

ON ASYMPTOTIC LINEARITY OF L-ESTIMATES

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*Dedicated to Professor Andrej Pázman on the occasion of his 70th birthday**(Communicated by Gejza Wimmer)*

ABSTRACT. A theorem on asymptotic linearity of L -estimates is proved under general set of regularity conditions, allowing the sampled distribution to be non-integrable. The main result is the improvement in the order of the remainder term in the formula for asymptotic linearity of L -statistic. It is shown that in the case of the integral coefficients this term $R_n = \mathcal{O}_P(\frac{1}{n})$ and the case of functional coefficients is also covered.

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1. Assumptions and main results

Suppose that $X_1, \dots, X_n$ is a random sample from the distribution of the random variable X and $X_n^{(1)} \leq \dots \leq X_n^{(n)}$ are the order statistics. The aim of the paper is to prove the asymptotic linearity of the L -estimate $\frac{1}{n} \sum_{i=1}^n J(\frac{i}{n+1}) X_n^{(i)}$ in a way useful mainly in the case when X is not integrable. Before stating a theorem on this topic and discussing its relation with another results we present the regularity conditions imposed on distribution of X and on the score function J .

(A1) The distribution function $F(t) = P(X \leq t)$ is continuous and strictly increasing on (d, D) , where $d = \inf\{t : F(t) > 0\}$, $D = \sup\{t : F(t) < 1\}$.

(A2) The function $J : (0, 1) \rightarrow E^1$ possesses the derivative J' on $(0, 1)$ and

$$\mu = \int_0^1 J(u) F^{-1}(u) \, du = E(J(F(X))x) \quad (1)$$

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is a real number.

(A3) There exist real numbers $\gamma = \gamma_d > -2$, $K > 0$ such that for each $u \in (0, \frac{1}{2})$

$$|J(u)| \leq Ku^{1+\gamma}, \quad |J'(u)| \leq Ku^\gamma.$$

There exist real numbers $\gamma = \gamma_D > -2$, $K > 0$ such that for each $u \in (\frac{1}{2}, 1)$

$$|J(u)| \leq K(1-u)^{1+\gamma}, \quad |J'(u)| \leq K(1-u)^\gamma.$$

(A4) There exist real numbers $\kappa_d < \gamma_d + 1$, $\kappa_D < \gamma_D + 1$ such that the integrals

$$\int_0^{1/2} u^{\kappa_d} dF^{-1}(u) = \int_{-\infty}^m F^{\kappa_d}(x) dx, \quad \int_{1/2}^1 (1-u)^{\kappa_D} dF^{-1}(u) = \int_m^{\infty} (1-F(x))^{\kappa_D} dx$$

are real numbers (here m denotes the median of F).

(A5) For every real number x the integrals

$$H(x) = \int_x^{+\infty} |J(F(y))| dy, \quad \int_{-\infty}^{+\infty} H(x) dF(x), \quad \int_{-\infty}^{+\infty} J(F(y))F(y) dy$$

are real numbers and $F(x)H(x) \rightarrow 0$ as $|x| \rightarrow +\infty$.

THEOREM 1. Suppose that (A1)–(A5) hold and put

$$\psi(x) = \int_{-\infty}^{+\infty} J(F(y))F(y) dy - \int_x^{+\infty} J(F(y)) dy. \quad (2)$$

(I) Let

$$\tilde{L}_n = \sum_{i=1}^n \tilde{c}_{ni} X_n^{(i)}, \quad \tilde{c}_{ni} = \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(u) du. \quad (3)$$

Then (cf. (1))

$$\tilde{L}_n = \mu + \frac{1}{n} \sum_{i=1}^n \psi(X_i) + \mathcal{O}_P\left(\frac{1}{n}\right). \quad (4)$$

(II) Put

$$L_n^* = \sum_{i=1}^n c_{ni} X_n^{(i)}, \quad c_{ni} = \frac{1}{n} J\left(\frac{i}{n+1}\right). \quad (5)$$

