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PSEUDO-RANK FUNCTIONS ON CERTAIN LATTICES

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Dedicated to Prof. G. R. Shendge on the occasion of his 70th birthday

(*Communicated by Jiří Rachůnek*)

ABSTRACT. Pseudo-rank functions on bounded lattices are introduced and their properties are studied. It is shown that if the set of all pseudo-rank functions on a Boolean lattice is nonempty then it is a Choquet simplex.

1. Introduction

A real valued function D on a lattice L is called a *dimension function*, if the range of D has either an upper bound or a lower bound and for all $a, b \in L$, $D(a \vee b) + D(a \wedge b) = D(a) + D(b)$, see von Neumann [12, p. 58]. The theory of dimension functions is studied in various structures. Von Neumann [12] introduced dimensionality in continuous geometries by using perspectivity, whereas Iwamura [6] used the concept of a relation called the p-relation.

Kaplansky [7], Murray and von Neumann [11] and others have introduced dimensionality in rings of operators by using equivalence of projections. Maeda [10] generalized the work of von Neumann [12] and Kaplansky [7] for a certain class of lattices. At the same time Loomis [8] gave an abstract setting to the Murray, von Neumann dimension theory by using complete orthocomplemented lattices. Goodearl [3] developed the dimension theory for a certain class of modules. Von Neumann [12, p. 231] has introduced the concept of a rank-function on a regular ring, which generalizes

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the dimension function. Goodearl [2], [4] has introduced and developed the study of *pseudo-rank functions* on regular rings, which is a generalization of rank functions.

In this paper we introduce and study the concept of a *pseudo-rank function* on a bounded lattice L on the lines of Goodearl [4]. In Section 2, we obtain some basic properties of pseudo-rank functions and the set of all pseudo-rank functions on L and show how new pseudo-rank functions can be constructed from some given ones. In Section 3 we consider the set $\mathbb{P}(L)$ of all pseudo-rank functions as a subset of the real vector space $\mathbb{R}^L$ equipped with the product topology. It is proved that $\mathbb{P}(L)$ is a compact convex subset of $\mathbb{R}^L$. We also show that if L is a Boolean lattice with nonempty $\mathbb{P}(L)$ then the convex cone in $\mathbb{R}^L$ with base $\mathbb{P}(L)$ is a lattice. A necessary and sufficient condition for two pseudo-rank functions to lie in disjoint faces of $\mathbb{P}(L)$ are given. Further it is shown that $\mathbb{P}(L)$ is a Choquet simplex.

2. Pseudo-rank functions

A lattice L is said to have the *zero element*, if there exists an element, denoted by 0, such that $0 \leq x$ for all $x \in L$. L is said to have the *unit element*, if there exists an element, denoted by 1, such that $x \leq 1$ for all $x \in L$. If L has both 0 and 1, then L is called a *bounded lattice*. A bounded lattice is called *complemented* if for any $a \in L$, there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. Then b is called a *complement* of a. A distributive complemented lattice is called a *Boolean lattice*. We say that $x_1, \dots, x_n \in L$ are *completely orthogonal*, if $x_i \wedge \bigvee\limits_{j \neq i} x_j = 0$ for each i. The undefined terms are from Grätzer [5] and Maeda and Maeda [9].

Definition 1. A pseudo-rank function f on a bounded lattice L is a mapping $f\colon L \to [0,1]$ such that

(a) $f(1) = 1$.

(b) $f(x) \leq f(y)$ for all $x, y \in L$ such that $x \leq y$.

(c) $f(x \vee y) = f(x) + f(y)$ for all $x, y \in L$ with $x \wedge y = 0$.

Clearly, $f(0) = 0$ and $f(\bigvee\limits_{i=1}^{n} x_i) = \sum\limits_{i=1}^{n} f(x_i)$ whenever $x_1, \dots, x_n \in L$ are completely orthogonal.

A pseudo-rank function f with the property $f(x) > 0$, for $x \neq 0$ is called a *rank function* on L.

The following proposition is evident.

Proposition 1. *Let L be a bounded lattice and f be a pseudo-rank function on L. Suppose that $\{x_1,\dots,x_n\}$ and $\{y_1,\dots,y_k\}$ are sets of completely orthogonal elements in L. If $x_1\vee\dots\vee x_n\le y_1\vee\dots\vee y_k$, then $\sum_{i=1}^{n} f(x_i)\le\sum_{i=1}^{k} f(y_i)$.*

A lattice L is called *relatively complemented*, if for $a,b,c\in L$ with $a\in[b,c]$ there exists $d\in L$ such that $a\wedge d=b$ and $a\vee d=c$. Then d is called a *relative complement of* a in $[b,c]$.

Proposition 2. *Let L be a relatively complemented bounded lattice and f be a pseudo-rank function on L. Suppose that $x_1,\dots,x_n\in L$. Then $f(x_1\vee\dots\vee x_n)\le \sum_{i=1}^{k} f(x_i)$.*

Proof. We prove the proposition for $n=2$. Let a and b be relative complements of $x_1\wedge x_2$ in $[0,x_1]$ and in $[0,x_2]$ respectively. Then $a\wedge b\le x_1\wedge x_2$. Taking meet with a, we get $a\wedge b=0$. Also, $0=a\wedge x_1\wedge x_2=a\wedge x_2$. Now, $f(x_1\vee x_2)=f(a\vee(x_1\wedge x_2)\vee x_2)=f(a\vee x_2)=f(a)+f(x_2)\le f(x_1)+f(x_2)$. By using induction, we get the result. □

Two elements $x,y\in L$ are called *perspective* (in notation $x\sim y$) if there exists $z\in L$ such that $x\wedge z=y\wedge z=0$ and $x\vee z=y\vee z=1$.

