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INTERVAL ESTIMATION OF THE MEAN OF A NORMAL DISTRIBUTION BASED ON QUANTIZED OBSERVATIONS

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Dedicated to Professor Andrej Pázman on the occasion of his 70th birthday

(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. We consider the problem of making statistical inference about the mean of a normal distribution based on a random sample of quantized (digitized) observations. This problem arises, for example, in a measurement process with errors drawn from a normal distribution and with a measurement device or process with a known resolution, such as the resolution of an analog-to-digital converter or another digital instrument. In this paper we investigate the effect of quantization on subsequent statistical inference about the true mean. If the standard deviation of the measurement error is large with respect to the resolution of the indicating measurement device, the effect of quantization (digitization) diminishes and standard statistical inference is still valid. Hence, in this paper we consider situations where the standard deviation of the measurement error is relatively small. By Monte Carlo simulations we compare small sample properties of the interval estimators of the mean based on standard approach (i.e. by ignoring the fact that the measurements have been quantized) with some recently suggested methods, including the interval estimators based on maximum likelihood approach and the fiducial approach. The paper extends the original study by Hannig *et al.* (2007).

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1. Introduction

The measurement results (of physical quantities) are affected by limited resolution and subsequent inherent digitization of the used instrumentation and/or by quantization during the analog-to-digital conversion. Typically, it is assumed that the unobservable additive measurement errors come from normal distribution with zero mean and common unknown variance, and further, the observable measurement results are represented by the quantized values at the given digital scale of the used measurement instrument. The effect of quantization (digitization) on statistical inference about the true mean, say μ, has been of great interest for a long time, especially in the engineering and metrological literature. For recent studies see e.g. [19], [6], [15], [16], [10], [24], [23], [18], [3], [22], [25] and [14]. It has been observed in many instances that the effect of quantization diminishes and the standard statistical inference (i.e. the tests and the confidence intervals about μ based on Student's t distribution of the pivotal statistic) is still valid, if the standard deviation of the measurement error is large with respect to the resolution of the measurement instrument. So, from the theoretical point of view, it is important to study the properties of the suggested methods (here we consider interval estimators only) if the error variance is relatively small.

In this paper we compare, by Monte Carlo simulations, small sample properties (empirical coverage probabilities and mean widths of the interval estimates) of selected interval estimators for the parameter μ based on the following four approaches:

i) the standard approach based on the Student's t distribution of the pivotal statistic (i.e. by ignoring the fact that the measurements have been quantized),

ii) the asymptotically correct confidence intervals based on maximum likelihood approach,

iii) the metrological approach based on a modified expression for squared standard uncertainty, recently suggested by Willink (2007),

iv) the method based on the generalized fiducial approach, recently suggested by Hannig *et al.* (2007).

For more details on the (generalized) fiducial inference see [7], [13], and also [12].

The rest of this paper is organized as follows. In Section 2 we introduce the model for quantized observations and the interval estimators, suggested as approximate confidence intervals for μ, based on the maximum likelihood approach, the standard Student's t confidence interval based on ignorance of the problem with limited resolution, as well as the version of the standard confidence interval based on a modified expression for squared standard uncertainty, recently suggested by Willink (2007). In Section 3 we describe the generalized fiducial approach and the algorithm for generating sample from the joint fiducial distribution of the parameters μ and σ as suggested by Hannig *et al.* (2007). Here we present a modified version of the interval estimator for μ based on the marginal fiducial distribution. In Section 4 we compare the small sample properties (coverage probabilities and the average lengths) of the suggested interval estimators for μ by Monte Carlo simulation study. Discussion and conclusions are presented in Section 5.

2. Model for quantized observations

The model for quantized observations is motivated by the measurement problem, frequently considered in metrology. Consider estimation of a quantity μ based on a set of independent quantized (digitized) measurements observed on a discrete scale with unit of resolution δ. The resolution δ is defined as the smallest difference between indications that can be meaningfully distinguished by the measurement instrument.