In addition to (A1)–(A5) suppose also that for some $t_d, t_D \in (d, D)$ and some positive real numbers β_d, β_D the inequalities

$$\begin{aligned} \sup\{|x|^{\beta_d} F(x) : d < x \leq t_d\} &< +\infty, \\ \sup\{|x|^{\beta_D} (1-F(x)) : t_D \leq x < D\} &< +\infty \end{aligned} \quad (6)$$

hold. If

$$\delta_1 = 2 + \gamma_d - \frac{1}{\beta_d} > 0, \quad \delta_2 = 2 + \gamma_D - \frac{1}{\beta_D} > 0, \quad (7)$$

then

$$L_n^* = \mu + \frac{1}{n} \sum_{i=1}^n \psi(X_i) + R_n, \quad (8)$$

where

$$R_n = \begin{cases} \mathcal{O}_P\left(\frac{\log n}{n}\right), & \min\{\delta_1, \delta_2\} = 1, \\ \mathcal{O}_P\left(\frac{1}{n^{\delta^*}}\right) & \text{otherwise.} \end{cases} \quad (9)$$

Here (cf. (7))

$$\delta^* = \min\{1, \delta_1, \delta_2\}. \quad (10)$$

Note that if $d < D$ from (A1) are real numbers, then (6), (7) hold, $F(x)H(x) \rightarrow 0$ as $|x| \rightarrow +\infty$ and (A3) implies that the integral (1) is finite.

Let the assumptions of the previous theorem be fulfilled. Then for the function (2) the equality $\int_{-\infty}^{+\infty} \psi(x) dF(x) = 0$ holds and if $V = \int_{-\infty}^{+\infty} \psi^2(x) dF(x) < +\infty$,

by means of the central limit theorem one obtains that $\sqrt{n}(\tilde{L}_n - \mu) \rightarrow N(0, V)$ in distribution as n tends to infinity. Moreover, if also for the remainder term R_n from (9) the equality $R_n = o_P(n^{-1/2})$ holds, then $\sqrt{n}(L_n^* - \mu) \rightarrow N(0, V)$ in distribution.

A review of results on the asymptotic normality of L -estimates can be found in the monograph of Serfling [7]. General results on this topic are proved by Chernoff, Gastwirth and Johns [1] under set of conditions, which are of general nature but may be not easy to verify. The asymptotic linearity of L -estimates with the remainder term $R_n = \mathcal{O}_P(\frac{1}{n})$, from which the asymptotic normality follows by CLT, has been proved in Section 4 of Jurečková and Sen [3]. But for the L -statistics with the integral coefficients (3) and the number $\beta < 1$ from (6) they assume in their Theorem 4.3.1 that the untrimmed score function J fulfils the Lipschitz condition of the order $\nu > 1$, which in typical cases is not fulfilled, and for the statistics with the functional coefficients (5) the exponent β is in their Theorem 4.3.2 assumed to be greater than 1. Govindarajulu and Mason [2] proved the asymptotic linearity of the L -statistics even in a setting allowing X not to be integrable, but in difference from the previous theorem only with the remainder term $R_n = o_P(\frac{1}{\sqrt{n}})$. The remainder term in this paper has better accuracy than this result both for the L -statistics with integral coefficients and with functional scores as well, and the conditions (A1)-(A5) can be applied also in cases not covered by the conditions from Govindarajulu and Mason. Another results for strong representation of L -statistics were proved by Mason and Shorack [4] and [5], but

again with $R_n = o_P(\frac{1}{\sqrt{n}})$. Thus the main contribution of the previous theorem is the improvement in remainder term in the formula for the asymptotic linearity of the L -estimates in the case when $\beta < 1$, which occurs in the case when X is not integrable.