We note that if f is a pseudo-rank function on L, then $x\sim y$ implies $f(x)=f(y)$. The proof of the following lemma is evident.

Lemma 1. *Let f be a pseudo-rank function on a bounded lattice L. Let $x\in L$ be such that $f(x)\neq 0$. Then the function $P(y)=f(y)/f(x)$ defines a pseudo-rank function on $(x]$.*

An element x in a lattice L is called a *dually distributive element*, if $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$ for all $y,z\in L$, see Grätzer [5, p. 181]. An element x in a bounded lattice L is called a *central element*, if there exist two lattices L_1 and L_2 and an isomorphism between L and the direct product L_1L_2 such that x corresponds to the element $[1_1,0_2]\in L_1L_2$. It is known that $z\in L$ is a central element if and only if z is a neutral element having a complement (see Maeda and Maeda [9, Theorem 4.13]).

Lemma 2. *Let f be a pseudo-rank function on a bounded lattice L. Let $x\in L$, be a dually distributive element in L such that $f(x)\neq 0$. The function $Q(y)=f(x\wedge y)/f(x)$ defines a pseudo-rank function on L.*

P r o o f. Clearly, $Q(1) = f(x \wedge 1)/f(x) = 1$. If $a, b \in L$ and $a \leq b$ then $x \wedge a \leq x \wedge b$ implies $Q(a) \leq Q(b)$. Moreover, if $a \wedge b = 0$ for some $a, b \in L$, then $f(a \vee b) = f(a) + f(b)$ and so $Q(a \vee b) = Q(a) + Q(b)$. Thus Q is a pseudo-rank function on L. □

Lemma 3. *Let f be a pseudo-rank function on a bounded lattice L. Let $x \in L$, be a central element in L such that $f(x) = 1$. Then $f(x \wedge y) = f(y)$ for all $y \in L$.*

P r o o f. Since x is a central element, there exists a complement x' of x. Clearly, $1 = f(1) = f(x \vee x') = f(x) + f(x') = 1 + f(x')$. Hence $f(x') = 0$. For any $y \in L$, $f(x' \wedge y) \leq f(x') = 0$. Hence $f(y) = f(y \wedge 1) = f[y \wedge (x \vee x')] = f[(y \wedge x) \vee (y \wedge x')] = f(y \wedge x) + f(y \wedge x') = f(y \wedge x)$. □

Lemma 4. *Let L be a bounded lattice and $\{x_1, \dots, x_n\}$ be a set of completely orthogonal elements in L. Suppose I, J are nonempty subsets of $\{1, \dots, n\}$. Let α_i and β_j be nonzero real numbers. For each $i \in I$ and $j \in J$, let P_i and Q_j be pseudo-rank functions on $(x_i]$ and $(x_j]$ respectively. If $\sum\limits_{i \in I} \alpha_i P_i(x_i \wedge y) = \sum\limits_{j \in J} \beta_j Q_j(x_j \wedge y)$ for every $y \in L$, then $I = J$, $\alpha_i = \beta_i$ and $P_i = Q_i$ for each i.*

P r o o f. Let $t \in J$. Then using $x_j \wedge x_t = 0$ for $j \neq t$, $Q_j(0) = 0$ and $Q_j(x_j) = 1$, we get $\sum\limits_{i \in I} \alpha_i P_i(x_i \wedge x_t) = \sum\limits_{j \in J} \beta_j Q_j(x_j \wedge x_t) = \beta_t \neq 0$. Hence $P_s(x_s \wedge x_t) \neq 0$ for some $s \in I$. This implies $x_s \wedge x_t \neq 0$ and so $s = t$, i.e. $t \in I$. Thus $J \subseteq I$. Similarly we get $I \subseteq J$.

Given $s \in I$, $y \in (x_s]$, then using $x_i \wedge y = 0$ for $i \neq s$ we get, $\alpha_s P_s(y) = \sum\limits_{i \in I} \alpha_i P_i(x_i \wedge y) = \sum\limits_{j \in J} \beta_j Q_j(x_j \wedge y)) = \beta_s Q_s(x_s \wedge y) = \beta_s Q_s(y)$. In particular, $\alpha_s = \alpha_s P_s(x_s) = \beta_s Q_s(x_s) = \beta_s$. Consequently, $P_s(y) = Q_s(y)$ for every $y \in (x_s]$. Thus $P_s = Q_s$. □

Lemma 5. *Let L be a bounded lattice and $\{x_1, \dots, x_n\}$ be a set of completely orthogonal central elements in L. Suppose I, J are nonempty subsets of $\{1, \dots, n\}$. Let α_i and β_j be nonzero real numbers. For each $i \in I$ and $j \in J$, let P_i and Q_j be pseudo-rank functions on L such that $P_i(x_i) = 1$ and $Q_j(x_j) = 1$. If $\sum\limits_{i \in I} \alpha_i P_i(y) = \sum\limits_{j \in J} \beta_j Q_j(y)$ for every $y \in L$, then $I = J$, $\alpha_i = \beta_i$ and $P_i = Q_i$ for each i.*