First, consider n mutually independent measurements, $n \geq 2$, that are registered by an instrument with perfect resolution ($\delta = 0$). The observed values, say $y_1, y_2, \ldots, y_n$, are realizations of independent random variables $Y_1, Y_2, \ldots, Y_n$. We assume that $Y_i = \mu + \sigma Z_i$ with $Z_i \sim N(0,1)$, i.e. the instrument has no systematic error and the additive measurement errors are normally distributed with zero mean and variance σ^2. Hence, the optimal estimators of μ and σ^2 are $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, and $S_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$, respectively. The optimal exact two-sided $(1-\alpha) \times 100\%$-confidence interval for μ is

$$\bar{Y} \mp \frac{S_Y}{\sqrt{n}} t_{n-1}(1-\alpha/2), \tag{1}$$

where $t_{n-1}(1-\alpha/2)$ is the $(1-\alpha/2)$-quantile of the Student's t distribution with $n-1$ degrees of freedom.

Now, let us consider the outcomes from the measurement instrument with limited resolution ($\delta > 0$), say $x_1, x_2, \ldots, x_n$, which are registrations of the rounded values (values at the given digital scale) of the inherent unobservable measurements $y_1, y_2, \ldots, y_n$. The observed values are $x_i = k_i\delta = \lfloor y_i/\delta + 0.5 \rfloor \delta = \lfloor (\mu + \sigma z_i)/\delta + 0.5 \rfloor \delta$, $i = 1, 2, \ldots, n$, with $k_i \in \{0, \pm 1, \pm 2, \ldots\}$, ($\lfloor a \rfloor$ denotes the floor function resulting in the greatest integer not greater than a).

So, in the case of independent quantized observations, the observed results of the measurement process are the quantized values $x_1, x_2, \ldots, x_n$ — the realizations of independent random variables $X_1, X_2, \ldots, X_n$, where

$$X_i = K_i\delta, \tag{2}$$

and $K_i = \lfloor(\mu + \sigma Z_i)/\delta + 0.5\rfloor$ with $Z_i \sim N(0,1)$, $i = 1, 2, \ldots, n$. The distribution of each X_i depends on the unknown parameters μ and σ and the known parameter δ. The probability mass function $\{P_{k\delta}\}$, $k = 0, \pm1, \pm2, \ldots$, where $P_{k\delta} = \Pr(X_i = k\delta)$ is given by

$$P_{k\delta} = \int_{(k-0.5)\delta}^{(k+0.5)\delta} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\} \mathrm{d}y. \tag{3}$$

Notice that, $X_i \sim N(\mu, \sigma^2)$ if $\delta \to 0$.

Based on this, Frenkel and Kirkup (2005) and Cordero *et al.* (2006) have characterized the effect of quantization by describing the (functional) relationship between the parameters, μ and σ^2, and the expectation and variance of X_i, i.e. $E(X_i) = \delta \sum_k kP_{k\delta}$ and $\mathrm{Var}(X_i) = \sum_k (k\delta - E(X_i))^2 P_{k\delta}$.

If a random sample $X_1, X_2, \ldots, X_n$ from distribution given by (3) is available, an asymptotically correct interval estimator for μ (i.e. an asymptotically exact confidence interval) could be derived based on the maximum likelihood (ML) approach, see e.g. [1] and/or [17]. Given $x_1, x_2, \ldots, x_n$, with $x_i = k_i\delta$, the log-likelihood function for the parameters μ and σ is

$$l(\mu, \sigma \mid x_1, \ldots, x_n) = \sum_{i=1}^{n} \ln(P_{k_i\delta}). \tag{4}$$

The realizations of the maximum likelihood estimators

$$(\hat{\mu}_{ML}, \hat{\sigma}_{ML})' = \operatorname*{argmax}_{(\mu,\sigma)\in\Theta} l(\mu, \sigma \mid X_1, \ldots, X_n),$$

where $\Theta = (-\infty, \infty) \times (0, \infty)$, could be evaluated numerically for any $x_1, x_2, \ldots$ $\ldots, x_n$, provided that they exist.

For sufficiently large n, the distribution of the ML estimators, $\hat{\mu}_{ML}$ and $\hat{\sigma}_{ML}$, can be approximated by

$$\begin{pmatrix} \hat{\mu}_{ML} \\ \hat{\sigma}_{ML} \end{pmatrix} \overset{\text{appr.}}{\sim} N\left(\begin{pmatrix} \mu \\ \sigma \end{pmatrix}, I_F^{-1}(\mu, \sigma)\right), \tag{5}$$

where $I_F(\mu, \sigma)$ is the Fisher's information matrix for the parameters μ and σ, i.e. the covariance matrix of the score vector $(\partial l(\mu,\sigma)/\partial\mu, \partial l(\mu,\sigma)/\partial\sigma)'$, defined by

