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VARIATIONAL MEASURES AND THE KURZWEIL-HENSTOCK INTEGRAL

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ABSTRACT. For a given continuous function F on a compact interval E in the set $\mathbb{R}$ of reals the problem is how to describe the "total change" of F on a set $M \subset E$. Full variational measures $W_F(M)$ and $V_F(M)$ (see Section 2) in the sense presented by B. S. Thomson are introduced in this work to this aim. They are generated by two slightly different interval functions, namely the oscillation of F over an interval and the value of the additive interval function generated by F, respectively. They coincide with the concept of classical total variation if M is an interval and they are zero if on the set M the function F is of negligible variation.

The Kurzweil-Henstock integration is shortly described and some of its properties are studied using the variational measure $W_F(M)$ for the indefinite integral F of an integrable function f.

1. Notations, divisions, tags, gauges

Let $-\infty < a < b < \infty$ and let the compact interval $E = [a, b]$ be fixed in the sequel. The topology on E is induced by the usual topology on the set $\mathbb{R}$ of reals.

We denote by $\mathrm{Int}(M)$ the interior of a set $M \subset E$ and $\overline{M}$ denotes the closure of a set $M \subset E$.

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In the next I and J always denote closed subintervals of E. The set of all closed subintervals of J will be denoted by $\mathrm{Sub}(J)$. The empty set $\emptyset$ is also assumed to belong to $\mathrm{Sub}(J)$.

If I is nonempty, then by $l(I)$, $r(I)$ we denote the left, right endpoint of I, respectively.

The number $|I| = r(I) - l(I)$ is the length of I.

For the purposes of this paper a mapping T from a set Γ into a set M will be sometimes called a *system of elements* of M.

The notation $T = \{V_j :\ j \in \Gamma\}$ means that $T(j) = V_j \in M$ for $j \in \Gamma$. A system $\{V_j :\ j \in \Gamma\}$ of elements of M is called finite if Γ is finite. The usual use of this are mostly the cases $\Gamma = \mathbb{N}$ or $\Gamma = \mathbb{N}_k$ where $\mathbb{N}$ is the set of natural numbers and $N_k = \{j \in \mathbb{N} :\ j \leq k\}$.

When we will deal with a system of elements belonging to $\mathrm{Sub}(E)$, we will speak simply *about a system* (*of intervals*).

The set of all finite unions of closed subintervals of E (i.e. unions of elements of all finite systems) is denoted by $\mathrm{Alg}(E)$.

The set $\mathrm{Alg}(E)$ is closed with respect to finite unions and intersections. Any set $M \in \mathrm{Alg}(E)$ is the union of elements of a finite system $\{I_j :\ j \in \Gamma\}$, where $I_j \cap I_k = \emptyset$ for $j \neq k$. If $M \in \mathrm{Alg}(E)$, then clearly also $\overline{E \setminus M} \in \mathrm{Alg}(E)$.

A *division* is a finite system $D = \{I_j :\ j \in \Gamma\}$ of intervals, where $\mathrm{Int}\,(I_j) \cap I_k = \emptyset$ for $j \neq k$. This means that the elements of a division do not overlap.

For a given set $M \subset E$ the division D is called a *division in* M if $M \supset \bigcup_{j\in\Gamma} I_j$ D is called a *division of* M if $M = \bigcup_{j\in\Gamma} I_j$; and the division D *covers* M if $M \subset \bigcup_{j\in\Gamma} I_j$.

A division of M exists if and only if $M \in \mathrm{Alg}(E)$.

A map τ from $\mathrm{Sub}(E)$ into E is called a *tag* if $\tau(I) \in I$ for $I \in \mathrm{Dom}(\tau)$. In the sequel only tags of this sort will be used.

A *tagged system* is a pair (D, τ), where $D = \{I_j :\ j \in \Gamma\}$ is a system and τ is a tag defined on the range of D, i.e. on all I_j, $j \in \Gamma$. In this case we write usually τ_j instead of $\tau(I_j)$.

The tagged system (D, τ) is called M*-tagged* for some set $M \subset E$ if $\tau_j \in M$ for $j \in \Gamma$.

Given a function $f\colon E \to \mathbb{R}$ and a set $M \subset E$ we denote

$$|f|_M = \sup_{x\in M} |f(x)|.$$

A *gauge* is any function on E with values in the set $\mathbb{R}^+$ of positive reals. The set of all gauges is denoted by $\Delta(E)$.

For $\delta_1, \delta_2 \in \Delta(E)$ we write $\delta_1 \leq \delta_2$ if $\delta_1(x) \leq \delta_2(x)$ for $x \in E$. In this way a partial ordering in $\Delta(E)$ is defined and any finite set in $\Delta(E)$ has an infimum with respect to this ordering.

If $\delta \in \Delta(E)$, then a tagged system (D, τ), where $D = \{I_j : j \in \Gamma\}$, is called *$\delta$-fine* if $|I_j| < \delta(\tau_j)$ for $j \in \Gamma$.

If $\delta_1, \delta_2 \in \Delta(E)$, $\delta_1 \leq \delta_2$, then every δ_1-fine tagged system is also δ_2-fine.

Remark. Let us note that for a given $M \subset E$ and a gauge $\delta \in \Delta(E)$ in some situations it can be helpful to use divisions $D = \{I_j : j \in \Gamma\}$ with the property

$$|I_j| \leq |\delta|_{I_j \cap M}, \qquad j \in \Gamma,$$

instead of δ-fine M-tagged divisions. Let us call divisions of this type *δ-fine* and *M-related.*

If $\{I_j : j \in \Gamma\}$ is δ-fine and M-related and $I_j \cap M = \emptyset$ then $|\delta|_{I_j \cap M} = 0$. Hence $|I_j| = 0$ and the element I_j of the division $D = \{I_j : j \in \Gamma\}$ can be neglected in many of the considerations.

If $(D, \tau) = (\{I_j : j \in \Gamma\}, \tau)$ is an M-tagged δ-fine system then $\tau(I_j) = \tau_j \in M \cap I_j$ and $|I_j| \leq \delta(\tau_j) \leq |\delta|_{I_j \cap M}$ and $D = \{I_j : j \in \Gamma\}$ is δ-fine and M-related.

If, conversely, $D = \{I_j : j \in \Gamma\}$ is δ-fine and M-related then it need not be possible to find $\tau_j \in M \cap I_j$ for $j \in \Gamma$ such that $|I_j| \leq \delta(\tau_j)$.

The following crucial statement is known as Cousin's lemma (see e.g. [5, 3.4 Lemma] or any other relevant text on Kurzweil-Henstock integration).

Proposition 1.1. *To any $\delta \in \Delta(E)$ and $I \in \mathrm{Sub}(E)$ there exists a δ-fine division of I.*

Cousin's lemma can be used in many different ways. We shall use the following statements.