P r o o f. Let $t \in J$. Then using Lemma 3 and $x_j \wedge x_t = 0$ for $j \neq t$ we get $Q_j(x_t) = Q_j(x_j \wedge x_t) = 0$. Hence $\sum\limits_{i \in I} \alpha_i P_i(x_t) = \sum\limits_{j \in J} \beta_j Q_j(x_t) = \beta_t Q_t(x_t) =$

$\beta_t \neq 0$. Hence $P_s(x_t) \neq 0$ for some $s \in I$. This implies $x_s \wedge x_t \neq 0$ and so $s = t$, i.e. $t \in I$. Thus $J \subseteq I$. Similarly we get $I \subseteq J$.

Given $s \in I$, $y \in L$ we have by using Lemma 3, $\alpha_s P_s(y) = \alpha_s P_s(x_s \wedge y) = \sum_{i\in I} \alpha_i P_i(x_i \wedge x_s \wedge y) = \sum_{i\in I} \alpha_i P_i(x_s \wedge y) = \sum_{j\in J} \beta_j Q_j(x_s \wedge y) = \sum_{j\in J} \beta_j Q_j(x_j \wedge x_s \wedge y) = \beta_s Q_s(x_s \wedge y) = \beta_s Q_s(y)$. In particular, $\alpha_s = \alpha_s P_s(x_s) = \beta_s Q_s(x_s) = \beta_s$. Consequently, $P_s(y) = Q_s(y)$ for every $y \in L$. Thus $P_s = Q_s$. □

Lemma 6. *Let L be a bounded lattice and $\{x_1, \ldots, x_n\}$ be a set of completely orthogonal central elements in L. Let f be a pseudo-rank function on L such that $\sum_{i=1}^{n} f(x_i) = 1$ and $f(x_i) \neq 0$ for all i.*

(a) *There exist unique pseudo-rank functions P_i on $(x_i]$ such that $f(y) = \sum_{i=1}^{n} f(x_i) P_i(x_i \wedge y)$ for all $y \in L$.*

(b) *There exist unique pseudo-rank functions Q_i on L such that $Q_i(x_i) = 1$ and $f(y) = \sum_{i=1}^{n} f(x_i) Q_i(y)$ for all $y \in L$.*

P r o o f. Since $\{x_1, \ldots, x_n\}$ is a set of completely orthogonal elements in L, $f(x_1 \vee \cdots \vee x_n) = \sum_{i=1}^{n} f(x_i) = 1$. As $x_1, \ldots, x_n$ are central elements, so is $x_1 \vee \cdots \vee x_n$ and hence $f(y) = f((x_1 \vee \cdots \vee x_n) \wedge y)$ for all $y \in L$ by Lemma 3. Clearly, the set $\{x_1 \wedge y, \ldots, x_n \wedge y\}$ is a completely orthogonal subset of L. Hence $f(y) = f((x_1 \wedge y) \vee \cdots \vee (x_n \wedge y)) = \sum_{i=1}^{n} f(x_i \wedge y)$.

(a) For each i, by Lemma 1, $P_i(y) = f(y)/f(x_i)$ defines a pseudo-rank function P_i on $(x_i]$. Given any $y \in L$, we then have $f(x_i \wedge y) = f(x_i) P_i(x_i \wedge y)$ for each i. Hence $f(y) = \sum_{i=1}^{n} f(x_i) P(x_i \wedge y)$. Uniqueness follows from Lemma 4.

(b) For each i, by Lemma 2, $Q_i(y) = f(x_i \wedge y)/f(x_i)$ defines a pseudo-rank function Q_i on L. We note that $Q_i(x_i) = 1$. Given any $y \in L$, we then have $f(x_i \wedge y) = f(x_i) Q_i(y)$ for all i. Hence $f(y) = \sum_{i=1}^{n} f(x_i) Q_i(y)$. Uniqueness follows from Lemma 5. □

Theorem 1. *Let L be a bounded lattice and $\{x_1, \ldots, x_n\}$ be a set of completely orthogonal central elements in L such that $x_1 \vee \cdots \vee x_n = 1$.*

(a) *Suppose I is a nonempty subsets of $\{1, \ldots, n\}$. Let α_i be positive real numbers such that $\sum_{i\in I} \alpha_i = 1$. For each $i \in I$, let P_i be a pseudo-rank* *function on* $(x_i]$. *Then* $f(y) = \sum_{i\in I} \alpha_i P_i(x_i \wedge y)$ *is a pseudo-rank function on* L.

(b) *Let* α_i, $1 \le i \le n$, *be nonnegative real numbers such that* $\sum_{i\in I} \alpha_i = 1$. *For each* $i \in I$, *let* P_i *be a pseudo-rank function on* $(x_i]$. *Then* $f(y) = \sum_{i=1}^{n} \alpha_i P_i(x_i \wedge y)$ *is a pseudo-rank function on* L.

