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STUDIES ON A MULTI-POINT BOUNDARY VALUE PROBLEM FOR SECOND ORDER DIFFERENTIAL EQUATION WITH p-LAPLACIAN

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ABSTRACT. The existence of at least one solution of the following multi-point boundary value problem

$$\begin{cases} [\phi(x'(t))]' = f(t,x(t),x'(t)), & t \in (0,1),\\ x(0) - \sum\limits_{i=1}^{m} \alpha_i x'(\xi_i) = 0, & \\ x'(1) - \sum\limits_{i=1}^{m} \beta_i x(\xi_i) = 0 & \end{cases}$$

is studied, where $\alpha_i \in \mathbb{R}$, $\beta_i \in \mathbb{R}$ for all $i = 1, \dots, m$. Examples are presented to illustrate the main result.

1. Introduction

In [10], applying a fixed point theorem in cones, sufficient conditions are given for the existence of a positive solution and multiple positive solutions of the following BVP

$$\begin{cases} [\phi(x'(t))]' + f(t,x(t)) = 0, & t \in (0,1),\\ x(0) - \alpha x'(\xi) = 0, & \\ x'(1) + \beta x(\eta) = 0. & \end{cases} \tag{1}$$

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Sun, Ge and Zhao [17], applying the monotone iterative technique and without the assumption of the existence of lower and upper solutions, obtained the existence of positive solutions for the BVP

$$\begin{cases} [\phi(x'(t))]' + q(t)f(t,x(t),x'(t)) = 0, & t \in (0,1), \\ x(0) - \alpha x'(\xi) = 0, \\ x'(1) + \beta x(\eta) = 0 \end{cases} \tag{2}$$

and the iterative schemes for approximating the solutions.

In [5], [6], by the use of fixed point index theory, Leray-Schauder degree and upper and lower solution method, some existence, nonexistence and multiplicity results of symmetric positive solutions of the BVP

$$\begin{cases} [\phi(x'(t))]' + \lambda a(t)f(t,x(t)) = 0, & t \in (0,1), \\ x(0) - \alpha x'(\xi) = 0, \\ x'(1) + \beta x(\eta) = 0. \end{cases} \tag{3}$$

are acquired.

Pang, Lian and Ge in [16] studied the following more generalized BVP

$$\begin{cases} [\phi(x'(t))]' + f(t,x(t),x'(t)) = 0, & t \in (0,1), \\ \alpha x(0) - \beta x'(\xi) = 0, \\ \gamma x'(1) + \delta x(\eta) = 0, \end{cases} \tag{4}$$

by using a generalization of Leggett-Williams fixed point theorem due to Avery and Peterson, sufficient conditions for the existence of at least three positive solutions of BVP(2) are established.

In [2], Feng and Ge, applying the fixed point theorem due to Avery and Peterson, studied the existence of at least three symmetric positive solutions to the following boundary value problem

$$\begin{cases} [\phi(x'(t))]' + q(t)f(t,x(t),x'(t)) = 0, & t \in (0,1), \\ x(0) - \sum\limits_{i=1}^{m} \mu_i x'(\xi_i) = 0, \\ x(1) - \sum\limits_{i=1}^{m} \mu_i x'(\eta_i) = 0, \end{cases} \tag{5}$$

where $0 < \xi_1 < \dots < \xi_m < \frac{1}{2}$ and $\xi_i + \eta_i = 1$ for all $i = 1, \dots, m$ and q, f are nonnegative continuous functions.

There is no paper concerned with the existence of solutions of multi-point Sturm-Liouville boundary value problems for p-Laplacian differential equations.

The purpose of this paper is to investigate the existence of solutions of the following multi-point boundary value problem of second order differential equation with p-Laplacian

$$\begin{cases} [\phi(x'(t))]' = f(t, x(t), x'(t)), \qquad t \in (0,1), \\ x(0) - \sum\limits_{i=1}^{m} \alpha_i x'(\xi_i) = 0, \\ x'(1) - \sum\limits_{i=1}^{m} \beta_i x(\xi_i) = 0, \end{cases} \tag{6}$$

where

- $0 < \xi_1 < \cdots < \xi_m < 1$;
- $\alpha_i, \beta_i \in \mathbb{R}$;
- f is continuous;
- ϕ is called p-Laplacian, $\phi(x) = |x|^{p-2}x$ for $x \neq 0$ and $\phi(0) = 0$ with $p > 1$, its inverse function is denoted by $\phi^{-1}(x)$ with $\phi^{-1}(x) = |x|^{q-2}x$ for $x \neq 0$ and $\phi^{-1}(0) = 0$ with $1/p + 1/q = 1$.