$$I_F(\mu,\sigma) = E\begin{pmatrix} \dfrac{\partial l(\mu,\sigma)}{\partial\mu}\dfrac{\partial l(\mu,\sigma)}{\partial\mu}, & \dfrac{\partial l(\mu,\sigma)}{\partial\mu}\dfrac{\partial l(\mu,\sigma)}{\partial\sigma} \\ \dfrac{\partial l(\mu,\sigma)}{\partial\sigma}\dfrac{\partial l(\mu,\sigma)}{\partial\mu}, & \dfrac{\partial l(\mu,\sigma)}{\partial\sigma}\dfrac{\partial l(\mu,\sigma)}{\partial\sigma} \end{pmatrix}, \tag{6}$$

where $l(\mu,\sigma) = l(\mu,\sigma \mid X_1,\ldots,X_n)$. Hence, it directly follows that $I_F(\mu,\sigma) = nJ(\mu,\sigma)$, where the elements of the matrix $J(\mu,\sigma)$ are given by

$$\begin{aligned} J_{1,1}(\mu,\sigma) &= \sum_{k=0,\pm1,\ldots}\left[\frac{\partial P_{k\delta}}{\partial\mu}\frac{1}{P_{k\delta}}\right]^2 P_{k\delta}, \\ J_{1,2}(\mu,\sigma) &= \sum_{k=0,\pm1,\ldots}\left[\frac{\partial P_{k\delta}}{\partial\mu}\frac{\partial P_{k\delta}}{\partial\sigma}\frac{1}{P_{k\delta}^2}\right] P_{k\delta} = J_{2,1}(\mu,\sigma), \\ J_{2,2}(\mu,\sigma) &= \sum_{k=0,\pm1,\ldots}\left[\frac{\partial P_{k\delta}}{\partial\sigma}\frac{1}{P_{k\delta}}\right]^2 P_{k\delta}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} P_{k\delta} &= \Phi(h) - \Phi(d), \\ \frac{\partial P_{k\delta}}{\partial\mu} &= -\frac{1}{\sigma}\left[\phi(h) - \phi(d)\right], \\ \frac{\partial P_{k\delta}}{\partial\sigma} &= -\frac{1}{\sigma}\left[h\phi(h) - d\phi(d)\right], \end{aligned} \tag{8}$$

with $h = [(k+0.5)\delta - \mu]/\sigma$, $d = [(k-0.5)\delta - \mu]/\sigma$. $\Phi(\cdot)$ denotes the cumulative distribution function and $\phi(\cdot)$ denotes the probability density function of $N(0,1)$ distribution.

Based on (5), the two-sided interval estimator for μ, suggested as an approximate $(1-\alpha)\times 100\%$-confidence interval (which is asymptotically exact), is given by

$$CI_\mu^{ML}(1-\alpha) := \hat{\mu}_{ML} \mp \sqrt{\frac{1}{n}\frac{\hat{J}_{2,2}}{\hat{J}_{1,1}\hat{J}_{2,2} - \hat{J}_{1,2}^2}}\, u(1-\alpha/2), \tag{9}$$

where $\hat{J}_{i,j}$, $i,j = 1,2$, are the elements of the estimated matrix $\hat{J} = J(\hat{\mu}_{ML},\hat{\sigma}_{ML})$ and $u(1-\alpha/2)$ is the $(1-\alpha/2)$-quantile of $N(0,1)$ distribution.

Notice, however, that the ML estimator does not exist if $X_{(n)} - X_{(1)} \leq \delta$. By $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ we denote the order statistics[1] . In order to avoid the undefined estimates in the subsequent simulation study, in such situations we

[1]Given any random variables $X_1, X_2, \ldots, X_n$, the order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are also random variables, defined by sorting $X_1, X_2, \ldots, X_n$ in increasing order, i.e. such that $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$.

forced the estimates to the pre-specified values: $\hat{\mu}_{ML} = \bar{x}$ and $\hat{\sigma}_{ML} = 0$, if $x_{(n)} - x_{(1)} = 0$, and $\hat{\sigma}_{ML} = \underset{\sigma\in(0,\infty)}{\operatorname{argmax}}\, l(\bar{x}, \sigma \mid x_1, \dots, x_n)$, if $x_{(n)} - x_{(1)} = \delta$.