(c) *Every pseudo-rank function on* L *may be uniquely obtained as in* (a). *Moreover, if there exists at least one pseudo-rank function on each* $(x_i]$, *then every pseudo-rank function on* L *may be obtained as in* (b).

P r o o f. (a) and (b) follow immediately from the definition of a pseudo-rank function.

(c) Let f be a pseudo-rank function on L. Let I be the set of those $i \in \{1, \dots, n\}$ for which $f(x_i) \neq 0$. Put $\alpha_i = f(x_i)$ for all $i \in I$. Then $\sum_{i\in I} \alpha_i = \sum_{i\in I} f(x_i) = f(x_1 \vee \dots \vee x_n) = f(1) = 1$. By Lemma 6 there exist pseudo-rank functions P_i on $(x_i]$ for each $i \in I$ such that $f(x) = \sum_{i\in I} \alpha_i P_i(x_i \wedge y)$ for all $y \in L$. Thus f has a representation as in (a).

Suppose that there exists at least one pseudo-rank function on each $(x_i]$. Put $\alpha_i = f(x_i)$ for all $i = 1, \dots, n$. For $i \in \{1, \dots, n\} - I$, let P_i be any pseudo-rank function on $(x_i]$. Then $f(x) = \sum_{i\in I} \alpha_i P_i(x_i \wedge y) = \sum_{i=1}^{n} \alpha_i P_i(x_i \wedge y)$ for all $y \in L$, which represents f as in (b). □

We recall some terms from B i r k h o f f [1, p. 5]. The *length* of a poset P is defined as the least upper bound of the lengths of the chains in P and it is denoted by $l(P)$. If $l(P)$ is finite, then the poset P is said to be of *finite length.* Let P be a poset of finite length with 0 and $a \in P$. The *height* of a, denoted by $h(a)$, is defined as the least upper bound of all chains in $[0, a]$.

It is known that in a modular lattice, $h(a \vee b) + h(a \wedge b) = h(a) + h(b)$.

Corollary 1. *Let* L *be a modular lattice of finite length. Let* x_i, $1 \le i \le n$, *be completely orthogonal nonzero central elements in* L *such that* $x_1 \vee \dots \vee x_n = 1$. *Let* α_i, $1 \le i \le n$, *be nonnegative real numbers such that* $\sum \alpha_i = 1$. *Then* $f(y) = \sum_{i=1}^{n} \alpha_i [h(x_i \wedge y)/h(x_i)]$ *defines a pseudo-rank function on* L.

P r o o f. Clearly, for each $y \in L$, $h(x_i \wedge y) \le h(x_i)$, so $f(y) \in [0, 1]$. It is clear that $f(1) = 1$ and if $y, z \in L$, $y \le z$ then $f(y) \le f(z)$. Let $y, z \in L$, $y \wedge z = 0$.

Then using successively that x_i are central and L is modular of finite length, we have $f(y \vee z) = \sum_{i=1}^{n} \alpha_i h[x_i \wedge (y \vee z)]/h(x_i) = \sum_{i=1}^{n} \alpha_i h[(x_i \wedge y) \vee (x_i \wedge z)]/h(x_i) = \sum_{i=1}^{n} \alpha_i [h(x_i \wedge y) + h(x_i \wedge z)]/h(x_i) = f(y) + f(z)$. □

The proof of the following lemma follows from the definition of a pseudo-rank function.

Lemma 7. *Let L_1, L_2 be two bounded lattices, $f\colon L_1 \to L_2$ be a* 0, 1*-homomorphism. If g is a pseudo-rank function on L_2, then $g \circ f$ is a pseudo-rank function on L_1.*

Lemma 8. *Let f be a pseudo-rank function on a relatively complemented bounded lattice L. The set $A = \{x \in L:\ f(x) = 0\}$ is an ideal of L.*

P r o o f. Since $f(0) = 0$, A is nonempty. Clearly, if $x \leq y$ and $y \in A$, then $x \in A$. Let $x, y \in A$. Then $f(x \vee y) \leq f(x) + f(y)$ shows that A is an ideal of L. □

Lemma 9. *If f, g are pseudo-rank functions on a complemented lattice L such that $f(x) \leq g(x)$ for all $x \in L$, then $f = g$.*

P r o o f. If $f \neq g$, then $f(x) < g(x)$ for some $x \in L$. We have $f(x') \leq g(x')$. Hence $1 = f(1) = f(x \vee x') = f(x) + f(x') < g(x) + g(x') = g(x \vee x') = g(1) = 1$, a contradiction. □

Remark 1. Let L be a lattice with 0, I be a prime ideal of L. Then the mapping $\phi\colon L \to (L - I) \cup \{0\}$ defined by $\phi(x) = x$, if $x \notin I$ and $\phi(x) = 0$, if $x \in I$ is a meet-homomorphism.