We establish sufficient conditions for the existence of at least one solution of BVP(6).

The methods used and the result in this paper are different from those in known papers [1], [2], [4]–[24]. Our theorems are new.

The following of this paper is organized as follows: the main result is presented in Section 2, and some examples are given in Section 3.

2. Main results

In this section, we present the main results in this paper.

Lemma 2.1. *$\sum\limits_{i=1}^{m} a_i^\sigma \leq K_\sigma^{m-1} \left(\sum\limits_{i=1}^{m} a_i \right)^\sigma$ for all $a_i \geq 0$ and $\sigma > 0$, where K_σ is defined by $K_\sigma = 1$ for $\sigma \geq 1$ and $K_\sigma = 2$ for $\sigma \in (0,1)$.*

Proof.
Case 1. $m = 2$.
Without loss of generality, suppose $a_1 \geq a_2$. Let $g(x) = K_\sigma(1+x)^\sigma - (1 + x^\sigma)$, $x \in [1, +\infty)$, then

$$g(1) = K_\sigma 2^\sigma - 2 = \begin{cases} 2^\sigma - 2 \geq 0, & \sigma \geq 1, \\ 2^{\sigma+1} - 2 \geq 0, & \sigma \in (0,1) \end{cases}$$

and for $x \in [1, \infty)$, we get

$$\begin{aligned} g'(x) &= \sigma x^{\sigma-1}[K_\sigma(1+1/x)^{\sigma-1} - 1] \\ &\geq \begin{cases} 0, & \sigma \geq 1, \\ \sigma x^{\sigma-1}[2(1+1/1)^{0-1} - 1] = 0, & \sigma \in (0,1). \end{cases} \end{aligned}$$

We get that $g(x) \geq g(1)$ for all $x \geq 1$ and so $1 + x^\sigma \leq K_\sigma(1+x)^\sigma$ for all $x \in [1, +\infty)$. Hence $a_1^\sigma + a_2^\sigma = a_2^\sigma[1 + (a_1/a_2)^\sigma] \leq K_\sigma a_2^\sigma[1 + a_1/a_2]^\sigma = K_\sigma(a_1 + a_2)^\sigma$.
Case 2. $m > 2$.
It is easy to see that

$$\begin{aligned}
\sum_{i=1}^m a_i^\sigma &= a_1^\sigma + a_2^\sigma + \sum_{i=3}^m a_i^\sigma \\
&\leq K_\sigma(a_1+a_2)^\sigma + \sum_{i=3}^m a_i^\sigma \\
&\leq K_\sigma\left((a_1+a_2)^\sigma + \sum_{i=3}^m a_i^\sigma\right) \\
&\leq K_\sigma\left((a_1+a_2)^\sigma + a_3^\sigma + \sum_{i=4}^m a_i^\sigma\right) \\
&\leq K_\sigma\left(K_\sigma(a_1+a_2+a_3)^\sigma + \sum_{i=4}^m a_i^\sigma\right) \\
&\leq K_\sigma^2\left((a_1+a_2+a_3)^\sigma + \sum_{i=4}^m a_i^\sigma\right) \\
&\leq \ldots \\
&\leq K_\sigma^{m-1}\left(\sum_{i=1}^m a_i\right)^\sigma.
\end{aligned}$$

The proof is complete. □

Remark 1. It is easy to see that

$$\sum_{i=1}^m \phi(a_i) \leq K_{p-1}^{m-1}\phi\left(\sum_{i=1}^m a_i\right), \qquad \sum_{i=1}^m \phi^{-1}(a_i) \leq K_{q-1}^{m-1}\phi^{-1}\left(\sum_{i=1}^m a_i\right).$$

Lemma 2.2. $\left(\sum_{i=1}^m a_i\right)^\sigma \leq L_\sigma^{m-1}\left(\sum_{i=1}^m a_i^\sigma\right)$ *for all* $a_i \geq 0$ *and* $\sigma > 0$, *where* L_σ *is defined by* $L_\sigma = 2^{\sigma-1}$ *for* $\sigma \geq 1$ *and* $L_\sigma = 1$ *for* $\sigma \in (0,1)$.