Lemma 1.2. *Let $I \in \mathrm{Sub}(E)$ and let A be a closed subset of I. Then to every $\delta \in \Delta(E)$ there is a δ-fine A-tagged division in I which covers A.*

Proof. Denote $\mathrm{dist}(x, A)$ the distance of a point $x \in \mathbb{R}$ from the set A. Let us set

$$\eta(x) = \begin{cases} \min\{\delta(x), \frac{1}{2}\,\mathrm{dist}(x, A)\} & \text{for } x \in I \setminus A, \\ \delta(x) & \text{for } x \in A \cup (E \setminus I). \end{cases}$$

It is easy to see that $\eta \in \Delta(E)$. Let $(\{I_j : j \in \Phi\}, \tau)$ be an η-fine division of I (it exists by Proposition 1.1) and set $\Gamma = \{j \in \Phi : \tau_j \in A\}$. Then $(\{I_j : j \in \Gamma\}, \tau)$ is a δ-fine A-tagged division which covers A. This follows from the definition of η for $x \notin A$ because for the tag $\tau_j \notin A$ the corresponding interval I_j does not intersect A by the definition of the gauge η. $\square$

Lemma 1.3. *Let A be a closed subset of E, $\delta \in \Delta(E)$ and let $(\{I_j:\ j \in \Gamma\}, \tau)$ be a δ-fine A-tagged division.*

Then there exists a set $\Phi \supset \Gamma$ a tag σ and a σ-fine A-tagged division $(\{I_j :\ j \in \Phi\}, \sigma)$ such that $\sigma_j = \tau_j$ for $j \in \Gamma$ and

$$A \subset \operatorname{Int}\Big(\bigcap_{j\in\Phi} I_j\Big).$$

P r o o f. Let $E \setminus \bigcup\limits_{j\in\Gamma} I_j = \bigcup\limits_{k\in\Psi} U_k$ where $\{\overline{U_k}:\ k \in \Psi\}$ is a pairwise disjoint finite system of closed intervals.

For any $k \in \Psi$ let $(\{I_j :\ j \in \Gamma_k\}, \tau^{(k)})$ be a δ-fine A-tagged division in $\overline{U_k}$ which covers $A \cap \overline{U_k}$. Now it suffices to set $\Phi = \Gamma \cup \Big(\bigcap\limits_{k\in\Psi} \Gamma_k\Big)$ and $\sigma(I_j) = \tau(I_j)$ for $j \in \Gamma$ and $\sigma(I_j) = \tau^{(k)}(I_j)$ for $j \in \Gamma_k$. □

Remark. Lemma 1.3 means that any δ-fine A-tagged division can be extended to a δ-fine A-tagged division which covers a closed set $A \subset E$.

2. The function W

Assume that $F\colon E \to \mathbb{R}$ is a real function defined on E. For $I \in \operatorname{Sub}(E)$ define the usual interval function

$$F[I] = F(r(I)) - F(l(I)).$$

Let us denote by $C(E)$ the set of all continuous real-valued functions on E.

The *oscillation* of $F \in C(E)$ on an interval $I \in \operatorname{Sub}(E)$ is defined in the usual way by

$$\omega(F, I) = \sup\{|F(x) - F(y)| :\ x, y \in I\} = \sup\{|F[J]| :\ J \in \operatorname{Sub}(I)\}.$$

The following simple properties of the oscillation of a function may be mentioned:

$$\omega(F, I) \ge 0, \tag{2.1}$$

$$\omega(F, I) = 0 \iff F \text{ is constant on } I, \tag{2.2}$$

$$\omega(\alpha F, I) = |\alpha|\omega(F, I) \quad \text{for } \alpha \in \mathbb{R}, \tag{2.3}$$

$$\omega\Big(\sum_{j\in\Phi} F_j, I\Big) \le \sum_{j\in\Phi} \omega(F_j, I) \quad \text{if } \Phi \text{ is finite}, \tag{2.4}$$

$$\omega\Big(F, \bigcup_{j\in\Phi} I_j\Big) \le \sum_{j\in\Phi} \omega(F, I_j) \quad \text{if } \Phi \text{ is finite and } \bigcup_{j\in\Phi} I_j \in \text{Sub }(E). \tag{2.5}$$

Definition 2.1. For $F \in C(E)$ and a division $D = \{I_j :\ j \in \Gamma\}$ let us set

$$\Omega(F, D) = \sum_{j\in\Gamma} \omega(F, I_j)$$

and

$$A(F, D) = \sum_{j\in\Gamma} |F[I_j]|.$$

If $F \in C(E)$ and $M \subset E$ then for any $\delta \in \Delta(E)$ set

$$W_\delta(F, M) = \sup\{\Omega(F, D) :\ D \text{ is } \delta\text{-fine}, M\text{-tagged}\}$$

and

$$V_\delta(F, M) = \sup\{A(F, D) :\ D \text{ is } \delta\text{-fine}, M\text{-tagged}\}$$

and put

$$W_F(M) = \inf\{W_\delta(F, M) :\ \delta \in \Delta(E)\}, \tag{2.6}$$

$$V_F(M) = \inf\{V_\delta(F, M) :\ \delta \in \Delta(E)\}, \tag{2.7}$$

Let us note that if $\delta_1, \delta_2 \in \Delta(E)$, $\delta_1 \le \delta_2$ then $W_{\delta_1}(F, M) \le W_{\delta_2}(F, M)$ and $V_{\delta_1}(F, M) \le V_{\delta_2}(F, M)$.

Therefore in the definition of $W_F(M)$ and $V_F(M)$ it suffices to take into account gauges which are less than some fixed gauge δ_0 only.

If $D = \{I_j :\ j \in \Gamma\}$ is a division then

$$|F[I_j]| \le \omega(F, I_j) \qquad \text{for} \qquad j \in \Gamma.$$

Therefore

$$A(F, D) \le \Omega(F, D)$$

and

$$V_F(M) \le W_F(M) \tag{2.8}$$

Let us recall the notion $V(F, I)$ of *total variation* of a function F over $I \in \mathrm{Sub}(E)$ which is defined by

$$V(F, I) = \sup\left\{\sum_{j\in\Gamma} |F[I_j]| :\ \{I_j :\ j \in \Gamma\} \text{ is a division of } I\right\}. \tag{2.9}$$

Note that $V(F, I) = 0$ for $I \in \mathrm{Sub}(E)$ if and only if the function F is constant on I and that $V(F, I) = V_F(I)$ for $I \in \mathrm{Sub}(E)$.

First let us show that in the simple situation of an interval $I \in \mathrm{Sub}(E)$ the values $W_F(I)$ and $V_F(I)$ have the classical meaning of the total variation of F over I.

Lemma 2.2. *Let $F \in C(E)$ and $I \in \mathrm{Sub}(E)$. Then*

$$W_F(I) = V_F(I) = V(F, I). \tag{2.10}$$

P r o o f. Assume that $\varepsilon > 0$ is given.

Since F is uniformly continuous on E there is a $\sigma > 0$ such that $|F[J]| < \frac{1}{2}\varepsilon$ provided $J \subset E$ and $|J| \le \sigma$.