Lemma 10. *Let f be a pseudo rank function on a relatively complemented bounded lattice L. Let I be a prime ideal of L, such that $I \subseteq \ker(f)$. Then there exists a unique pseudo-rank function g on $(L - I) \cup \{0\}$ such that $g \circ \phi = f$. Further g is a rank function iff $I = \ker(f)$.*

P r o o f. Suppose $x, y \in L$ and $\phi(x) = \phi(y)$. We claim that $f(x) = f(y)$. If $x, y \in I$, then $\phi(x) = \phi(y) = 0$ and $f(x) = f(y) = 0$. If $x, y \in L - I$, then $\phi(x) = \phi(y)$ implies $x = y$ and $f(x) = f(y)$. If $x \in I$ and $y \notin I$, then $\phi(x) = 0$ and $\phi(y) = y$. Hence $\phi(x) = \phi(y)$ implies $y = 0$ which contradicts $y \notin I$.

Define a map $g\colon (L - I) \cup \{0\} \to [0, 1]$, by $g(x) = f(x)$. As $g(x) = f(x)$ for all $x \in L - I$, g is a rank function iff $f(x) > 0$ for all $x \in L - I$, which is equivalent to $I = \ker(f)$. □

Theorem 2. *Let L be a modular lattice of finite length. Then the function N on L defined by $N(x) = h(x)/h(1)$ is a pseudo-rank function on L, where $h(x)$ denotes the height of x.*

Proof. Clearly $N(1) = 1$ and for $x, y \in L$, $x \leq y$ implies $N(x) \leq N(y)$.

Let $x, y \in L$ be such that $x \wedge y = 0$. Using L is modular we get $N(x \vee y) = h(x \vee y)/h(1) = [h(x) + h(y)]/h(1) = N(x) + N(y)$.

Thus N is a pseudo-rank function on L. □

3. Properties of the set of pseudo-rank functions

Definition 2. For a bounded lattice L, we denote the set of all pseudo-rank functions on L by $\mathbb{P}(L)$ (this set might be empty). We consider it as a subset of the real vector space $\mathbb{R}^L = \{f : f\colon L \to \mathbb{R}\}$ equipped with the product topology.

We note that $\mathbb{R}^L$ is a Hausdorff topological vector space. The topology on $\mathbb{R}^L$ can be described in terms of convergence of nets. Given a net $\{f_i\}$ in $\mathbb{R}^L$, and some $f \in \mathbb{R}^L$, we have $f_i \to f$ if and only if $f_i(x) \to f(x)$ for all $x \in L$. A partial order can be defined on $\mathbb{R}^L$ by $f \leq g$ iff for each $x \in L$, $f(x) \leq g(x)$.

Definition 3. Let E be a real vector space. A *convex combination* of points $x_1, \dots, x_n \in E$, is any linear combination of the form $\alpha_1 x_1 + \dots + \alpha_n x_n$ such that $\alpha_i \in \mathbb{R}$, $\alpha_i \geq 0$, $\sum\limits_{i=1}^{n} \alpha_i = 1$. A *convex subset* of E is any subset $K \subseteq E$ such that for $0 \leq \alpha \leq 1$, and any $x, y \in K$, $\alpha x + (1-\alpha)y \in K$. A *convex cone* in E is a subset $C \subseteq E$, such that $0 \in C$ and $\alpha x + \beta y \in C$ for all $x, y \in C$ and nonnegative real numbers α and β. A convex cone C is called *strict*, if $C \cap (-C) = 0$. A subset A of E is called an *affine subspace*, if it is closed under linear combinations of the form $\sum\limits_{i=1}^{n} \alpha_i x_i$, where $x_i \in A$ and $\sum \alpha_i = 1$. A *hyperplane* in E is an affine subspace of the form $V + x$, such that V is a linear subspace of E of codimension 1. A *base* for a strict cone in C is a convex $K \subseteq C$ such that K is contained in a hyperplane not containing the origin and $C = \{\alpha x : x \in K \text{ and } \alpha \geq 0\}$.

Proposition 3. *For a bounded lattice L, the set $\mathbb{P}(L)$ is a compact convex subset of $\mathbb{R}^L$.*

Proof. Clearly, $\mathbb{P}(L)$ is a convex set.

We note that $\mathbb{P}(L)$ is contained in $W = [0,1]^L$, which is compact by Tichonoff's theorem. Thus it is sufficient to show that $\mathbb{P}(L)$ is closed in W. Let N_i be a net in $\mathbb{P}(L)$, which converges to some $N \in W$. Since $N_i(1) \to N(1)$, we have $N(1) = 1$. For $x, y \in L$ with $x \le y$, we have $N_i(x) \le N_i(y)$ and so $N(x) \le N(y)$. Similarly, if $x \wedge y = 0$, then $N(x \vee y) = N(x) + N(y)$. Thus $N \in \mathbb{P}(L)$, so $\mathbb{P}(L)$ is closed in W. □

A convex subset F of a convex set K is called a *face* of K if for $x, y \in K$ and $0 < \alpha < 1$, $\alpha x + (1-\alpha)y \in F$ implies $x, y \in F$.