Proof.
Case 1. $m = 2$.
Without loss of generality, suppose $a_1 \geq a_2$. Let

$$g(x) = (1+x)^\sigma - L_\sigma(1+x^\sigma), \qquad x \in [1, +\infty),$$

then

$$g(1) = 2^\sigma - 2L_\sigma = \begin{cases} 2^\sigma - 2^\sigma = 0, & \sigma \geq 1, \\ 2^{\sigma+1} - 2 \leq 0, & \sigma \in (0,1) \end{cases}$$

and for $x \in [1, \infty)$, we get

$$\begin{aligned} g'(x) &= \sigma x^{\sigma-1}[(1+1/x)^{\sigma-1} - L_\sigma] \\ &= \begin{cases} \sigma x^{\sigma-1}[(1+1/x)^{\sigma-1} - 2^{\sigma-1}] \leq 0, & \sigma \geq 1, \\ \sigma x^{\sigma-1}[(1+1/x)^{\sigma-1} - 1] \leq 0, & \sigma \in (0,1). \end{cases} \end{aligned}$$

We get that $g(x) \leq g(1)$ for all $x \geq 1$ and so $(1+x)^\sigma \leq L_\sigma(1+x^\sigma)$ for all $x \in [1, +\infty)$. Hence $(a_1+a_2)^\sigma = a_2^\sigma(1+(a_1/a_2))^\sigma \leq L_\sigma a_2^\sigma[1+(a_1/a_2)^\sigma) = L_\sigma(a_1^\sigma + a_2^\sigma)$.

Case 2. $m > 2$.

It is easy to see that

$$\begin{aligned} \left(\sum_{i=1}^m a_i\right)^\sigma &= \left(a_1 + \sum_{i=2}^m a_i\right)^\sigma \\ &\leq L_\sigma\left(a_1^\sigma + \left(\sum_{i=2}^m a_i\right)^\sigma\right) \\ &\leq L_\sigma\left[a_1^\sigma + L_\sigma\left(a_2^\sigma + \left(\sum_{i=3}^m a_i\right)^\sigma\right)\right] \\ &\leq L_\sigma^2\left(a_1^\sigma + a_2^\sigma + \left(\sum_{i=3}^m a_i\right)^\sigma\right) \\ &\leq \ldots\ldots \\ &\leq L_\sigma^{m-1}\left(\sum_{i=1}^m a_i^\sigma\right). \end{aligned}$$

The proof is complete. □

Remark 2. It is easy to see that

$$\phi\left(\sum_{i=1}^m a_i\right) \leq L_{p-1}^{m-1}\sum_{i=1}^m \phi(a_i), \qquad \phi^{-1}\left(\sum_{i=1}^m a_i\right) \leq L_{q-1}^{m-1}\sum_{i=1}^m \phi^{-1}(a_i).$$

The following assumptions are supposed in this section.

$(\mathbf{H_1})$ There exist continuous nonnegative functions a, b and c so that

$$|f(t,x,y)| \leq a(t) + b(t)\phi(|x|) + c(t)\phi(|y|), \qquad (t,x,y) \in [0,1] \times \mathbb{R}^2;$$

($\mathbf{H_2}$) $\alpha_i, \beta_i \in \mathbb{R}$ for all $i = 1, \dots, m$, the inequality holds:

$$\left(L_{p-1}^{m-1}\sum_{i=1}^{m}\phi(|\beta_i|) + \int_0^1 b(s)\,\mathrm{d}s\right)\phi\left(\sum_{i=1}^{m}|\alpha_i| + 1\right) + \int_0^1 c(s)\,\mathrm{d}s < 1.$$

Consider the following boundary value problem

$$\begin{cases} x'(t) = \phi^{-1}(y(t)), \quad t \in (0,1), \\ y'(t) = f(t, x(t), \phi^{-1}(y(t))), \quad t \in (0,1), \\ x(0) = \sum\limits_{i=1}^{m} \alpha_i \phi^{-1}(y(\xi_i)), \\ y(1) = \phi\left(\sum\limits_{i=1}^{m} \beta_i x(\xi_i)\right). \end{cases} \tag{7}$$

It is easy to show that if $(x,y)^T$ is a solution of BVP(7), then x is a solution of BVP(6).