As mentioned before, it has been observed that the effect of quantization diminishes and the standard statistical inference is valid if the inherent measurement standard error σ is relatively large (with respect to δ). This could be taken as a rationale for using the standard confidence interval of the form (1) even in the case of quantized observations. Hence, the natural approximate two-sided interval estimator for μ, suggested as an approximate $(1-\alpha) \times 100\%$-confidence interval, is given by

$$CI_{\mu}^{St}(1-\alpha) := \bar{X} \mp \frac{S_X}{\sqrt{n}}\, t_{n-1}\,(1-\alpha/2)\,, \tag{10}$$

where $S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Notice, however, that the confidence interval (10) degenerates to a singleton if all X_i, $i = 1, \dots, n$, result at the same value, i.e. $X_{(1)} = X_{(n)}$.

If σ is small with respect to δ the effect of quantization could be critical on the properties of the approximate confidence interval (10). For illustration, consider $n = 100$ independent measurements taken by an instrument with known resolution $\delta = 1$, and assume that the true parameters are $\mu = 0.47$ and $\sigma = 0.01$. Then the probability of observing the value 0 is $P_0 = 0.99865$ and the probability of observing the value 1 is $P_1 = 0.00135$. It is likely that all independent observations will result at the unique value 0 (with probability 0.8736). If this is true, $\bar{X} = 0$ and $S_X^2 = 0$, and the degenerated approximate confidence interval (10) does not cover the true value $\mu = 0.47$.

This problem was addressed by Willink (2007), who suggested that the variable

$$\hat{\sigma}_{\bar{X}}^2 = \begin{cases} \dfrac{c_n\delta^2}{12}, & \text{if } X_{(1)} = X_{(n)}, \\ \max\left\{\dfrac{S_X^2}{n}, \dfrac{[(X_{(1)} + X_{(n)})/2 - \bar{X}]^2}{3}\right\}, & \text{if } X_{(n)} - X_{(1)} = \delta, \\ \dfrac{S_X^2}{n}, & \text{otherwise,} \end{cases} \tag{11}$$

could take the role of S_X^2/n in the estimation of μ. The quantity c_n is a specific correction constant depending on sample size n ($c_2 = 6.4$, $c_3 = 1.3$ and $c_n = 1$ for $n \geq 4$). Based on (11), the approximate two-sided interval estimator for μ, suggested as an approximate $(1-\alpha) \times 100\%$-confidence interval, is given by

$$CI_{\mu}^{W}(1-\alpha) := \bar{X} \mp \hat{\sigma}_{\bar{X}}\, t_{n-1}\,(1-\alpha/2)\,. \tag{12}$$

3. Inference based on the fiducial approach

The fiducial argument, as stated by R. A. Fisher, see e.g. [7], was an attempt to make rigorous probability statements about the unknown parameters of the population from which the observational data are a random sample, without the assumption of any knowledge respecting their probability *a priori* — which Fisher saw as a shortcoming of the Bayesian inference. Although the idea is appealing, and closely related to another widely accepted approach to statistical inference introduced by Fisher, namely, that based on the likelihood function, Fisher did not succeed in building a concise theory of the fiducial inference. Moreover, in many particular situations, the fiducial approach led to unexplainable inconsistencies, especially, in comparison with classical frequentist solutions. On the other hand, in many other situations, the fiducial distribution of the parameters could be seen as a Bayesian posterior distribution based on improper prior. The Fisher's fiducial argument started a long and rather controversial discussion in statistical literature, for such examples see e.g. [21], [8], [2], [4], [9], [20], [26], [5], and [11]. However, according to Bradley Efron's words, *if fiducial inference is an error it certainly has been a fertile one.*

Recently, J. Hannig re-formulated the principles of the fiducial inference and generalized it to the vector parameter of continuous as well as discrete distributions, see [12]. Moreover, Hannig identified the sources of inherent non-uniqueness of fiducial inference (similar to the non-uniqueness of Bayesian inference due to the choice of a prior) and showed that the generalized fiducial distribution has good asymptotic properties. As Hannig argues, the fiducial inference appears to be a good tool for deriving statistical procedures and should not be ignored by the statistical community.

The fiducial argument is based on idea of switching the role of the parameters and the data. Here we present basic building blocks for derivation of the fiducial distribution for the vector parameter, as suggested by Hannig (2009). Let $X = G(U, \xi)$ be a known structural equation (a data generating mechanism) which relates the random vector of (possibly discrete) observations X with the vector parameter $\xi \in \Xi$ and a random vector U (a noise process) with completely known distribution (functionally independent of ξ). For any fixed (observed) value x of X and any fixed (however unobservable) value u of U consider a set-valued function $Q(x, u) = \{\xi : x = G(u, \xi)\}$. Notice that, $Q(X, U)$ represents 'the inverse' of G. For any fixed x and u, $Q(x, u)$ defines either empty set, unique value of the parameter ξ, or a more complex set of possible values of parameters ξ, depending on a particular relationship between x and u, and on G.