Lemma 11. *Let L be a bounded lattice and $X \subseteq L$. Then the set $F = \{N \in \mathbb{P}(L) :\ N(x) = 0$ for all $x \in X\}$ is a closed face of $\mathbb{P}(L)$.*

Proof. Let N_i be a net in F converging to some $N \in \mathbb{P}(L)$. Clearly, for all $x \in L$, $N_i(x) = 0$ for each i and so $N(x) = 0$. Thus $N \in F$, i.e. F is a closed subset of $\mathbb{P}(L)$. If $0 < \alpha < 1$, then for any $P, Q \in F$, $[\alpha P + (1-\alpha)Q](x) = 0$. Thus F is convex. Suppose that for some α, $0 < \alpha < 1$, and for some $P, Q \in \mathbb{P}(L)$, $\alpha P + (1-\alpha)Q = N \in F$. For all $x \in X$, we have $P(x) \le \alpha^{-1}\big([\alpha P + (1-\alpha)Q](x)\big) = \alpha^{-1}N(x) = 0$. Thus $P(x) = 0$, i.e. $P \in F$. Similarly, $Q \in F$. Therefore, F is a face of $\mathbb{P}(L)$. □

A lattice L is called *irreducible* if 0 and 1 are the only central elements in L.

Lemma 12. *Let L be a relatively complemented bounded lattice. Let $f \in \mathbb{P}(L)$ be such that* $\ker(f) = \{0\}$*. If f is an extreme point of $\mathbb{P}(L)$ then L is an irreducible lattice.*

Proof. Let $z \in L$ be a central element such that $z \neq 0$. By [9, Theorem 4.13, Remark 4.14], z has a complement z', which is also a central element. We have $f(z) > 0$, $f(z') > 0$ and $f(z) + f(z') = 1$. By Lemma 6, there exist pseudo-rank functions g_1 and g_2 on L such that $g_1(z) = 1$, $g_2(z') = 1$ and $f(y) = f(z)g_1(y) + f(z')g_2(y)$ for all $y \in L$. Since $g_1(z) = 1$ implies $g_1(z') = 0$, we get $g_1 \neq g_2$. Thus f is a convex combination of distinct pseudo-rank functions in $\mathbb{P}(L)$. This contradicts the assumption that f is an extreme point. □

Proposition 4. *Let L be a relatively complemented bounded lattice such that $\mathbb{P}(L)$ is nonempty. Let V be a convex cone in $\mathbb{R}^L$ with base $\mathbb{P}(L)$.*

(a) *A mapping $f \in \mathbb{R}^L$ belongs to V if and only if $f \ge 0$, and $f(x) = f(y) + f(z)$ for all $x, y, z \in L$ such that $x = y \vee z$ and $y \wedge z = 0$.*
(b) *For $f, g \in V$, we have $f \le g$ iff $g - f \in V$.*

P r o o f.

(a) If $f \in V$, then $f = \alpha N$ for some nonnegative real number α and for some $N \in \mathbb{P}(L)$. Clearly, $f \geq 0$. If $x, y, z \in L$, such that $x = y \vee z$ and $y \wedge z = 0$, then $N(x) = N(y) + N(z)$ shows that $f(x) = f(y) + f(z)$.

Conversely, suppose $f \geq 0$ and $f(x) = f(y) + f(z)$, whenever $x = y \vee z$ and $y \wedge z = 0$. Let $a, b \in L$ with $a \leq b$ and z be the relative complement of a in $[0, b]$. Then $f(b) = f(a) + f(z)$ shows that $f(a) \leq f(b)$. We note that $0 \leq f(x) \leq 1$ for all $x \in L$. Thus if $f(1) = 0$, then $f = 0$. In this case, $f = 0N$ for any $N \in \mathbb{P}(L)$. Now suppose, $f(1) > 0$. Put $N(x) = f(x)/f(1)$. Then N satisfies the properties of a pseudo-rank function on L and we have $f = f(1)N \in V$.

(b) If $g - f \in V$, then $g - f \geq 0$ and so $f \leq g$. Conversely, suppose $f \leq g$. We note that $g - f \geq 0$. Let $x, y \in L$ be such that $x \wedge y = 0$ Then by (a), $f(x \vee y) = f(x) + f(y)$ and $g(x \vee y) = g(x) + g(y)$. Hence $(g - f)(x \vee y) = (g - f)(x) + (g - f)(y)$. Hence $g - f \in V$. □

The following proposition is from [4, p. 336].

Proposition 5. *Let K be a convex subset of a real vector space. Let $X \subseteq K$ and $x \in K$. Then x lies in the face generated by X in K iff there exists $y \in K$ and real number α, $0 < \alpha < 1$, such that $\alpha x + (1 - \alpha)y$ lies in the convex hull of X.*

Corollary 2. *Let L be a relatively complemented bounded lattice. Let $P \in \mathbb{P}(L)$, $X \subseteq \mathbb{P}(L)$. Then P lies in the face generated by X in $\mathbb{P}(L)$ iff $P \leq \alpha Q$ for some positive real number α and some Q in the convex hull of X.*

P r o o f. Suppose that P lies in the face generated by X. Then by Proposition 5, there exists $P' \in \mathbb{P}(L)$, $\beta \in \mathbb{R}$, $0 < \beta < 1$, such that $Q = \beta P + (1 - \beta)P'$ belongs to the convex hull of X. Then $\beta P \leq Q$. Thus $P \leq \beta^{-1}Q$.