2. Proofs

In accordance with the assumptions of Theorem 1 throughout this section we assume that the assumptions (A1)–(A5) hold and we use the notation $F^{-1}(u) = \inf\{x : F(x) \geq u\}$. Further, $U_1, \dots, U_n$ denotes a random sample from the uniform distribution on $(0, 1)$ and

$$U_n(t) = \frac{1}{n} \sum_{i=1}^n \chi_{\langle 0, t \rangle}(U_i)$$

its empirical distribution function. In the proofs we shall use the function

$$\phi(s) = - \int_s^1 J(u) \, du + \int_0^1 u J(u) \, du, \quad s \in \langle 0, 1 \rangle, \quad (11)$$

and the fact, that under the validity of (A2) the equality $\phi'(s) = J(s)$ holds. The symbol K will denote the generic constant, i.e., it will not depend on n but even though the symbol remains the same, it may denote various values.

Proof of Theorem 1(I). Since the condition (A4) holds, the integrals

$$\int_{-\infty}^t F(x)^{\gamma_d+2} \, dx, \quad \int_t^{+\infty} (1 - F(x))^{\gamma_D+2} \, dx$$

are finite for every real number $t \in (d, D)$. Thus for the empirical distribution function

$$F_n(x) = \frac{1}{n} \sum_{j=1}^n \chi_{(-\infty, x)}(X_j)$$

the integral $\int_{-\infty}^{+\infty} (\phi(F_n(x)) - \phi(F(x))) \, dx$ is finite, and making use of the integration by parts and proceeding similarly as described in [3, p. 144], one obtains that

$$\tilde{L}_n - \mu - \frac{1}{n} \sum_{i=1}^n \psi(X_i) = \int_0^1 V_n(s) \, dF^{-1}(s),$$

where

$$|V_n(s)| = |\phi(U_n(s)) - \phi(s) - \phi'(s)(U_n(s) - s)|.$$

By [6, Theorem 2.11.10] given $\eta \in (0, \frac{1}{2})$ there exists a constant $C(\eta)$ such that for every real number $M > 0$

$$P\left(\sup_{0 < s < 1} \frac{|\sqrt{n}(U_n(s) - s)|}{(s(1-s))^{1/2-\eta}} > M\right) \leq \frac{C(\eta)}{M^2}$$

for all integers $n > 1$. Hence if $\varepsilon > 0$ and $\eta \in (0, 1/2)$ then there exists a number $M > 0$ such that with probability at least $1 - \varepsilon$

$$\sup_{0 < s < 1} \frac{|\sqrt{n}(U_n(s) - s)|}{(s(1-s))^{1/2-\eta}} \leq M. \quad (12)$$

Further, employing the Daniels theorem from [8, p. 345], one obtains that

$$\sup_{s \in (0,1)} \frac{U_n(s)}{s} = \mathcal{O}_P(1), \quad \sup_{s \in (0,1)} \frac{1 - U_n(s)}{1 - s} = \mathcal{O}_P(1),$$

the Wellner-Shorack inequality from [8, p. 415] implies that

$$\begin{aligned} \sup \left\{ \frac{s}{U_n(s)} : U_n(s) > 0, s < 1 \right\} &= \mathcal{O}_P(1), \\ \sup \left\{ \frac{1-s}{1-U_n(s)} : U_n(s) < 1, s > 0 \right\} &= \mathcal{O}_P(1). \end{aligned}$$

Hence given $\varepsilon > 0$ there exist positive constants a_1, a_2, b_1, b_2 such that for all n with probability at least $1 - \varepsilon$

$$U_n(s) > 0, s \in (0, 1) \implies a_1 s < U_n(s) < a_2 s, \quad (13)$$

$$U_n(s) < 1, s \in (0, 1) \implies b_1(1-s) < 1 - U_n(s) < b_2(1-s). \quad (14)$$

Thus it is sufficient to prove that for a suitably chosen $\eta \in (0, \frac{1}{2})$ under the validity of (12), (13) and (14)

$$\int_0^1 |V_n(s)| dF^{-1}(s) = \mathcal{O}\left(\frac{1}{n}\right). \quad (15)$$

In proving this we shall utilize the fact, that for any $0 \leq \alpha \leq 1$ and positive real numbers c_1, c_2 the inequality

$$(\alpha c_1 + (1 - \alpha)c_2)^\gamma \leq c_1^\gamma + c_2^\gamma \quad (16)$$

holds.

