and if $f_a(x) = (x_1, x_2)$, then $x = x_1 \oplus x_2$ for any $x \in E$.

The *center* $C(E)$ is the set of all central elements in pseudoeffect algebra E.

Proposition 3.3. ([7]) *If* $a \in C(E)$*, then:*

(1) $f_a(a^{\sim}) = (0, a^{\sim})$

(2) *If* $x \leqslant a$*, then* $f_a(x) = (x, 0)$*.*

(3) $a \wedge a^{\sim} = 0$.
(4) *If* $y \leqslant a^{\sim}$, *then* $f_a(y) = (0, y)$.
(5) $a^{-} = a^{\sim}$.
(6) *For any* $x \in E$, $x \wedge a \in E$ *and* $x \wedge a^{\sim} \in E$, *and* $f_a(x) = (x \wedge a, x \wedge a^{\sim})$.
(7) *If* $f_a(x) = (x_1, x_2)$, *then* $x = x_1 \vee x_2$, $x_1 \wedge x_2 = 0$, *and* $x_2 \oplus x_1 = x$.

The following important result shows that the center represent the "classical part" in the quantum structures.

Theorem 3.3. ([7]) *Let* E *be a pseudoeffect algebra. If* $e, f \in C(E)$, *then* $e \wedge f \in E$ *and* $e \wedge f \in C(E)$, *and* $C(E) = (C(E); \wedge, \vee, ', 0, 1)$ *is a Boolean algebra.*

Now, we will give some characterizations of central elements in pseudoeffect algebras.

Proposition 3.4. *Let* E *be a pseudoeffect algebra. Suppose* $a \in E$ *satisfies the following conditions:*

(1) *for all* $p \in E$, *there exist* $q, r \in E$ *such that* $q \leqslant a, r \leqslant a^{\sim}$, *and* $p = q \oplus r$;
(2) a *and* $a^{\sim}$ *are principal;*
(3) *for all* $b, c \in E$, $b \leqslant a, c \leqslant a^{\sim}$, $b \oplus c$, $c \oplus b$ *exist and* $b \oplus c = c \oplus b$.

Then we have the following statements (for every $p \in P$*):*

(4) $a^{-} = a^{\sim}$. *We write* a' *for* $a^{-} = a^{\sim}$.
(5) $a \wedge p$ *and* $a' \wedge p$ *exist in* E.
(6) $p = (a \wedge p) \oplus (a' \wedge p)$.
(7) $a \oplus p$ *exists if and only if* $a \wedge p = 0$ *if and only if* $p \oplus a$ *exists.*

Proof. By (3), we have $a \oplus a^{\sim} = a^{\sim} \oplus a = 1$, thus $a^{-} = a^{\sim}$.

We can assume $a \neq 0, 1$, so a, a' form an orthogonal sequence of nonzero principal elements satisfying the condition (C1). Applying (1) and Theorem 3.2 to $e_1 := a$ and $e_2 := a'$, we obtain (5) and (6). Part (7) is a consequence of (6). □

Proposition 3.5. *Let* E *be a pseudoeffect algebra. An element* a *is a central element of* E *if and only if* a *satisfies the conditions* (1), (2) *and* (3) *of* Proposition 3.4.

Proof. Suppose a is central element of E. Then we have $a^{\sim} = a^{-}$ by Proposition 3.3(5). By Proposition 3.3(6) and (7), we have that (1) and (3) hold. Let $x, y \leqslant a$ and $x \oplus y$ exists in E. Then $f_a(x \oplus y) = f_a(x) \oplus f_a(y) = (x, 0) \oplus (y, 0) =$

$(x \oplus y, 0)$, thus $f_a(x \oplus y) \leqslant f_a(a)$, hence, $x \oplus y \leqslant a$. Then a is principal. Similarly, $a^\sim$ is principal.