Conversely, suppose that $P \leq \alpha Q$ for some positive real number α and some Q in the convex hull of X. Since, P and αQ lie in the convex cone in $\mathbb{R}^L$, with base $\mathbb{P}(L)$, by Proposition 4, $\alpha Q - P$ lies in this cone also. Hence $\alpha Q - P = \beta P'$ for some $P' \in \mathbb{P}(L)$ and some nonnegative real number β. Then $P + \beta P' = \alpha Q$ and $1 + \beta = P(1) + \beta P'(1) = \alpha Q(1) = \alpha$. Hence $\gamma = \alpha^{-1}$ is a real number such that $0 < \gamma \leq 1$ and $\gamma P + (1 - \gamma)P' = Q$. Hence by Proposition 5, P lies in the face generated by X. □

Proposition 6. *Let L be a Boolean lattice such that $\mathbb{P}(L)$ is nonempty. Let V be the convex cone in $\mathbb{R}^L$ with base $\mathbb{P}(L)$. Then V is a lattice. For $f, g \in V$ and any $w \in L$, we have*

(a) $(f \wedge g)(w) = \inf\Big\{\sum \min\{f(a_i), g(a_i)\} :\ w = a_1 \vee \dots \vee a_n,$ *where* $a_1, \dots, a_n$ *are completely orthogonal elements in* $L\Big\} = \inf\Big\{\sum_{i=1}^{2} \min\{f(a_i), g(a_i)\} :$ $w = a_1 \vee a_2,$ *where* $a_1 \wedge a_2 = 0\Big\}.$

(b) $(f \vee g)(w) = \inf\Big\{\sum \max\{f(a_i), g(a_i)\} :\ w = a_1 \vee \dots \vee a_n,$ *where* $a_1, \dots, a_n$ *are completely orthogonal elements in* $L\Big\} = \sup\Big\{\sum_{i=1}^{2} \max\{f(a_i), g(a_i)\} :$ $w = a_1 \vee a_2,$ *where* $a_1 \wedge a_2 = 0\Big\}.$

Proof. For all $w \in L$, let $h_1(w) = \inf\Big\{\sum \min\{f(a_i), g(a_i)\} :\ w = a_1 \vee \dots \vee a_n$ where $a_1, \dots, a_n$ are completely orthogonal elements in $L\Big\}$ and $h_2(w) = \inf\Big\{\sum_{i=1}^{2} \min\{f(a_i), g(a_i)\} :\ w = a_1 \vee a_2$, where $a_1 \wedge a_2 = 0\Big\}$.

We have $h_1(w) \le h_2(w)$. If $w = a_1 \vee \dots \vee a_n$, where $a_1, \dots, a_n$ are completely orthogonal elements in L, then we may renumber so that there is an index k with $f(a_i) \le g(a_i)$ for all $i \le k$ and $f(a_i) \ge g(a_i)$ for all $i > k$. Put $x = a_1 \vee \dots \vee a_k$ and $y = a_{k+1} \vee \dots \vee a_n$. By distributivity of L, $x \wedge y = 0$. Also $f(x) \le g(x)$ and $f(y) \ge g(y)$. Hence $h_2(w) \le \min\{f(x), g(x)\} + \min\{f(y), g(y)\} = f(x) + g(y) = \sum \min\{f(a_i), g(a_i)\}$. Thus $h_2(w) \le h_1(w)$ and so $h_1 = h_2$.

Clearly, $h_1 \ge 0$ and $h_1 \le f$, $h_1 \le g$. Suppose, $k \in V$ and $k \le f$, $k \le g$. Then $k = \alpha r$ for some $\alpha \ge 0$ and for some pseudo-rank function r. Let $x \in L$ and $x = a_1 \vee \dots \vee a_n$ where $a_1, \dots, a_n$ are completely orthogonal. Hence $k(x) = \alpha r(x) = \alpha \sum_{i=1}^{n} r(a_i) = \sum_{i=1}^{n} \alpha r(a_i) = \sum_{i=1}^{n} k(a_i) \le \sum \min\{f(a_i), g(a_i)\}$. Thus $k(x) \le h_1(x)$. Therefore $h_1 = f \wedge g$ if $h_1 \in V$.

To show that $h_1 \in V$, it is sufficient to show that $h_1(x) = h_1(y) + h_1(z)$ whenever, $x = y \vee z$ and $y \wedge z = 0$. By the definition of h_1, it follows that for a given $\varepsilon > 0$, there exist completely orthogonal elements $y_1, \dots, y_n$ such that $y = y_1 \vee \dots \vee y_n$ and $\sum \min\{f(y_i), g(y_i)\} < h_1(y) + \varepsilon/2$. Similarly, $\sum \min\{f(z_i), g(z_i)\} < h_1(z) + \varepsilon/2$ for some completely orthogonal elements $z_1, \dots, z_m$ satisfying $z = z_1 \vee \dots \vee z_m$. Since $y \wedge z = 0$, it follows by distributivity of L that $y_1, \dots, y_n, z_1, \dots, z_m$ are completely orthogonal. Then $x = y_1 \vee \dots \vee y_n \vee z_1 \vee \dots \vee z_m$ shows that $h_1(x) < h_1(y) + h_1(z) + \varepsilon$. Hence $h_1(x) \le h_1(y) + h_1(z)$.