Choose $X = C^0[0,1] \times C^0[0,1]$. Denote the norm of $(x,y)^T \in X$ by

$$\|(x,y)^T\| = \max\left\{\|x\| = \max_{t\in[0,1]}|x(t)|,\ \|y\| = \max_{t\in[0,1]}|y(t)|\right\}.$$

Let $Y = X \times \mathbb{R}^2$. Denote the norm of $(x,y,a,b)^T \in Y$ by

$$\|(x,y,a,b)^T\| = \max\left\{\max_{t\in[0,1]}|x(t)|,\ \max_{t\in[0,1]}|y(t)|,\ \ |a|,\ \ |b|\right\}.$$

Then X and Y are real Banach spaces. Let

$$D(L) = \{(x,y)^T \in C^1(0,1) \times C^1(0,1)\} \subset X.$$

Define the linear operator $L\colon X \to Y$ and the nonlinear operator $N\colon X \to Y$ by

$$L\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} x'(t) \\ y'(t) \\ x(0) \\ y(1) \end{pmatrix}, \qquad (x,y) \in D(L)$$

and

$$N\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} \phi^{-1}(y(t)) \\ f(t, x(t), \phi^{-1}(y(t))) \\ \sum\limits_{i=1}^{m} \alpha_i \phi^{-1}(y(\xi_i)) \\ \phi\left(\sum\limits_{i=1}^{m} \beta_i x(\xi_i)\right) \end{pmatrix}, \qquad (x,y) \in X,$$

respectively. It is easy to see that $(x,y)^T$ is a solution of BVP(7) if and only if $(x,y)^T$ is a solution of the operator equation $L(x,y)^T = N(x,y)^T$.

Lemma 2.3. *Suppose that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. Then*

$$\Omega_1 = \{(x,y)^T : \ L(x,y)^T = \lambda N(x,y)^T, \ (x,y)^T \in D(L) \text{ for some } \lambda \in (0,1)\}$$

is bounded if

$$1 > \left(L_{p-1}^{m-1} \sum_{i=1}^{m} \phi(|\beta_i|) + \int_0^1 b(s)\,\mathrm{d}s \right) \phi\left(\sum_{i=1}^{m} |\alpha_i| + 1 \right) + \int_0^1 c(s)\,\mathrm{d}s.$$

Proof. For $(x,y)^T \in \Omega_1$, we get $L(x,y)^T = \lambda N(x,y)^T$, i.e.,

$$\begin{cases} x'(t) = \lambda\phi^{-1}(y(t)), \quad t \in (0,1), \\ y'(t) = \lambda f(t, x(t), \phi^{-1}(y(t))), \quad t \in (0,1), \\ x(0) = \lambda \sum\limits_{i=1}^{m} \alpha_i \phi^{-1}(y(\xi_i)), \\ y(1) = \lambda\phi\left(\sum\limits_{i=1}^{m} \beta_i x(\xi_i) \right). \end{cases} \tag{8}$$

It follows from Lemma 2.1 and Lemma 2.2 that

$$\begin{aligned} |y(1)| = \lambda \left| \phi\left(\sum_{i=1}^{m} \beta_i x(\xi_i) \right) \right| &\le L_{p-1}^{m-1} \sum_{i=1}^{m} \phi(|\beta_i|)\phi(|x(\xi_i)|) \\ &\le L_{p-1}^{m-1} \sum_{i=1}^{m} \phi(|\beta_i|)\phi(\|x\|). \end{aligned}$$

Then

$$\begin{aligned} |y(t)| &= \left| y(1) - \int_t^1 y'(s)\,\mathrm{d}s \right| \le |y(1)| + \int_0^1 |y'(s)|\,\mathrm{d}s \\ &\le L_{p-1}^{m-1} \sum_{i=1}^{m} \phi(|\beta_i|)\phi(\|x\|) + \int_0^1 |f(s, x(s), \phi^{-1}(y(s)))|\,\mathrm{d}s \\ &\le L_{p-1}^{m-1} \sum_{i=1}^{m} \phi(|\beta_i|)\phi(\|x\|) + \int_0^1 (a(s) + b(s)\phi(|x(s)|) + c(s)|y(s)|)\,\mathrm{d}s \\ &\le L_{p-1}^{m-1} \sum_{i=1}^{m} \phi(|\beta_i|)\phi(\|x\|) + \int_0^1 a(s)\,\mathrm{d}s + \int_0^1 b(s)\,\mathrm{d}s\phi(\|x\|) + \int_0^1 c(s)\,\mathrm{d}s\|y\| \\ &= \left(L_{p-1}^{m-1} \sum_{i=1}^{m} \phi(|\beta_i|) + \int_0^1 b(s)\,\mathrm{d}s \right) \phi(\|x\|) + \int_0^1 a(s)\,\mathrm{d}s + \int_0^1 c(s)\,\mathrm{d}s\|y\|. \end{aligned}$$