If $\delta(x) = \sigma$ for $x \in E$ then for any δ-fine I-tagged division $\{I_j : \ j \in \Gamma\}$ we have $\Omega(F, D) = \sum\limits_{j\in\Gamma} \omega(F, I_j) = \sum\limits_{j\in\Gamma} |F[J_j]|$ where $J_j \in \mathrm{Sub}(I_j)$, $j \in \Gamma$, is such that $|F[J_j]| = \omega(F, I_j)$.

Define $\Gamma_1 = \{j \in \Gamma : \ I_j \subset I\}$ and $\Gamma_2 = \Gamma \setminus \Gamma_1$. Since I is an interval, the set Γ_2 consists of at most two elements. Hence

$$\Omega(F, D) = \sum_{j\in\Gamma} |F[J_j]| = \sum_{j\in\Gamma_1} |F[J_j]| + \sum_{j\in\Gamma_2} |F[J_j]| < V(F, I) + \varepsilon$$

and therefore also

$$W_F(I) \le V(F, I) + \varepsilon$$

and

$$W_F(I) \le V(F, I) \tag{2.11}$$

since $\varepsilon > 0$ can be taken arbitrarily small.

Further let $\{I_j : \ j \in \mathbb{N}_k\}$ be a division of I, for which

$$V(F, I) < \sum_{j=1}^{k} |F[I_j]| + \frac{\varepsilon}{2}.$$

Let $\delta \in \Delta(E)$ be arbitrary and let $D_j = \{J_i^j : \ i \in \Phi_j\}$ be a δ-fine division of I_j. Then

$$|F[I_j]| \le \sum_{i\in\Phi_j} |F[J_i^j]|$$

and

$$V(F, I) < \frac{\varepsilon}{2} + \sum_{j=1}^{k} \sum_{i\in\Phi_j} |F[J_i^j]|.$$

Let us set $D = \{J_i^j : \ j = 1, \ldots, k, \ \ i \in \Phi_j\}$. Then D is a δ-fine division of I and therefore

$$\sum_{j=1}^{k} \sum_{i\in\Phi_j} |F[J_i^j]| \le V_\delta(F, I).$$

This yields then $V(F, I) < \frac{\varepsilon}{2} + V_\delta(F, I)$ and also $V(F, I) < \varepsilon + V_F(I)$, i.e. we get

$$V(F, I) \le V_F(I).$$

Using (2.8), (2.11) we obtain

$$V_F(I) \le W_F(I) \le V(F,I) \le V_F(I)$$

and this finishes the proof. □

The following simple assertion will be also useful.

Lemma 2.3. *Let $F \in C(E)$, $I \in \mathrm{Sub}(E)$ and $\tau \in I$.*

Then there exists $J \in \mathrm{Sub}(I)$ such that $\tau \in J$ and

$$\omega(F,I) \le 2|F[J]|.$$

Proof. Since $F \in C(E)$, there is $\widetilde{I} \in \mathrm{Sub}(I)$ such that $|F[\widetilde{I}]| = \omega(F,I)$.

If $\tau \in \widetilde{I}$, then we may take $J = \widetilde{I}$.

If $\tau \notin \widetilde{I}$, then we have two intervals $J_1, J_2 \in \mathrm{Sub}(I)$, where the endpoints of J_1 are τ and $l(\widetilde{I})$ and J_2, where the endpoints of J_2 are τ and $r(\widetilde{I})$. We have evidently $\omega(F,I) \le |F[J_1]| + |F[J_2]|$. To get the statement we put $J = J_1$ if $|F[J_1]| \ge |F[J_2]|$ or $J = J_2$ if $|F[J_1]| < |F[J_2]|$. □

Corollary 2.4. *Assume that $F \in C(E)$. If $M \subset E$ then*

$$V_F(M) \le W_F(M) \le 2V_F(M).$$

(This implies e.g. that $V_F(M) = 0$ if and only if $W_F(M) = 0$.)

Given a function $F \in C(E)$ by $W_F(M)$ and $V_F(M)$ two set functions are given. Using the terms presented by B. S. Thomson in [10] we identify $W_F(M)$ and $V_F(M)$ as the *full variational measures* generated by the continuous interval functions given for $I \in \mathrm{Sub}(E)$ by $\omega(F,I)$, $F[I]$, respectively.

By [10, Theorem 3.7], $W_F(\cdot)$ and $V_F(\cdot)$ are metric outer measures. This means that the following holds.

Proposition 2.5. *Assume that $F \in C(E)$.*

1. *If $M, M_1, M_2, M_3, \ldots$ is a sequence of sets in E for which $M \subset \bigcup_{i=1}^{\infty} M_i$ then*

$$W_F(M) \le \sum_{i=1}^{\infty} W_F(M_i)$$

and

$$V_F(M) \le \sum_{i=1}^{\infty} V_F(M_i).$$

2. *If $M_1, M_2 \subset E$ are such that there are open sets G_1, G_2 with $M_1 \subset G_1$, $M_2 \subset G_2$ and $G_1 \cap G_2 = \emptyset$, then*

$$W_F(M_1) + W_F(M_2) = W_F(M_1 \cup M_2)$$

and

$$V_F(M_1) + V_F(M_2) = V_F(M_1 \cup M_2).$$

From the second part of this proposition we obtain immediately the following.

Corollary 2.6. *If $F \in C(E)$ and $A_1, A_2 \subset E$ are closed sets with $A_1 \cap A_2 = \emptyset$, then*

$$W_F(A_1 \cup A_2) = W_F(A_1) + W_F(A_2)$$

and

$$V_F(A_1 \cup A_2) = V_F(A_1) + V_F(A_2)$$

Since $\omega(F, I)$ and $F[I]$ are continuous interval functions for the case $F \in C(E)$, by [10, Theorem 3.10], the outer measures $W_F(\cdot)$ and $V_F(\cdot)$ have the increasing sets property presented in the following statement.

Proposition 2.7. *If $F \in C(E)$ and M_i is a sequence of sets with $M_i \subset M_{i+1}$ then*

$$W_F\Big(\bigcap_{i=1}^{\infty} M_i\Big) = \lim_{n\to+\infty} W_F(M_n)$$

and similarly

$$V_F\Big(\bigcap_{i=1}^{\infty} M_i\Big) = \lim_{n\to+\infty} V_F(M_n).$$

Let us recall another known concept.

Definition 2.8. Let $F \in C(E)$ and $M \subset E$. The function F is called to be of *negligible variation on the set M* if for any $\varepsilon > 0$ there is a $\delta \in \Delta(E)$ such that

$$\Big|\sum_{j\in\Gamma} F[I_j]\Big| < \varepsilon \tag{2.12}$$

for any δ-fine M-tagged division $(\{I_j : j \in \Gamma\}, \tau)$.

Remark. Let us mention that if M is countable then every $F \in C(E)$ is of negligible variation on M.

It is easy to see that the notion of negligible variation on a set M for a function $F \in C(E)$ remains unchanged if (2.12) is replaced by

$$\sum_{j\in\Gamma} |F[I_j]| < \varepsilon$$

in Definition 3.8.