Let $x = x_1 \vee \dots \vee x_n$ for some completely orthogonal elements $x_1, \dots, x_n$ and $x = y \vee z$ for some $y, z \in L$ with $y \wedge z = 0$. By distributivity of L, $y =$

$y_1 \vee \cdots \vee y_n$ for completely orthogonal elements $y_i = x_i \wedge y$, $i = 1, \ldots, n$. Similarly $z = z_1 \vee \cdots \vee z_n$ for completely orthogonal elements $z_i = x_i \wedge z$, $i = 1, \ldots, n$ and $x_i = y_i \vee z_i$. Then $h_1(y) + h_1(z) \leq \sum \min\{f(y_i), g(y_i)\} + \sum \min\{f(z_i), g(z_i)\} = \sum\bigl(\min\{f(y_i), g(y_i)\} + \min\{f(z_i), g(z_i)\}\bigr) \leq \sum \min\{f(y_i) + f(z_i), g(y_i) + g(z_i)\} = \sum \min\{f(x_i), g(x_i)\}$. Therefore $h_1(y) + h_1(z) \leq h_1(x)$.

Thus $h_1(x) = h_1(y) + h_1(z)$ for all $x, y, z \in L$ with $x = y \vee z$. Since $h_1 \geq 0$, by Proposition 4, $h_1 \in V$.

Similarly, we can prove (b). □

We now give a characterization for two elements $P, Q \in \mathbb{P}(L)$ to be in disjoint faces of $\mathbb{P}(L)$.

Proposition 7. *Let L be a Boolean lattice. Let $P, Q \in \mathbb{P}(L)$ and V be the convex cone in $\mathbb{R}^L$, with base $\mathbb{P}(L)$. Then the following statements are equivalent.*

(a) *P and Q lie in disjoint faces of $\mathbb{P}(L)$.*

(b) *$P \wedge Q = 0$ in V.*

(c) *For any $x \in L$ and any positive real number ε, there exists a decomposition $x = y \vee z$, $y \wedge z = 0$ such that $P(y) + Q(z) < \varepsilon$.*

Proof. Let F be the face generated by P in $\mathbb{P}(L)$ and let G be the face generated by Q in $\mathbb{P}(L)$.

(a) $\implies$ (b):
If $P \wedge Q \neq 0$, then $P \wedge Q = \alpha N$ for some $N \in \mathbb{P}(L)$ and some positive real number α. Then $N \leq \alpha^{-1}P$ and $N \leq \alpha^{-1}Q$. Hence by Corollary 2, $N \in F \cap G$, a contradiction.

(b) $\implies$ (a):
If there is some $N \in F \cap G$, then by Corollary 2, there exists a positive real number α such that $N \leq \alpha P$ and $N \leq \alpha Q$. Then we get $\alpha^{-1}N \in V$ such that $0 < \alpha^{-1}N \leq P \wedge Q$, which is impossible.

(b) $\implies$ (c):
If $P(x) = 0$ then we take $y = x$ and $z = 0$. If $Q(x) = 0$, then take $y = 0$ and $z = x$. Suppose, $P(x) > 0$ and $Q(x) > 0$. Then by decreasing ε if necessary, we may also assume that $\varepsilon < P(x)$ and $\varepsilon < Q(x)$.

Since $(P \wedge Q)(x) = 0$, by Proposition 6, there exists a decomposition $x = y \vee z$ with $y \wedge z = 0$ such that $[\min\{P(y), Q(y)\}] + \min[\{P(z), Q(z)\} < \varepsilon$. If $P(y) \leq Q(y)$ and $P(z) \leq Q(z)$, then it follows that $P(x) = P(y) + P(z) < \varepsilon < P(x)$, which is impossible. Similarly, we cannot have both $Q(y) \leq P(y)$ and $Q(z) \leq P(z)$. Thus by interchanging y and z if necessary, we may assume that $P(y) < Q(y)$ and $Q(z) < P(z)$. Hence $P(y) + Q(z) < \varepsilon$.

(c) $\implies$ (b):
Follows from Proposition 6. □

Definition 4. If A, B are convex subsets of a real vector space E, a mapping $f\colon A \to B$ is called an *affine map* if f preserves convex combinations. A simplex in a real vector space E is a convex subset $K \subseteq E$ such that K is affinely isomorphic to a base for a lattice cone in some real vector space. A *Choquet simplex* is a compact simplex in a locally convex, Hausdorff topological vector space.

Theorem 3. *If L is a Boolean lattice such that $\mathbb{P}(L)$ is nonempty, then $\mathbb{P}(L)$ is a Choquet simplex.*

Proof. We know that $\mathbb{R}^L$ is a locally convex, Hausdorff topological vector space. It follows from Proposition 3 that $\mathbb{P}(L)$ is a closed, convex subset of $\mathbb{R}^L$. Moreover, $\mathbb{P}(L)$ is contained in the closed hyperplane $\{f \in \mathbb{R}^L : \ f(1) = 1\}$, which misses the origin. Let V be the convex cone in $\mathbb{R}^L$, with base $\mathbb{P}(L)$, and let $\leq_V$ be the partial order on $\mathbb{R}^L$ induced by V. For any $f, g \in V$, Proposition 4 shows that $f \leq_V g$ iff $f \leq g$. Consequently, it follows from Proposition 6 that V is a lattice cone. Therefore $\mathbb{P}(L)$ is a base for a lattice cone. Therefore $\mathbb{P}(L)$ is a Choquet simplex. □

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