On the other hand, we get

$$|x(0)| = \left|\lambda \sum_{i=1}^{m} \alpha_i \phi^{-1}(y(\xi_i))\right| \leq \sum_{i=1}^{m} |\alpha_i| \phi^{-1}(\|y\|).$$

It follows that

$$\begin{aligned}
|x(t)| &= \left|x(0) + \int_0^t x'(s)\,\mathrm{d}s\right| \leq |x(0)| + \int_0^1 |x'(s)|\,\mathrm{d}s \\
&\leq \sum_{i=1}^{m} |\alpha_i| \phi^{-1}(\|y\|) + \int_0^1 \phi^{-1}(|y(s)|)\,\mathrm{d}s \\
&\leq \left(\sum_{i=1}^{m} |\alpha_i| + 1\right) \phi^{-1}(\|y\|).
\end{aligned}$$

Hence

$$\phi(\|x\|) \leq \phi\left(\left(\sum_{i=1}^{m} |\alpha_i| + 1\right) \phi^{-1}(\|y\|)\right) = \phi\left(\sum_{i=1}^{m} |\alpha_i| + 1\right) \|y\|.$$

It follows that

$$\begin{aligned}
|y(t)| \leq \left(L_{p-1}^{m-1} \sum_{i=1}^{m} \phi(|\beta_i|) + \int_0^1 b(s)\,\mathrm{d}s\right) \phi\left(\sum_{i=1}^{m} |\alpha_i| + 1\right) \|y\| + \int_0^1 a(s)\,\mathrm{d}s \\
+ \int_0^1 c(s)\,\mathrm{d}s \|y\|.
\end{aligned}$$

We get

$$\begin{aligned}
\|y\| \leq \left(L_{p-1}^{m-1} \sum_{i=1}^{m} \phi(|\beta_i|) + \int_0^1 b(s)\,\mathrm{d}s\right) \phi\left(\sum_{i=1}^{m} |\alpha_i| + 1\right) \|y\| + \int_0^1 a(s)\,\mathrm{d}s \\
+ \int_0^1 c(s)\,\mathrm{d}s \|y\|.
\end{aligned}$$

So

$$\left[1-\left(L_{p-1}^{m-1}\sum_{i=1}^{m}\phi(|\beta_i|)+\int_0^1 b(s)\,\mathrm{d}s\right)\phi\left(\sum_{i=1}^{m}|\alpha_i|+1\right)-\int_0^1 c(s)\,\mathrm{d}s\right]\|y\| \le \int_0^1 a(s)\,\mathrm{d}s.$$

Then

$$\|y\| \le \frac{\int_0^1 a(s)\,\mathrm{d}s}{1-\left(L_{p-1}^{m-1}\sum_{i=1}^{m}\phi(|\beta_i|)+\int_0^1 b(s)\,\mathrm{d}s\right)\phi\left(\sum_{i=1}^{m}|\alpha_i|+1\right)-\int_0^1 c(s)\,\mathrm{d}s} =: M_1.$$

Hence

$$\|x\| \le \left(\sum_{i=1}^{m}|\alpha_i|+1\right)\phi^{-1}(\|y\|) \le \left(\sum_{i=1}^{m}|\alpha_i|+1\right)\phi^{-1}(M_1) =: M_2.$$

It follows that

$$\|(x,y)^T\| \le \max\{M_1,\ M_2\}.$$

It follows that Ω_1 is bounded. □

Lemma M. ([3]) *Let X and Y be Banach spaces. Suppose $L\colon D(L)\subset X\to Y$ is a Fredholm operator of index zero with* $\operatorname{Ker} L=\{0\}$*, $N\colon X\to Y$ is L-compact on any open bounded subset of X. If $0\in\Omega\subset X$ is a open bounded subset and $Lx\neq\lambda Nx$ for all $x\in D(L)\cap\partial\Omega$ and some $\lambda\in[0,1]$, then there is at least one $x\in\Omega$ so that $Lx=Nx$.*

Theorem L1. *Suppose that* $(\mathbf{H_1}),(\mathbf{H_2})$ *hold. Then $BVP(6)$ has at least one solution.*

P r o o f. Let Ω be a non-empty open bounded subset of X such that $\Omega\supset\Omega_1$ centered at zero. It is easy to see that $\operatorname{Ker} L=\{0\}$, L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. Thus Lemma M implies that $L(x,y)^T=N(x,y)^T$ has at least one solution $(x,y)^T\in D(L)\cap\overline{\Omega}$, So x is a solution of BVP(6). The proof is complete. □

Now, suppose

$(\mathbf{H_3})$ $\alpha_i\ge 0$, $\beta_i\le 0$ for all $i=1,\dots,m$.