Conversely, suppose a satisfies the conditions (1), (2) and (3) in the proposition. Then a and $a^\sim$ are principal. Let $e_1 = a$ and $e_2 = a^\sim$. By (1), (3) and Theorem 3.2, there is an isomorphism $\phi\colon [0, e_1] \times [0, e_2] \to E$, where $\phi(x_1, x_2) = x_1 \oplus x_2$. Let $f_a = \phi^{-1}$. Then f_a is an isomorphism from E onto $[0, a] \times [0, a^\sim]$ such that $f_a(a) = (a, 0)$. Hence, a is a central element of E. □

Proposition 3.6. *Let E be a pseudoeffect algebra. Then $a \in C(E)$ if and only if there is an orthogonal sequence $e_1, e_2, \dots, e_n$ of principal elements in E satisfying the condition* (C1) *such that, for all $p \in E$, there are elements $p_j \leqslant e_j$ for $j = 1, 2, \dots, n$ with $p = p_1 \oplus p_2 \oplus \dots \oplus p_n$, and a is an orthogonal sum of a subsequence of $e_1, e_2, \dots, e_n$.*

P r o o f. If $a \in C(E)$, let $e_1 := a$ and $e_2 := a'$. Conversely, by Corollary 3.1, we have that $e_1 \oplus e_2 \oplus \dots \oplus e_n = 1$. Without loss of generality, we can assume that $a = e_1 \oplus \dots \oplus e_k$ and $a^\sim = e_{k+1} \oplus \dots \oplus e_n$. If $p \in E$, then there exist $p_j \leqslant e_j$ for $j = 1, 2, \dots, n$ with $p = p_1 \oplus p_2 \oplus \dots \oplus p_n$. Then let $q := p_1 \oplus \dots \oplus p_k \leqslant a$ and $r := p_{k+1} \oplus \dots \oplus p_n \leqslant a^\sim$, we have $p = q \oplus r = r \oplus p$, by the assumption that the sequence $e_1, e_2, \dots, e_n$ satisfies the condition (C1). Then a is a central element by Proposition 3.5. □

Proposition 3.7. *Let E be a pseudoeffect algebra. Then $a \in E$ is central if and only if the following conditions hold:*

(1) *a and $a^\sim$ are principal.*

(2) *$b = 0$ whenever $b \wedge a = b \wedge a^\sim = 0$.*

(3) *$[0, a] \cap [0, b]$, $[0, a^\sim] \cap [0, b]$ have maximal elements for every $b \in E$.*

(4) *For all $b, c \in E$, $b \leqslant a, c \leqslant a^\sim$, $b \oplus c$, $c \oplus b$ exist and $b \oplus c = c \oplus b$.*

P r o o f. Let a be a central element. It suffices to prove the conditions (2) and (3). Obviously, (2) follows from Proposition 3.4(4) and (6). (3) follows from Proposition 3.4(4) and (5).

Conversely, let a satisfy the conditions (1), (2), (3) and (4). By (4), we have that $a^- = a^\sim$. Let $b \in E$. Assume b_1 is a maximal element of $[0, a] \cap [0, b]$ and b_2 is a maximal element of $[0, a^\sim] \cap [0, b \ominus_l b_1]$. Let $c = (b \ominus_l b_1) \ominus_l b_2$. We assert that $a \wedge c = a^\sim \wedge c = 0$. In fact, if $d \leqslant c, a$, then $d \leqslant (b \ominus_l b_1) \ominus_l b_2$ and $d \leqslant b \ominus_l b_1$. Since $d, b_1 \leqslant a$ and a is principal, $b_1 \oplus d \leqslant a$. Since b_1 is maximal element of $[0, a] \cap [0, b]$, $b_1 \oplus d = b_1$ and therefore $d = 0$, i.e., $c \wedge a = 0$. Similarly, we have $a^\sim \wedge c = 0$. Hence, by assumption (2), $c = 0$. Therefore, $b = b_1 \oplus b_2$. Hence, a is a central element by Proposition 3.5. □

Remark 3.3. The condition (2) of Proposition 3.7 is a weak form of distributivity [15].

3.3. Central elements of atomic pseudoeffect algebras

In this subsection, we discuss central elements in atomic pseudoeffect algebras. Firstly, we further discuss the weak distributivity in atomic pseudoeffect algebras.

Definition 3.5. ([15]) An *atom* of a pseudoeffect algebra E is a minimal element of $E - \{0\}$. A *coatom* of a pseudoeffect algebra is the orthosupplement of an atom (i.e., a is a coatom iff both a^- and $a^\sim$ are atoms). A pseudoeffect algebra is *atomic*, if every nonzero element dominates an atom (i.e., there is an atom less than or equal to it). A pseudoeffect algebra is *atomistic*, if for any nonzero element a, there is an atom subset A of E such that $a = \bigvee A$.

Lemma 3.2. *Let E be a pseudoeffect algebra. Let us consider the following conditions:*

(1) *For all $a, b \in E$, if $a \wedge b = a \wedge b^- = 0$, then $a = 0$.*

(2) *For all $a, b \in E$, if a is an atom, then either $a \leqslant b$ or $a \leqslant b^-$.*

(3) *For all $a, b \in E$, if a, b are atoms and $a \neq b$, then $a \oplus b$ exists.*

The condition (1) *implies the condition* (2) *which implies the condition* (3).