The next statement indicates where the function W_F might be important. It shows that the concept of negligible variation can be characterized by W_F.

Lemma 2.9. *Let $F \in C(E)$ and $M \subset E$. Then F is of negligible variation on M if and only if $W_F(M) = V_F(M) = 0$.*

Proof. Let $\varepsilon > 0$ be given and let $\delta \in \Delta(E)$ be such that (2.12) is satisfied in the case that F is of negligible variation on M.

Assume that $(\{I_j : j \in \Gamma\}, \tau)$ is a δ-fine M-tagged division and let $\Gamma_+ = \{j \in \Gamma : F[I_j] \geq 0\}$ and $\Gamma_- = \Gamma \setminus \Gamma_+$. Then $(\{I_j : j \in \Gamma_+\}, \tau)$ and $(\{I_j : j \in \Gamma_-\}, \tau)$ are again δ-fine M-tagged divisions and this implies that

$$\sum_{j \in \Gamma} |F[I_j]| = \sum_{j \in \Gamma_+} F[I_j] - \sum_{j \in \Gamma_-} F[I_j] < 2\varepsilon$$

holds. By Lemma 2.3 for any $j \in \Gamma$ there is an interval J_j for which $\tau_j \in J_j \subset I_j$ and $\omega(F, I_j) \leq 2|F[J_j]|$ for $j \in \Gamma$. Hence

$$\sum_{j \in \Gamma} \omega(F, I_j) \leq 2 \sum_{j \in \Gamma} |F[J_j]| < 4\varepsilon,$$

because $(\{J_j : j \in \Gamma\}, \tau)$ is also a δ-fine M-tagged division. The last inequality gives $W_\delta(F, M) \leq 4\varepsilon$ and this yields $W_F(M) \leq 4\varepsilon$ for any $\varepsilon > 0$. Hence $W_F(M) = 0$.

If $W_F(M) = 0$ then by definition to every $\varepsilon > 0$ there is a $\delta \in \Delta(E)$ such that $W_\delta(F, M) < \varepsilon$. Hence for every δ-fine M-tagged division $D = (\{I_j : j \in \Gamma\}, \tau)$ we have $\Omega(F, D) < \varepsilon$ and this yields the other implication because $|F[I_j]| \leq \omega(F, I_j)$ for every $j \in \Gamma$.

The quantity $V_F(M)$ appears in the result simply by using Corollary 2.4. □

The basic properties of the function W are summarized in the following statement.

Theorem 2.10. *Let $F, F_j \in C(E)$ and $M, M_j \subset E$, $j \in \mathbb{N}$.*

Then

$$0 \leq W_F(M_1) \leq W_F(M_2) \qquad \text{if} \quad M_1 \subset M_2, \tag{2.13}$$

$$W_F\Big(\bigcap_{j \in \Phi} M_j\Big) \leq \sum_{j \in \Phi} W_F(M_j) \qquad \text{if} \quad \Phi \text{ is at most countable}, \tag{2.14}$$

$$W(\alpha F, I) = |\alpha| W_F(I) \qquad \text{for} \quad \alpha \in \mathbb{R}, \tag{2.15}$$

$$W_{\sum_{j \in \Phi} F_j}(M) \leq \sum_{j \in \Phi} W_{F_j}(M) \qquad \text{if} \quad \Phi \text{ is finite.} \tag{2.16}$$

Proof. The items (2.13), (2.14), (2.16) are easy to prove. (2.14) follows from Proposition 2.5. □

Remark. The problem under what conditions the equality holds in (2.14), i.e. when

$$W_F\Big(\bigcap_{j\in\Phi} M_j\Big) = \sum_{j\in\Phi} W_F(M_j)$$

if Φ is at most countable, will be important. We give a result of this type in Theorem 2.14 below.

For a given set $M \subset E$ denote by $\mu(M)$ the *Lebesgue measure* of M.

Definition 2.11. By $C^*(E)$ we denote the set of all continuous functions on E which are of negligible variation on sets of Lebesgue measure zero, i.e.

$$C^*(E) = \{F \in C(E):\ W_F(N) = 0 \text{ whenever } \mu(N) = 0\}. \tag{2.17}$$

(See Lemma 2.9.)

It should be mentioned that functions $F \in C^*(E)$ are called in the literature also functions satisfying the *strong Luzin condition* on E (see e.g. [7, Definition 4.1.1]).

If $E = [0,1]$ and $F\colon E \to \mathbb{R}$ is the well known Cantor function (cf. [3, Theorem 1.21]) then $F \in C(E)$ but $F \notin C^*(E)$.

The following well known assertion will be also needed in the sequel.

Proposition 2.12. *Let M be a (Lebesgue) measurable subset of E. Then there exists a sequence $\{A_j :\ j \in \mathbb{N}\}$ of closed sets, for which $A_j \subset A_{j+1} \subset M$ for $j \in \mathbb{N}$ and*

$$\mu\Big(M \setminus \bigcup_{j=1}^{\infty} A_j\Big) = 0. \tag{2.18}$$

This statement means that there is an F_σ set F such that $F \subset M$ and $\mu(M \setminus F) = 0$. (See e.g. [3, Theorem 1.12].)

Lemma 2.13. *Let $F \in C^*(E)$, M a measurable subset of E and assume that $\{A_j,\ j \in \mathbb{N}\}$ is a sequence of closed sets, for which $A_j \subset A_{j+1} \subset M$ for $j \in \mathbb{N}$ and*

$$\mu\Big(M \setminus \bigcup_{j=1}^{\infty} A_j\Big) = 0.$$

Then

$$W_F(M) = \lim_{j\to\infty} W_F(A_j).$$

P r o o f. Clearly

$$M = \Big(M \setminus \bigcup_{j=1}^{\infty} A_j\Big) \cup \bigcup_{j=1}^{\infty} A_j.$$

Since $F \in C^*(E)$, we have $W_F\Big(M \setminus \bigcup\limits_{j=1}^{\infty} A_j\Big) = 0$. This yields by (2.14) in Theorem 2.10 and by Proposition 2.7

$$\begin{aligned} W_F(M) &\le W_F\Big(M \setminus \bigcup_{j=1}^{\infty} A_j\Big) + W_F\Big(\bigcap_{j=1}^{\infty} A_j\Big) \\ &= W_F\Big(\bigcap_{j=1}^{\infty} A_j\Big) = \lim_{j\to\infty} W_F(A_j). \end{aligned}$$

On the other hand, by (2.13) in Theorem 2.10 we have

$$W_F(A_j) \le W_F(A_{j+1}) \le W_F(M)$$

for every $j \in \mathbb{N}$ and therefore

$$\lim_{j\to\infty} W_F(A_j) \le W_F(M).$$

This together with the previous inequality gives the statement of the lemma. □

Theorem 2.14. *Assume that $F \in C^*(E)$ and that $\{M_k : k \in \mathbb{N}\}$ is a sequence of measurable subsets of E.*

If $M_k \cap M_n = \emptyset$ for $k \ne n$, then

$$W_F\Big(\bigcap_{k=1}^{\infty} M_k\Big) = \sum_{k=1}^{\infty} W_F(M_k).$$

P r o o f. Let $M_k \cap M_n = \emptyset$ for $k, n \in \mathbb{N}$ and $k \ne n$.