Let $0 < \delta < 1/2$ be a fixed real number, $s \in (0, \delta)$ and $\gamma = \gamma_d$. Assume that $U_n(s) > 0$. An application of Taylor theorem, (A3), (12), (13) and (16) yields that

$$\begin{aligned} |V_n(s)| &= \left| J' \left(\alpha s + (1 - \alpha) U_n(s) \right) \right| \frac{(U_n(s) - s)^2}{2} \\ &\leq \frac{K}{n} \left(\alpha s + (1 - \alpha) U_n(s) \right)^\gamma s^{2(1/2 - \eta)} \\ &\leq \frac{K}{n} \left(s^\gamma + U_n(s)^\gamma \right) s^{1 - 2\eta} \\ &\leq \frac{K}{n} s^{\gamma + 1 - 2\eta}. \end{aligned}$$

Similarly if $U_n(s) = 0$, then

$$\begin{aligned} |V_n(s)| &= |\phi(0) - \phi(s) + \phi'(s)s| \\ &\leq \int_0^s |J(u)| \, du + K s^{\gamma + 2} \\ &\leq K \int_0^s u^{\gamma + 1} \, du + K s^{\gamma + 2} \leq K s^{\gamma + 2} \\ &= K s^\gamma (s - U_n(s))^2 \leq \frac{K}{n} s^{\gamma + 1 - 2\eta}. \end{aligned}$$

Thus owing to (A4)

$$\int_0^\delta |V_n(s)| \, dF^{-1}(s) \leq \int_0^\delta \frac{K}{n} s^{\gamma + 1 - 2\eta} \, dF^{-1}(s) \leq \frac{K}{n}.$$

Since for $s \in (\delta, 1)$ one can proceed similarly, (15) is proved. $\square$

In the rest of the section we assume that in addition to (A1)–(A5) also the inequalities (6), (7) hold for some positive real β_d, β_D and some $t_d, t_D \in (d, D)$.

The proof of the assertion (II) of the theorem from the previous section will be based on the following auxiliary assertions.

LEMMA 1. *Let $U_n^{(j)}$ denotes the j th order statistic from $U_1, \dots, U_n$. Then for every positive real number c*

$$\lim_{n \rightarrow \infty} P \left(U_n^{(1)} < \frac{c}{n} \right) = 1 - e^{-c}, \quad \lim_{n \rightarrow \infty} P \left(U_n^{(n)} > 1 - \frac{c}{n} \right) = 1 - e^{-c}.$$

LEMMA 2. *For each $u \in (0, 1)$*

$$|F^{-1}(u)| \leq \max \left\{ \frac{K}{u^{1/\beta_d}}, \frac{K}{(1 - u)^{1/\beta_D}} \right\}.$$

LEMMA 3. Suppose that the number (cf. (7))

$$\delta = \min\{\delta_1, \delta_2\} \quad (17)$$

is positive. For $c > 0$ put

$$h_n(c) = \frac{c}{n}, \quad H_n(c) = 1 - \frac{c}{n}.$$

(I) The equality

$$\int_0^{h_n(c)} |J(u)F^{-1}(u)| du + \int_{H_n(c)}^1 |J(u)F^{-1}(u)| du = \mathcal{O}\left(\frac{1}{n^\delta}\right) \quad (18)$$

holds.