$(\mathbf{H_4})$ $f(t,x,y)\le 0$ for all $t\in[0,1]$, $(x,y)^T\in\mathbb{R}^2$, and there exist exist continuous nonnegative functions a, b and c so that

$$|f(t,x,y)|\le a(t)+b(t)\phi(|x|)+c(t)\phi(|y|),\qquad (t,x,y)\in[0,1]\times\mathbb{R}^2;$$

Lemma 2.4. *If* $(\mathbf{H_3}), (\mathbf{H_4})$ *hold and* (x, y) *is a solution of BVP*(8)*, then there exists a* $\xi \in [0, 1]$ *such that* $y(\xi) = 0$.

Proof.
Case 1. $y(t) > 0$ for all $t \in [0, 1]$.
At this case, $(\mathbf{H_4})$ implies that $y(t)$ is decreasing on $[0, 1]$. It follows that $x(t)$ is increasing on $[0, 1]$. Then (8) and $(\mathbf{H_3})$ implies that

$$x(0) = \lambda \sum_{i=1}^{m} \alpha_i \phi^{-1}(y(\xi_i)) \geq \lambda \sum_{i=1}^{m} \alpha_i \phi^{-1}(y(1)) \geq 0. \tag{9}$$

Then $x(t) > 0$ for all $t \in [0, 1]$. On the other hand, one sees from $(\mathbf{H_3})$ that

$$y(1) = \lambda \phi^{-1} \left(\sum_{i=1}^{m} \beta_i x(\xi_i) \right) \leq \lambda \phi^{-1} \left(\sum_{i=1}^{m} \beta_i \right) \phi^{-1}(x(0)) \leq 0, \tag{10}$$

a contradiction.
Case 2. $y(t) < 0$ for all $t \in [0, 1]$.
At this case, $(\mathbf{H_4})$ implies that $y(t)$ is decreasing on $[0, 1]$. It follows that $x(t)$ is decreasing on $[0, 1]$. Then (8) and $(\mathbf{H_3})$ implies that

$$x(0) = \lambda \sum_{i=1}^{m} \alpha_i \phi^{-1}(y(\xi_i)) \leq \lambda \sum_{i=1}^{m} \alpha_i \phi^{-1}(y(0)) \leq 0. \tag{11}$$

On the other hand, one sees that

$$y(1) = \lambda \phi^{-1} \left(\sum_{i=1}^{m} \beta_i x(\xi_i) \right) \geq \lambda \phi^{-1} \left(\sum_{i=1}^{m} \beta_i \right) \phi^{-1}(x(0)) \geq 0, \tag{12}$$

a contradiction. $\square$

It follows from Case 1 and Case 2 that there exists a $\xi \in [0, 1]$ such that $y(\xi) = 0$.

Lemma 2.5. *Suppose that* $(\mathbf{H_3})$ *and* $(\mathbf{H_4})$ *hold. Then*

$$\Omega_1 = \{(x, y)^T : \ L(x, y)^T = \lambda N(x, y)^T, \ (x, y)^T \in D(L) \text{ for some } \lambda \in (0, 1)\}$$

is bounded if

$$1 - \int_0^1 b(s) \, \mathrm{d}s \phi \left(\sum_{i=1}^{m} \alpha_i + 1 \right) - \int_0^1 c(s) \, \mathrm{d}s > 0.$$

Proof. For $(x,y)^T \in \Omega_1$, we get $L(x,y)^T = \lambda N(x,y)^T$, then one sees (8). It follows from Lemma 2.4 that there is $\xi \in [0,1]$ such that $y(\xi) = 0$. Then

$$\begin{aligned}
|y(t)| &= \left| y(\xi) + \int_\xi^t y'(s)\,\mathrm{d}s \right| \le \int_0^1 |y'(s)|\,\mathrm{d}s \\
&\le \int_0^1 |f(s, x(s), \phi^{-1}(y(s)))|\,\mathrm{d}s \\
&\le \int_0^1 (a(s) + b(s)\phi(|x(s)|) + c(s)|y(s)|)\,\mathrm{d}s \\
&\le \int_0^1 a(s)\,\mathrm{d}s + \int_0^1 b(s)\,\mathrm{d}s\phi(\|x\|) + \int_0^1 c(s)\,\mathrm{d}s\|y\|.
\end{aligned}$$