First let us show that

$$W_F(M_1 \cup M_2) = W_F(M_1) + W_F(M_2)$$

holds.

If $\{A_j : j \in \mathbb{N}\}$ and $\{B_j : j \in \mathbb{N}\}$ are sequences of closed sets such that $A_j \subset A_{j+1} \subset M_1$, $B_j \subset B_{j+1} \subset M_2$ for $j \in \mathbb{N}$ and

$$\mu\Big(M_1 \setminus \bigcup_{j=1}^{\infty} A_j\Big) = 0, \quad \mu\Big(M_2 \setminus \bigcup_{j=1}^{\infty} B_j\Big) = 0,$$

(cf. Proposition 2.12) then by Lemma 2.13 we have

$$W_F(M_1) = \lim_{j\to\infty} W_F(A_j),\ W_F(M_2) = \lim_{j\to\infty} W_F(B_j).$$

Further clearly

$$\mu\Big((M_1 \cup M_2) \setminus \bigcup_{j=1}^{\infty} (A_j \cup B_j)\Big) = 0$$

and again by Lemma 2.13 we get

$$\begin{aligned} W_F(M_1 \cup M_2) &= \lim_{j\to\infty} W_F(A_j \cup B_j) \\ &= \lim_{j\to\infty} W_F(A_j) + \lim_{j\to\infty} W_F(B_j) W_F(M_1) + W_F(M_2) \end{aligned}$$

because

$$W_F(A_j \cup B_j) = W_F(A_j) + W_F(B_j)$$

for every $j \in \mathbb{N}$ by Corollary 3.6.

This easily implies that

$$W_F\Big(\bigcap_{k=1}^{n} M_k\Big) = \sum_{k=1}^{n} W_F(M_k)$$

holds for every $n \in \mathbb{N}$. By (2.13) we have

$$W_F\Big(\bigcap_{k=1}^{n} M_k\Big) \le W_F\Big(\bigcap_{k=1}^{\infty} M_k\Big)$$

for every $n \in \mathbb{N}$ and therefore

$$\sum_{k=1}^{\infty} W_F(M_k) \le W_F\Big(\bigcap_{k=1}^{\infty} M_k\Big).$$

From (2.14) in Theorem 2.10 we have

$$W_F\Big(\bigcap_{k=1}^{\infty} M_k\Big) \le \sum_{k=1}^{\infty} W_F(M_k)$$

and the assertion follows. $\square$

Theorem 2.14 shows that if $F \in C^*(E)$ then the variational measure $W_F(\cdot)$ generated by F is countably additive on the σ-algebra of measurable subsets of E.

3. The Kurzweil-Henstock integral K

Let us start with the basic definition of the integral.

Definition 3.1. K denotes the set of all pairs (f,γ), where f is a function on E and $\gamma \in \mathbb{R}$, for which to any $\varepsilon > 0$ there exists a gauge δ such that

$$\Big|\sum_{j\in\Gamma} f(\tau_j)\Big|I_j| - \gamma| < \varepsilon$$

for any δ-fine division $(\{I_j:\ j\in\Gamma\},\tau)$ of the interval E.

The value $\gamma \in \mathbb{R}$ is called the *Kurzweil-Henstock integral* of f over E and it will be denoted by $K(f)$ or $(K)\int\limits_E f$.

K is in fact a mapping from a set of functions on E into $\mathbb{R}$ (a functional).

Denote by $\mathrm{Dom}(K)$ the set of all f for which the functional K is defined.

If $f \in \mathrm{Dom}(K)$ then f is called *K-integrable over E.*

Denote the *characteristic function* of a set $M \subset E$ by $\chi(M)$, i.e. $\chi(M) = 1$ on M and $\chi(M) = 0$ on $E \setminus M$.

The characteristic function of the empty set $\emptyset$ may be denoted simply by 0 if no confusion can arise.

If the product $f\cdot\chi(M)$ belongs to $\mathrm{Dom}(K)$, then $K(f,M)$ (or $(K)\int\limits_M f$) denotes the value of the functional K on $f\cdot\chi(M)$, i.e. $K(f,M) = K(f\cdot\chi(M))$ and of course $K(f,E) = K(f)$.

Definition 3.2. If $f \in \mathrm{Dom}(K)$, then a function $F\colon E\to\mathbb{R}$ is called a *K-primitive* (or the *indefinite K-integral*) to f provided

$$F[I] = K(f,I)$$

holds for every $I \in \mathrm{Sub}(E)$.

Now we present a collection of basic properties of the Kurzweil-Henstock integral which will be used in the framework of this paper and in subsequent work.

Proposition 3.3.

$$0 \in \mathrm{Dom}(K) \quad \textit{and} \quad K(0) = 0. \tag{3.1}$$

If $c \in [a,b] = E$ and $I_1 = [a,c], I_2 = [c,b]$ then $f \in \mathrm{Dom}(K)$ if and only if $f\cdot\chi(I_1), f\cdot\chi(I_2) \in \mathrm{Dom}(K)$ and

$$K(f) = K(f,I_1) + K(f,I_2). \tag{3.2}$$

If $f = 0$ almost everywhere (with respect to the Lebesgue measure) then

$$f \in \mathrm{Dom}(K) \quad \textit{and} \quad K(f) = 0. \tag{3.3}$$

$$\textit{If}\ \ f \in \mathrm{Dom}(K)\ \textit{and}\ \ F\ \textit{ is a }\ K\textit{-primitive to }\ f\ \textit{ then }\ F \in C^*(E). \tag{3.4}$$

$$\textit{If } f \in \operatorname{Dom}(K) \textit{ then } f \textit{ is (Lebesgue) measurable.} \tag{3.5}$$

K is a linear functional, i.e. if $f, g \in \operatorname{Dom}(K)$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g \in \operatorname{Dom}(K)$ and

$$K(\alpha f + \beta g) = \alpha K(f) + \beta K(g). \tag{3.6}$$

Proof. The properties (3.1), (3.2) and (3.6) are easy to prove.

In [3, Theorem 9.5] it is shown that (3.3) holds.

In [7, Theorem 3.9.2] it is proved that a K-primitive function F to $f \in \operatorname{Dom}(K)$ is continuous and of negligible variation on sets of zero (Lebesgue) measure and this means that (3.4) is satisfied (cf. Definition 2.11).

The Lebesgue measurability of every $f \in \operatorname{Dom}(K)$ is proved e.g. in [3, Theorem 9.12]). □

Let us mention that a K-primitive function to $f \in \operatorname{Dom}(K)$ always exists (e.g. $F(x) = K(f, [a, x])$ for $x \in E = [a, b]$ is a K-primitive to f) and it is determined uniquely up to a constant.