(II) Define the function $J_n(u)$ on $(0, 1)$ by the formula

$$J_n(u) = J\left(\frac{i}{n+1}\right) \quad \text{if} \quad \frac{i-1}{n} < u \leq \frac{i}{n}, \quad i = 1, \dots, n, \quad (19)$$

and put

$$R_n^{(1)} = \int_{h_n(c)}^{1/2} |J_n(u) - J(u)| |F^{-1}(u)| du, \quad R_n^{(2)} = \int_{1/2}^{H_n(c)} |J_n(u) - J(u)| |F^{-1}(u)| du. \quad (20)$$

Then with the notation from (7)

$$R_n^{(1)} = \begin{cases} \mathcal{O}\left(\frac{1}{n^{\delta_1^*}}\right), & \delta_1^* = \min\{1, \delta_1\} & \delta_1 \neq 1, \\ \mathcal{O}\left(\frac{\log n}{n}\right) & & \delta_1 = 1, \end{cases} \quad (21)$$

$$R_n^{(2)} = \begin{cases} \mathcal{O}\left(\frac{1}{n^{\delta_2^*}}\right), & \delta_2^* = \min\{1, \delta_2\} & \delta_2 \neq 1, \\ \mathcal{O}\left(\frac{\log n}{n}\right) & & \delta_2 = 1. \end{cases} \quad (22)$$

Proof. The proof of (I) easily follows from (A3), (6) and (7). If $\lambda \in (0, 1)$, then one can prove by means of (A3) and (16) that for each $u \in (\frac{c}{n}, \lambda)$ the inequalities

$$|J_n(u) - J(u)| \leq \frac{K}{n} \left(\left(\frac{[nu] + 1}{n+1} \right)^{\gamma_d} + u^{\gamma_d} \right) \leq \frac{K}{n} u^{\gamma_d}, \quad K = K(c, \lambda) \quad (23)$$

hold (here $[a]$ denotes the largest integer not exceeding a). Employing (6) and (7) after some computation one obtains the formula (21), (22) can be proved analogously. $\square$

LEMMA 4. Suppose that I_s denotes for $s \in (0, 1)$ the interval with the endpoints $s, U_n(s)$, i.e., $I_s = (s, U_n(s))$ if $s < U_n(s)$ and $I_s = (U_n(s), s)$ otherwise. Then in the notation from the previous lemmas

$$\int_{U_n^{(1)}}^{U_n^{(n)}} \left(\int_{I_s} |J_n(u) - J(u)| du \right) dF^{-1}(s) = \mathcal{O}_P\left(\frac{1}{n}\right).$$

Proof. Lemma 1 implies that given $\varepsilon > 0$ there exists a positive constant c such that for all sample sizes n sufficiently large

$$\frac{1}{2} > U_n^{(1)} \geq \frac{c}{n}, \quad \frac{1}{2} < U_n^{(n)} \leq 1 - \frac{c}{n} \quad (24)$$

with probability at least $1 - \varepsilon$. Therefore we may assume that the inequalities (24), (13) and (14) are fulfilled. Further, according to the Glivenko-Cantelli theorem we may assume that for all $n \geq n_0$ and $s \in \langle 0, \frac{1}{2} \rangle$ the inequalities $U_n(s) \leq \frac{1}{2} + \varepsilon^* < 1$ hold. Thus employing (24) and (13) we obtain the validity of (23) on the interval I_s for each $s \in (U_n^{(1)}, \frac{1}{2})$, and the repeated use of (13) yields that

$$\int_{U_n^{(1)}}^{1/2} \left(\int_{I_s} |J_n(u) - J(u)| du \right) dF^{-1}(s) \leq \int_{U_n^{(1)}}^{1/2} \frac{K}{n} s^{\gamma_d+1} dF^{-1}(s) = \mathcal{O}_P\left(\frac{1}{n}\right),$$

where the last equality follows from (A4). Since the statement

$$\int_{1/2}^{U_n^{(n)}} \left(\int_{I_s} |J_n(u) - J(u)| du \right) dF^{-1}(s) = \mathcal{O}_P\left(\frac{1}{n}\right)$$

can be verified similarly, the lemma is proved. $\square$

Proof of Theorem 1(II). Let F_n denote the empirical distribution function of $X_1, \dots, X_n$ and (cf. (19))

$$\phi_n(s) = - \int_s^1 J_n(u) du + \int_0^1 u J_n(u) du, \quad s \in \langle 0, 1 \rangle.$$