If the pseudoeffect algebra E is atomic, then the condition (2) *implies the condition* (1).

If the pseudoeffect algebra E is atomistic, then the condition (3) *implies the condition* (2).

Proof.

(1) implies (2). Let $a, b \in E$ and let a be an atom. If $a \nleqslant b$, then $a \wedge b = 0$. If $a \nleqslant b^-$, then $a \wedge b^- = 0$. Thus, $a = 0$ by (1), contradicting $a \neq 0$.

(2) implies (3). Let $a, b \in E$ be distinct atoms. Then $a \nleqslant b$ and, according to condition (2), $a \leqslant b^-$, thus $a \oplus b$ exists.

(2) implies (1). Assume pseudoeffect algebra E is atomic. Suppose that there are elements $a, b \in E$, $a \neq 0$, such that $a \wedge b = a \wedge b^- = 0$. Since E is atomic, there is an atom $c \in E$ such that $c \leqslant a$. Hence, $c \wedge b = c \wedge b^- = 0$, and therefore $c \nleqslant b$ and $c \nleqslant b^-$ which contradicts the assumption.

(3) implies (2). Assume pseudoeffect algebra E is atomistic. Let $a, b \in E$ and let a be an atom. Suppose that $a \nleqslant b$. If $b = 0$ then $b^- = 1 \geqslant a$. Let $b \neq 0$. Since E is atomistic, there is a set $A_b \subseteq E$ of atoms such that $b = \bigvee A_b$. Since $a \notin A_b$, we obtain, according to the condition (3), that $a \leqslant c^-$ and $c \leqslant a^\sim$ for every element $c \in A_b$. Hence, $b = \bigvee A_b \leqslant a^\sim$, $a \leqslant b^-$. □

Similarly to Lemma 3.2, we have the following proposition.

Lemma 3.3. *Let E be a pseudoeffect algebra. Let us consider the following conditions:*

(1) *For all $a, b \in E$, if $a \wedge b = a \wedge b^{\sim} = 0$, then $a = 0$.*

(2) *For all $a, b \in E$, if a is an atom, then either $a \leqslant b$ or $a \leqslant b^{\sim}$.*

(3) *For all $a, b \in E$, if a, b are atoms and $a \neq b$, then $b \oplus a$ exists.*

The condition (1) *implies the condition* (2) *which implies the condition* (3).

If the pseudoeffect algebra E is atomic, then the condition (2) *implies the condition* (1).

If the pseudoeffect algebra E is atomistic, then the condition (3) *implies the condition* (2).

By Proposition 3.7, Lemma 3.2 and Lemma 3.3, we can get the following result which give some conditions that guarantee that an element in pseudoeffect algebra is central.

Theorem 3.4. *Let E be an atomic pseudoeffect algebra satisfying the condition* (2) *of* Lemma 3.2 *or an atomistic pseudoeffect algebra satisying the condition* (3) *of* Lemma 3.2. *Then $a \in E$ is central if and only if a satisfies the conditions* (1), (3) *and* (4) *of* Proposition 3.7.

In the following part of this section, we will study when an atom is central in pseudoeffect algebras. And these results generalize those in effect algebras in [15, 16].

Lemma 3.4. *Let E be a pseudoeffect algebra and let $a \in E$ be an atom.*

(1) *The atom a is principal if and only if $a \oplus a$ does not exist.*

(2) *If the coatom a^{-} is principal, then for every element $b \in E$ either $a \nleqslant b$ or $a \nleqslant b^{\sim}$.*

(2′) *If the coatom $a^{\sim}$ is principal, then for every element $b \in E$ either $a \nleqslant b$ or $a \nleqslant b^{-}$.*

Proof.

(1) If a is principal, then $a \oplus a$ does not exist. Conversely, if $a \oplus a$ does not exist, and if $x \oplus y$ exists for $x, y \leqslant a$, then $x = y = 0$ and thus $x \oplus y = 0 \leqslant a$ or $\{x, y\} = \{0, a\}$ and then $x \oplus y = a \leqslant a$.

(2) If $a \leqslant b$ and $a \leqslant b^{\sim}$, then $b^{-} \leqslant a^{-}$ and $b \leqslant a^{-}$. Hence, $b^{-} \oplus b = 1 \leqslant a^{-}$ and therefore $a^{-} = 1$ contradicts $a \neq 0$.