If $M \in \operatorname{Alg}(E)$ and $\{I_j : j \in \Gamma\}$ is a division of M, then $f \cdot \chi(M) \in \operatorname{Dom}(K)$ if and only if $f \cdot \chi(I_j) \in \operatorname{Dom}(K)$ for all $j \in \Gamma$ and

$$K(f, M) = \sum_{j \in \Gamma} K(f, I_j).$$

In connection with the property (3.4) from Proposition 3.3 the following beautiful descriptive characterization of the Kurzweil-Henstock integral presented by Bongiorno, Di Piazza an Skvortsov in [1, Theorem 3] should be mentioned.

Theorem 3.4. *A function $F\colon E \to \mathbb{R}$ is a K-primitive function to some $f\colon E \to \mathbb{R}$ if and only if $F \in C^*(E)$.*

In other words the class of all functions $F\colon E \to \mathbb{R}$ which are K-primitive to some f coincides with the class of all $F \in C(E)$ for which $W_F(N) = 0$ if $N \subset E$ and $\mu(N) = 0$.

For more detail see [1] and also [8], [9].

From Gordon's book [3] it is known that a function $F\colon E \to \mathbb{R}$ is K-primitive to some $f\colon E \to \mathbb{R}$ if and only if F is an ACG_* function on E. This leads immediately to the conclusion of [1, Theorem 4] which says that the class of all ACG_* functions on E coincides with the class $C^*(E)$ of functions satisfying the strong Luzin condition.

Similar problems are dealt with also in the posthumous paper [2] of Vasile Ene in connection with an older result of Jarník and Kurzweil from [4].

The following assertion known as the Saks-Henstock lemma plays an important role in the theory (see e.g. [3, Lemma 9.11], [5, Lemma 5.3], etc.).

Proposition 3.5. *Let $f \in \mathrm{Dom}(K)$. Then to any $\varepsilon > 0$ there is a gauge δ such that for any δ-fine tagged division $(\{I_j : j \in \Gamma\}, \tau)$ in E the inequality*

$$\Big|\sum_{j\in\Gamma} f(\tau_j)|I_j| - K\Big(f, \bigcup_{j\in\Gamma} I_j\Big)\Big| < \varepsilon \tag{3.7}$$

holds.

In other words (F being the K-primitive to f) we have

$$\Big|\sum_{j\in\Gamma} f(\tau_j)\Big|I_j| - \sum_{j\in\Gamma} F[I_j]| < \varepsilon. \tag{3.8}$$

In [3, Theorem 9.21] the following is presented.

Theorem 3.6 (**Hake**). *Let $f\colon E \to \mathbb{R}$ be given. Suppose that $f \cdot \chi([c,d]) \in \mathrm{Dom}(K)$ for each $[c,d] \subset E$, $a < c < d < b$. If $K(f,[c,d])$ has a finite limit as $c \to a+$ and $d \to b-$ then $f \in \mathrm{Dom}(K)$ and*

$$K(f) = \lim_{c\to a+,\ d\to b-} K(f,[c,d]).$$

Now we give another property of the Kurzweil-Henstock integral.

Lemma 3.7. *Assume that $f \in \mathrm{Dom}(K)$ and let F be its K-primitive function. Then*

$$W_F(M) \le 2|E||f|_M \tag{3.9}$$

holds for $M \subset E$.

P r o o f. Let $\varepsilon > 0$ be given. Let $\delta \in \Delta(E)$ be such that (3.8) holds. Assume that $(\{I_j : j \in \Gamma\}, \tau)$ is a δ-fine M-tagged division and let $J_j \subset I_j$ be such that $\tau_j \in J_j$ and $\omega(F, I_j) \le 2|F[J_j]|$ for $j \in \Gamma$ (see Lemma 2.3).

Assume that $\Gamma_1 = \{j \in \Gamma : F[J_j] \ge 0\}$ and set $\Gamma_2 = \Gamma \setminus \Gamma_1$. Evidently $(\{I_j : j \in \Gamma_1\}, \tau)$ and $(\{I_j : j \in \Gamma_2\}, \tau)$ are δ-fine divisions in E.

We have

$$\sum_{j\in\Gamma} \omega(F, I_j) \le 2\sum_{j\in\Gamma} |F[J_j]| = 2\Big|\sum_{j\in\Gamma_1} F[J_j]\Big| + 2\Big|\sum_{j\in\Gamma_2} F[J_j]\Big|$$

and by (3.8)

$$\begin{aligned}\sum_{j\in\Gamma_1} |F[J_j]| &= \sum_{j\in\Gamma_1} F[J_j] = \sum_{j\in\Gamma_1} f(\tau_j)|J_j| + \sum_{j\in\Gamma_1} (F[J_j] - f(\tau_j)|J_j|)\\ &\le \Big|\sum_{j\in\Gamma_1} f(\tau_j)|J_j|\Big| + \Big|\sum_{j\in\Gamma_1} (F[J_j] - f(\tau_j)|J_j|)\Big| < \sum_{j\in\Gamma_1} |f(\tau_j)|J_j| + \varepsilon.\end{aligned}$$

Similarly

$$\sum_{j\in\Gamma_2}|F[J_j]| = -\sum_{j\in\Gamma_2}F[J_j] = \sum_{j\in\Gamma_2}f(\tau_j)|J_j| - \sum_{j\in\Gamma_2}(F[J_j]-f(\tau_j)|J_j|)$$
$$\le \Big|\sum_{j\in\Gamma_2}f(\tau_j)|J_j|\Big| + \Big|\sum_{j\in\Gamma_2}(F[J_j]-f(\tau_j)|J_j|)\Big| < \sum_{j\in\Gamma_1}|f(\tau_j)||J_j| + \varepsilon.$$

Therefore

$$\sum_{j\in\Gamma}\omega(F,I_j) < 2\sum_{j\in\Gamma}|f(\tau_j)||J_j| + 4\varepsilon$$
$$\le 2|f|_M\sum_{j\in\Gamma}|J_j| + 4\varepsilon \le 2|f|_M|E| + 4\varepsilon$$

and

$$W_\delta(F,M) < 2|f|_M|E| + 4\varepsilon.$$

Hence

$$W_F(M) < 2|f|_M|E| + 4\varepsilon$$

for every $\varepsilon > 0$ and this implies (3.9). □

Definition 3.8. If $I \in \mathrm{Sub}(E)$ and $A \subset E$ is closed then $\mathrm{Comp}(I,A)$ denotes the set of all (maximal and nonempty) connected components of the set $I \setminus A$.

The set $\mathrm{Comp}(I,A)$ is always at most countable and any element

$$U \in \mathrm{Comp}(I,A)$$

is an interval, i.e. $\overline{U} \in \mathrm{Sub}(E)$.

Lemma 3.9. *Let $A \subset E$ be a closed set, $f, F\colon E \to \mathbb{R}$.*

Assume that

1) *$f = 0$ on A,*
2) *for every $[c,d] \subset U \in \mathrm{Comp}(E,A)$ we have $f\cdot\chi([c,d]) \in \mathrm{Dom}(K)$ and*
$$K(f,[c,d]) = F(d) - F(c),$$
3) *$F \in C(E)$,*
4) *$W_F(A) = 0$.*

Then $f \in \mathrm{Dom}(K)$ and F is a K-primitive to f.