Then for the statistic (5) the equality

$$L_n^* = \int_{-\infty}^{+\infty} x d\phi_n(F_n(x)) \quad (25)$$

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holds. Put $U_i = F(X_i)$, $i = 1, \dots, n$. As the set having the probability not exceeding ε can be neglected, according to Lemma 1 we may assume that for properly chosen positive $c_1 < c_2$

$$F^{-1}\left(\frac{c_1}{n}\right) < X_n^{(1)} < F^{-1}\left(\frac{c_2}{n}\right), \quad F^{-1}\left(1 - \frac{c_2}{n}\right) < X_n^{(n)} < F^{-1}\left(1 - \frac{c_1}{n}\right), \quad (26)$$

$$X_i = F^{-1}(F(X_i)), \quad i = 1, \dots, n.$$

Put

$$\mu_n = \int_{U_n^{(1)}}^{U_n^{(n)}} J_n(u) F^{-1}(u) \, du.$$

By means of the continuity of F

$$\mu_n = \int_{F^{-1}(U_n^{(1)})}^{F^{-1}(U_n^{(n)})} J_n(F(x)) x \, dF(x) = \int_{F^{-1}(U_n^{(1)})}^{F^{-1}(U_n^{(n)})} x \, d\phi_n(F(x)),$$

because for $a < b$

$$\phi_n(F(b)) - \phi_n(F(a)) = \int_{F(a)}^{F(b)} J_n(u) \, du = \int_a^b J_n(F(x)) \, dF(x).$$

Since the product of right-continuous functions of bounded variation has also this property, the function $G(x) = x[\phi_n(F_n(x)) - \phi_n(F(x))]$ induces a signed measure ν_G . Thus making use of the integration by parts one obtains

$$L_n^* - \mu_n = \int_{\langle X_n^{(1)}, X_n^{(n)} \rangle} x \, d[\phi_n(F_n(x)) - \phi_n(F(x))] = \nu_G(\langle X_n^{(1)}, X_n^{(n)} \rangle) - T_2,$$

where the second term

$$\begin{aligned} T_2 &= \int_{X_n^{(1)}}^{X_n^{(n)}} [\phi_n(F_n(x)) - \phi_n(F(x))] \, dx \\ &= \int_{F(X_n^{(1)})}^{F(X_n^{(n)})} [\phi_n(F_n(F^{-1}(s))) - \phi_n(F(F^{-1}(s)))] \, dF^{-1}(s) \\ &= \int_{U_n^{(1)}}^{U_n^{(n)}} [\phi_n(U_n(s)) - \phi_n(s)] \, dF^{-1}(s). \end{aligned}$$

Hence if we show that

$$\nu_G(\langle X_n^{(1)}, X_n^{(n)} \rangle) = \mathcal{O}_P\left(\frac{1}{n^\delta}\right), \quad (27)$$

where δ is defined in (17), we obtain that

$$L_n^* - \mu_n = \mathcal{O}_P\left(\frac{1}{n^\delta}\right) - \int_{U_n^{(1)}}^{U_n^{(n)}} \left(\phi_n(U_n(s)) - \phi_n(s)\right) dF^{-1}(s). \quad (28)$$

But

$$G(X_n^{(n)}) = X_n^{(n)}[\phi_n(1) - \phi_n(F(X_n^{(n)}))] = F^{-1}(U_n^{(n)}) \int_{U_n^{(n)}}^1 J_n(u) du, \quad (29)$$

$$\begin{aligned} G(X_n^{(1)})^- &= \lim_{z \nearrow X_n^{(1)}} z[\phi_n(0) - \phi_n(F(z))] = X_n^{(1)}[\phi_n(0) - \phi_n(F(X_n^{(1)}))] \\ &= -F^{-1}(U_n^{(1)}) \int_0^{U_n^{(1)}} J_n(u) du. \end{aligned} \quad (30)$$