It is easy to get

$$\|x\| \le \left(\sum_{i=1}^m \alpha_i + 1\right) \phi^{-1}(\|y\|).$$

Thus

$$\|y\| \le \int_0^1 a(s)\,\mathrm{d}s + \int_0^1 b(s)\,\mathrm{d}s\phi\left(\sum_{i=1}^m \alpha_i + 1\right)\|y\| + \int_0^1 c(s)\,\mathrm{d}s\|y\|.$$

It follows that

$$\left(1 - \int_0^1 b(s)\,\mathrm{d}s\phi\left(\sum_{i=1}^m \alpha_i + 1\right) - \int_0^1 c(s)\,\mathrm{d}s\right)\|y\| \le \int_0^1 a(s)\,\mathrm{d}s.$$

Since

$$1 - \int_0^1 b(s)\,\mathrm{d}s\phi\left(\sum_{i=1}^m \alpha_i + 1\right) - \int_0^1 c(s)\,\mathrm{d}s > 0$$

we get

$$\|y\| \le \frac{\int_0^1 a(s)\,\mathrm{d}s}{1 - \int_0^1 b(s)\,\mathrm{d}s\phi\left(\sum_{i=1}^m \alpha_i + 1\right) - \int_0^1 c(s)\,\mathrm{d}s} =: M_1.$$

Then

$$\|x\| \le \left(\sum_{i=1}^m \alpha_i + 1\right)\phi^{-1}(M_1) =: M_2.$$

Hence $\|(x,y)^T\| \le \max\{M_1,\ M_2\}$. So Ω_1 is bounded. □

Theorem L2. *Suppose that* $(\mathbf{H_3}), (\mathbf{H_4})$ *hold. Then BVP*(6) *has at least one solution if*

$$\int_0^1 b(s)\,\mathrm{d}s\phi\left(\sum_{i=1}^{m}|\alpha_i|+1\right)+\int_0^1 c(s)\,\mathrm{d}s<1.$$

P r o o f. Let Ω be a non-empty open bounded subset of X such that $\Omega \supset \Omega_1$ centered at zero. It is easy to see that $\operatorname{Ker} L = \{0\}$, L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. Thus Lemma M implies that $L(x,y)^T = N(x,y)^T$ has at least one solution $(x,y)^T \in D(L) \cap \overline{\Omega}$, So x is a solution of BVP(6). The proof is complete. □

3. Examples

Now, we present some examples to illustrate the main results.

Example 1. Consider the following BVP

$$\begin{cases} x''(t) = a(t) + b(t)x(t) + c(t)x'(t), & t \in (0,1),\\ x(0) = \frac{1}{2}x'(1/2), \quad x'(1) = \frac{1}{2}x(\frac{1}{2}), \end{cases} \tag{13}$$

where a, b, c are continuous functions. We find from Theorem L1 that if

$$\left(\frac{1}{2}+\int_0^1 |b(s)|\,\mathrm{d}s\right)\frac{3}{2}+\int_0^1 |c(s)|\,\mathrm{d}s<1,$$

then BVP(13) has at least one solution.

Example 2. Consider the BVP problem

$$\begin{cases} (\phi_3(x'))' = -[a(t) + b(t)\phi_3(x(t)) + c(t)\phi_3(|x'(t)|)], & t \in (0,1),\\ x(0) = \frac{1}{2}x'(1/2), \quad x'(1) = \frac{1}{4}x(1/4) + \frac{1}{3}x(1/2), \end{cases} \tag{14}$$

where a, b, c are nonnegative continuous functions and $\phi_3(x) = x^2$ for $x \ge 0$ and $= -x^2$ for $x < 0$. We find $p = 3$ and $q = 3/2$. Then by application of Theorem L2, BVP(14) has at least one solution if

$$\int_0^1 b(s)\,\mathrm{d}s\phi_3\left(\frac{3}{2}\right)+\int_0^1 c(s)\,\mathrm{d}s<1.$$

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