Proof. By 4) to any $\varepsilon > 0$ there is a $\delta_0 \in \Delta(E)$ such that

$$\Omega(F,D) = \sum_{j\in\Gamma}\omega(F,I_j) < \varepsilon$$

for every δ_0-fine A-tagged division $(\{I_j:\ j\in\Gamma\},\tau)$. Therefore

$$\Bigl|\sum_{j\in\Gamma}F[I_j]\Bigr|\le\sum_{j\in\Gamma}|F[I_j]|\le\sum_{j\in\Gamma}\omega(F,I_j)<\varepsilon$$

for every δ_0-fine A-tagged division $(\{I_j:\ j\in\Gamma\},\tau)$.

The conditions 2) and 3) together with Hake's Theorem 3.6 yield

$$f\cdot\chi(\overline{U})\in\operatorname{Dom}(K)$$

for every $U\in\operatorname{Comp}(E,A)$ and

$$K(f,\overline{U})=F[\overline{U}]=F(r(\overline{U}))-F(l(\overline{U}))$$

by the continuity of F which is required by 3).

$\operatorname{Comp}(E,A)$ is at most countable, $\operatorname{Comp}(E,A)=\{U_j:\ j\in\mathbb{N}\}$, because A is closed.

Since $f\cdot\chi(\overline{U_j})\in\operatorname{Dom}(K)$ for every $j\in\mathbb{N}$ and $K(f,I)=F[I]$ for every $I\in\operatorname{Sub}(\overline{U_j})$, there is a $\delta_j\in\Delta(\overline{U_j})$ such that

$$\Bigl|\sum_{l\in\Gamma_j}(f(\tau_l)|I_l|-F[I_l])\Bigr|<\frac{\varepsilon}{2^j}$$

holds for every δ_j-fine division $(\{I_l:\ l\in\Gamma_j\},\tau)$ in $\overline{U_j}$, $j\in\mathbb{N}$. This follows from the Saks-Henstock lemma 3.5.

Define

$$\delta(t)=\begin{cases}\min\{\delta_j(t),\frac{1}{2}\operatorname{dist}(t,A)\} & \text{for } t\in\overline{U_j},\ \ j\in\mathbb{N},\\ \delta_0(t) & \text{for } t\in A.\end{cases}$$

Clearly $\delta\in\Delta(E)$. Assume that $(\{J_k:\ k\in\Phi\},\tau)$ is a δ-fine division of E. Denote $\Gamma_0=\{k\in\Phi:\ \tau_k\in A\}$, $\Gamma_j=\{k\in\Phi:\ \tau_k\in\overline{U_j}\}$. By the definition of $\delta\in\Delta(E)$ we have $J_k\subset U_j$ for $k\in\Gamma_j$ and

$$\begin{aligned}\Bigl|\sum_{k\in\Phi}f(\tau_k)\Bigr|J_k|-F[E]|&=\Bigl|\sum_{k\in\Phi}(f(\tau_k)|J_k|-F[J_k])\Bigr|\\&\le\Bigl|\sum_{k\in\Gamma_0}F[J_k]\Bigr|+\sum_{j\in\mathbb{N}}\Bigl|\sum_{k\in\Gamma_j}(f(\tau_k)|J_k|-F[J_k])\Bigr|\\&<\varepsilon+\sum_{j\in\mathbb{N}}\frac{\varepsilon}{2^j}=2\varepsilon.\end{aligned}$$

Hence $f\in\operatorname{Dom}(K)$ and $K(f)=F[E]$.

If $I\in\operatorname{Sub}(E)$ then the same procedure can be used for the interval I and the closed set $A\cap I\subset E$ to show that $f\cdot\chi([I])\in\operatorname{Dom}(K)$ and that $K(f,I)=F[I]$. This yields the statement. $\square$

Corollary 3.10. *Let $A \subset E$ be a closed set, $f, F\colon E \to \mathbb{R}$.*

Assume that

1) *$f = 0$ on A,*
2) *for every interval $I = [c, d] \subset U \in \operatorname{Comp}(E, A)$ we have $f \cdot \chi(I) \in \operatorname{Dom}(K)$ and*

$$K(f, I) = F[I] = F(d) - F(c),$$

3) *$F \in C(E)$.*

Then $f \in \operatorname{Dom}(K)$ and F is a K-primitive to f if and only if $W_F(A) = 0$.

Proof. Lemma 3.9 gives one of the implications and therefore it suffices to show that if $f \in \operatorname{Dom}(K)$ and F is a K-primitive to f then $W_F(A) = 0$. But this is clear by (3.9) from Lemma 3.7 because by 1) we have $|f|_A = 0$. □

Theorem 3.11. *Let $A \subset E$ be a closed set, $g, F\colon E \to \mathbb{R}$.*

Assume that

1) *$g \cdot \chi(A) \in \operatorname{Dom}(K)$,*
2) *for every interval $I \subset U \in \operatorname{Comp}(E, A)$ we have $g \cdot \chi(I) \in \operatorname{Dom}(K)$ and*

$$K(g, I) = F[I],$$

3) *$F \in C(E)$.*

Then $g \in \operatorname{Dom}(K)$ and

$$K(g) = K(g, A) + F[E] = K(g, A) + F(b) - F(a)$$

if and only if $W_F(A) = 0$.

Proof. Let us set $f = g - g \cdot \chi(A)$. Then clearly $f = 0$ on A and $f = g$ on every $U \in \operatorname{Comp}(E, A)$. By 2) we obtain that $f \cdot \chi(I) \in \operatorname{Dom}(K)$ for every $I \subset U \in \operatorname{Comp}(E, A)$ and

$$K(f, I) = F[I].$$

This together with 3) implies by Corollary 3.10 that $f \in \operatorname{Dom}(K)$ if and only if $W_F(A) = 0$ and F is a K-primitive to f. This implies also $K(f) = F[E]$.

By (3.6) and by the definition of f we obtain $g \in \operatorname{Dom}(K)$ if and only if $W_F(A) = 0$ and

$$K(g) = K(g \cdot \chi(A)) + K(f) = K(g, A) + F[E].$$

The theorem is proved. □

Remark. Let us mention that if G is a K-primitive to $g \cdot \chi(A) \in \mathrm{Dom}(K)$, then $G + F$ is a K-primitive to g.

In other words, if $W_F(A) = 0$ then the function

$$K(g \cdot \chi(A), [a, x]) + F(x) - F(a), \qquad x \in E$$

is a K-primitive to g.

In [7, Theorem 3.4.1] the following statement was proved.

Theorem 3.12. *If g is K-integrable over $I \in \mathrm{Sub}(E)$ and G is its K-primitive then $|g|$ is K-integrable over I if and only if $V(G, I) < \infty$ and*

$$V(G, I) = K(|g|, I).$$

In this situation we have $G \in C(E)$ and using Lemma 2.2 we get the following.