Assume without the loss of generality that $c_2 > 2$. Then by (A3)

$$\begin{aligned} \int_{U_n^{(n)}}^1 |J_n(u)| du &\leq \int_{1-\frac{c_2}{n}}^1 |J_n(u)| du \leq \sum_{i=n-[c_2]}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} |J_n(u)| du \\ &\leq \frac{K}{n} \sum_{i=n-[c_2]}^n \left(1 - \frac{i}{n+1}\right)^{1+\gamma_D} \\ &\leq \frac{K}{n} \sum_{i=1}^{[c_2]+1} \frac{i^{1+\gamma_D}}{n^{1+\gamma_D}} \leq \frac{K}{n^{2+\gamma_D}}. \end{aligned}$$

This together with (29), (26) and Lemma 2 means, that

$$|G(X_n^{(n)})| \leq |F^{-1}(U_n^{(n)})| \int_{U_n^{(n)}}^1 |J_n(u)| du \leq \frac{K}{(1 - U_n^{(n)})^{\frac{1}{\beta_D}}} \frac{1}{n^{2+\gamma_D}} \leq \frac{K}{n^{2+\gamma_d-1/\beta_D}}. \quad (31)$$

Further, since according to (26) the inequality $U_n^{(1)} \leq \frac{c_2}{n}$ holds, by means of (A2)

$$\int_0^{U_n^{(1)}} |J_n(u)| du \leq \int_0^{\frac{[c_2]+1}{n}} |J_n(u)| du \leq \frac{K}{n} \sum_{i=1}^{[c_2]+1} \left(\frac{i}{n+1}\right)^{1+\gamma_d} \leq \frac{K}{n^{2+\gamma_d}}$$

Combining this with (30), (26) and Lemma 2 one obtains

$$|G(X_n^{(1)})^-| \leq |F^{-1}(U_n^{(1)})| \int_0^{U_n^{(1)}} |J_n(u)| \, du \leq \frac{K}{(U_n^{(1)})^{\frac{1}{\beta_d}}} \frac{1}{n^{2+\gamma_d}} \leq \frac{K}{n^{2+\gamma_d-\frac{1}{\beta_d}}}. \quad (32)$$

Obviously, (31) and (32) imply (27). Further, $\tilde{L}_n = \int x \, d\phi(F_n(x))$, where ϕ is defined in (11). This together with (18), (26) and integration by parts similarly as in (28) means that

$$\tilde{L}_n - \mu = \mathcal{O}_P\left(\frac{1}{n^\delta}\right) - \int_{U_n^{(1)}}^{U_n^{(n)}} [\phi(U_n(s)) - \phi(s)] \, dF^{-1}(s). \quad (33)$$

Taking into account (28), (33) and Lemma 4 one obtains that

$$\begin{aligned} |(L_n^* - \mu_n) - (\tilde{L}_n - \mu)| &\leq \mathcal{O}_P\left(\frac{1}{n^\delta}\right) + \int_{U_n^{(1)}}^{U_n^{(n)}} \left[\int_{I_s} |J_n(u) - J(u)| \, du \right] \, dF^{-1}(s) \\ &= \mathcal{O}_P\left(\frac{1}{n^{\delta^*}}\right), \end{aligned} \quad (34)$$

where δ^* is defined in (10). But by means of (26) and Lemma 3

$$|\mu_n - \mu| \leq \int_{U_n^{(1)}}^{U_n^{(n)}} |J_n(u) - J(u)| |F^{-1}(u)| \, du + \mathcal{O}\left(\frac{1}{n^\delta}\right) \leq R_n^{(1)} + R_n^{(2)} + \mathcal{O}\left(\frac{1}{n^\delta}\right), \quad (35)$$

where $R_n^{(1)}$, $R_n^{(2)}$ are defined by (20) with $c = c_1$, and (8)–(10) can be obtained from (34), (35), (21), (22) and (4). $\square$

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