(2′) is similar. □

LEMMA 3.5. *Let E be a pseudoeffect algebra, $a, b \in E$ and a be an atom. Consider the following conditions:*

(1) *There are $b_1, b_2 \in E$ such that $b_1 \leqslant a, b_2 \leqslant a^{\sim}$ and $b = b_1 \oplus b_2$.*

(2) *$a \leqslant b$ or $a \leqslant b^{-}$.*

(1′) *There are $b_1, b_2 \in E$ such that $b_1 \leqslant a^{-}, b_2 \leqslant a$ and $b = b_1 \oplus b_2$.*

(2′) *$a \leqslant b$ or $a \leqslant b^{\sim}$.*

Then (1) *and* (2) *are equivalent;* (1′) *and* (2′) *are also equivalent.*

Proof. We only prove that (1) and (2) are equivalent.

(1) implies (2). If $a \nleqslant b$ and $b = b_1 \oplus b_2$, then $b_1 = 0$ and $b = b_2$. Hence, $b \leqslant a^{\sim}$ and $a \leqslant b^{-}$.

(2) implies (1). If $a \leqslant b$, then there exists b_2 such that $a \oplus b_2 = b$ and $b_2 \leqslant a^{\sim}$. If $a \leqslant b^{-}$, then $b \leqslant a^{\sim}$ and $b = 0 \oplus b$. □

Now, we get the conditions ensuring that an atom is central in a pseudoeffect algebra.

PROPOSITION 3.8. *Let E be a pseudoeffect algebra. An atom a of E is central iff the following conditions hold:*

(1) *$a^{-} = a^{\sim}$ and a^{-} is principal.*

(2) *For any $b \in E$, if $a \oplus b$ exists, then $a \oplus b = b \oplus a$.*

(3) *For any $b \in E$, $a \leqslant b$ or $a \leqslant b^{-}$.*

Proof. Suppose that a is central. By Proposition 3.3(5) and Proposition 3.5(2), we have (1) holds. Since a is an atom, we get (2) holds by Proposition 3.5(3). By Proposition 3.5(1) and Lemma 3.5, we have (3) holds.

Conversely, by (3) and Lemma 3.5, we have Proposition 3.5(1) holds. Hence, by Proposition 3.5, a is central. □

PROPOSITION 3.9. *If E is an atomic pseudoeffect algebra such that every nonempty subset of E has a maximal element, then every nonzero element of E is a finite sum of atoms.*

Proof. Let us suppose that an element $b \in E - \{0\}$ is not a finite sum of atoms and seek a contradiction. Since E is atomic, there is an atom $a_1 \in E$ such that $a_1 \leqslant b$. Since b is not a finite sum of atoms, we have $b \ominus_l a_1 \neq 0$ and therefore there is an atom $a_2 \leqslant b \ominus_l a_1$ such that $(b \ominus_l a_1) \ominus_l a_2 \neq 0$ and therefore there is an atoms a_3 such that $a_3 \leqslant (b \ominus_l a_1) \ominus_l a_2$. Continuing in this procedure, we obtain a sequence of atoms $a_1, a_2, \dots \in E$ such that $a_1 < a_1 \oplus a_2 < a_1 \oplus a_2 \oplus a_3 < \dots < b$, hence the set $\{a_1, a_1 \oplus a_2, a_1 \oplus a_2 \oplus a_3, \dots\}$ does not have a maximal element — a contradiction. □

By Proposition 3.8, Proposition 3.9 and Theorem 3.3, we can get the following result.

THEOREM 3.5. *Let E be a pseudoeffect algebra satisfying the following conditions:* (1), (2), (3) *and* (4). *Then E is a Boolean algebra.*

(1) *E is atomic and every non-empty subset of E has a maximal element.*

(2) *For every atom, $a^- = a^\sim$ and a^- is principal.*

(3) *For every atom $a \in E$ and every $b \in E$, $a \leqslant b$ or $b \leqslant a^-$.*

(4) *For every atom $a \in E$ and every $b \in E$, if $a \oplus b$ exists, then $a \oplus b = b \oplus a$.*

Proof. By (2), (3) and (4), we have that every atom in E is central. By Proposition 3.9 and (1), we have that E is atomistic. Then we have that $C(E)= E$. Hence, by Theorem 3.3, we have that E is a Boolean algebra. □

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