Lemma 3.13. *If g is K-integrable over $I \in \mathrm{Sub}(E)$ and G is its K-primitive then $|g|$ is K-integrable over I if and only if $W(G, I) < \infty$ and*

$$W_G(I) = K(|g|, I)$$

in this case.

Lemma 3.14. *Let $M \subset E$ and assume that $f, g = f \cdot \chi(M) \in \mathrm{Dom}(K)$ with F, G being their K-primitives, then*

$$W_F(M) = W_G(M) \tag{3.10}$$

Proof. Since $f - g \in \mathrm{Dom}(K)$ and $F - G$ is a K-primitive to $f - g$ we have by (3.9) in Lemma 3.7

$$W_{F-G}(M) \leq 2|E||f - g|_M = 0.$$

Hence by (2.14) from Theorem 2.10 we get

$$W_F(M) = W_{F-G+G}(M) \leq W_{F-G}(M) + W_G(M) = W_G(M).$$

Similarly also $W_G(M) \leq W_F(M)$ and (3.10) holds. $\square$

Lemma 3.15. *Assume that $f \in \mathrm{Dom}(K)$ with F being its K-primitive, $M \subset E$ (Lebesgue) measurable and $g = |f| \cdot \chi(M) \in \mathrm{Dom}(K)$ with the K-primitive G. Then*

$$W_F(M) = K(|f|, M) = K(g). \tag{3.11}$$

Proof. By (3.5) f is measurable and therefore $f \cdot \chi(M)$ is measurable as well.

Since $|f \cdot \chi(M)| = |f| \cdot \chi(M) \in \mathrm{Dom}(K)$ we have $f \cdot \chi(M) \in \mathrm{Dom}(K)$ (see e.g. [7, Theorem 3.11.2]).

Hence by Lemma 3.14 we have $W_F(M) = W_G(M)$.

Since $M \subset E$ we have $W_G(M) \le W_G(E)$ by (2.13) and on the other hand by (2.14) we get

$$W_G(E) \le W_G(M) + W_G(E \setminus M) = W_G(M)$$

because by Lemma 3.7 we have $W_G(E \setminus M) \le 2|E||g|_{E\setminus M} = 0$. This yields $W_G(M) = W_G(E)$ and therefore

$$W_F(M) = W_G(E).$$

By Lemma 3.13 we have

$$W_G(E) = K(g) = K(|f| \cdot \chi(M)) = K(|f|, M)$$

because $g = |g|$ and (3.11) is proved. □

For $f \in \mathrm{Dom}(K)$, $M \subset E$ measurable, denote

$$\overline{K}(|f|, M) = K(|f|, M) \qquad \text{if} \quad |f| \cdot \chi(M) \in \mathrm{Dom}(K),$$
$$\overline{K}(|f|, M) = \infty \qquad \text{otherwise.}$$

Using Lemma 3.15 we have

$$W_F(M) \le \overline{K}(|f|, M) \tag{3.12}$$

for every $f \in \mathrm{Dom}(K)$ with F being its K-primitive.

Proposition 3.16. *If $f \in \mathrm{Dom}(K)$, F a K-primitive to f and $M \subset E$ measurable, then*

$$W_F(M) = \overline{K}(|f|, M). \tag{3.13}$$

Proof. Since (3.12) holds, the equality (3.13) is valid for the case when $W_F(M) = \infty$.

Assume that $W_F(M) < \infty$. By (3.12) for proving (3.13) it suffices to show that

$$\overline{K}(|f|, M) \le W_F(M). \tag{3.14}$$

Denote $g = |f| \cdot \chi(M)$ and assume that $\varepsilon > 0$ is given.

Since $f \in \mathrm{Dom}(K)$, by the Saks-Henstock lemma (Proposition 3.5) there is a $\delta_1 \in \Delta(E)$ such that

$$\Big|\sum_{j\in\Gamma} (f(\tau_j)|I_j| - F[I_j])\Big| < \varepsilon \tag{3.15}$$

for any δ_1-fine division $(\{I_j, j \in \Gamma\}, \tau)$ in E. By the definition of $W_F(M)$ assume further that $\delta_2 \in \Delta(E)$ is such that

$$\sum_{j\in\Gamma} \omega(F, I_j) < W_F(M) + \varepsilon \tag{3.16}$$

for every δ_2-fine M-tagged division $(\{I_j : j \in \Gamma\}, \tau)$ in E and put

$$\delta = \min\{\delta_1, \delta_2\}.$$

Let $(\{I_j : j \in \Gamma\}, \tau)$ be an arbitrary δ-fine division in E. Denote $\widetilde{\Gamma} = \{j \in \Gamma : g(\tau_j) \neq 0\}$. For $j \in \widetilde{\Gamma}$ we have clearly $\tau_j \in M$ and $\{I_j, j \in \widetilde{\Gamma}\}$ forms an M-tagged division in E which is both δ_1- and δ_2-fine.

Then

$$\begin{aligned}\sum_{j\in\Gamma} g(\tau_j)|I_j| = \sum_{j\in\widetilde{\Gamma}} g(\tau_j)|I_j| &= \sum_{j\in\widetilde{\Gamma}} f(\tau_j)|I_j| \\ &= \sum_{j\in\Gamma_+} f(\tau_j)|I_j| - \sum_{j\in\Gamma_-} f(\tau_j)|I_j|\end{aligned}$$

where $\Gamma_+ = \{j \in \widetilde{\Gamma} : f(\tau_j) > 0\}$, $\Gamma_- = \{j \in \widetilde{\Gamma} : f(\tau_j) < 0\}$.

Hence by (3.15) and (3.16) we get

$$\begin{aligned}\sum_{j\in\Gamma} g(\tau_j)|I_j| \leq & \Big|\sum_{j\in\Gamma_+} f(\tau_j)|I_j| - F[I_j]\Big| + \Big|\sum_{j\in\Gamma_-} f(\tau_j)|I_j| - F[I_j]\Big| \\ & + \Big|\sum_{j\in\Gamma_+} F[I_j]\Big| + \Big|\sum_{j\in\Gamma_-} F[I_j]\Big| \\ & < 2\varepsilon + \sum_{j\in\widetilde{\Gamma}} |F[I_j]| \leq 2\varepsilon + \sum_{j\in\widetilde{\Gamma}} \omega(F, I_j) < W_F(M) + 3\varepsilon.\end{aligned}$$

Since all the integral sums corresponding to the nonnegative function $g = |f| \cdot \chi(M)$ and to the δ_1-fine tagged division $(\{I_j : j \in \Gamma\}, \tau)$ are bounded by $W_F(M) + 3\varepsilon$ we obtain that the integral $K(g) = K(|f|, M)$ exists and satisfies the estimate

$$K(g) = K(|f|, M) < W_F(M) + 3\varepsilon$$

for an arbitrary $\varepsilon > 0$. Hence

$$K(|f|, M) \leq W_F(M)$$

and (3.14) holds. □

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