Further, let $V(\cdot)$ define any predetermined random mechanism on measurable sets, such that for any set S, $V(S)$ is a random element from the closure of S. Then, the particular choice (depending on G, Q, and V) of the generalized fiducial distribution of ξ, say $F_\xi(\xi^\star)$ for fixed $\xi^\star$, is defined as any version of the distribution of $V(Q(x, U^\star))$ on the condition that $Q(x, U^\star) \neq \emptyset$, i.e.

$$F_\xi(\xi^\star) = \Pr\left(V(Q(x, U^\star)) \leq \xi^\star \mid Q(x, U^\star) \neq \emptyset\right), \tag{13}$$

where x is the observed value of X and $U^\star$ is a random variable independent of U with the same distribution. For simplicity, by $V(\cdot) \leq \xi^\star$ we represent $(V_1(\cdot) \leq \xi_1^\star, \dots, V_r(\cdot) \leq \xi_r^\star)$, where r is the dimension of the parameter vector $\xi \in \Xi$. The derivation of the explicit form of the fiducial distribution could be too difficult or intractable. However, notice that the suggested approach gives direct recipe for sampling from that distribution.

To illustrate this, consider mutually independent measurements, $Y_1, Y_2, \dots, Y_n$, where $Y_i = \mu + \sigma Z_i$ with $Z_i \sim N(0,1)$, registered by an instrument with perfect resolution ($\delta = 0$). Consider now a structural equation based on the sufficient statistics $\bar{Y}$ and S_Y^2. Note that $\bar{Y} = \mu + (\sigma/\sqrt{n})Z$, with $Z \sim N(0,1)$, and $S_Y^2 = \sigma^2 W/(n-1)$, with $W \sim \chi^2_{(n-1)}$, are independent random variables. From that, the structural equation has the standard form $X = G(U, \xi)$, where $\xi = (\mu, \sigma)$, $U = (Z, W)$, $X = \big(\bar{Y}, S_Y^2\big)$, and $G(U, \xi) = \big(\mu + (\sigma/\sqrt{n})Z, \sigma^2 W/(n-1)\big)$. The 'inverse' function $Q(x, u)$ is defined by

$$Q(x, u) = Q\left((\bar{y}, s_Y^2), (z, w)\right) = \left\{\left(\bar{y} - \sqrt{\frac{(n-1)s_Y^2}{nw}}\, z, \sqrt{\frac{(n-1)s_Y^2}{w}}\right)\right\}, \tag{14}$$

where $\bar{y}$ is observed value of $\bar{Y}$ and s_Y^2 is observed value of S_Y^2. As $Q(x, u)$ always defines a non-empty and unique solution from the parameter space, the fiducial distribution does not depend on any random mechanism $V(\cdot)$ and is unconditionally given by

$$F_{(\mu,\sigma)}((\mu^\star, \sigma^\star)) = \Pr\left(\bar{y} - \sqrt{\frac{(n-1)s_Y^2}{nW^\star}}\, Z^\star \leq \mu^\star, \sqrt{\frac{(n-1)s_Y^2}{W^\star}} \leq \sigma^\star\right). \tag{15}$$

This is equivalent with the fiducial distribution derived by F i s h e r (1935). Equation (15) also describes the distribution obtained in a Bayesian analysis with the usual improper priors. Moreover, the numerical interval derived for μ (or σ^2) with a certain level of probability coincides with that obtained in a frequentist analysis for the same level of confidence. For more details, and different solutions based on other possible choices of the structural equation, see [12].

The joint fiducial distribution for the parameters μ and σ based on a random sample of quantized observations, i.e. in the case with $\delta > 0$, have been derived by H a n n i g *et al.* (2007). We note that this distribution is different from the Bayesian posterior distribution, as suggested e.g. in [6]. Here we follow the basic steps of the suggested algorithm for sampling from the fiducial distribution:

Consider a random sample $X = (X_1, X_2, \dots, X_n)$ of quantized observations and let $x = (x_1, x_2, \dots, x_n)$ be vector of the observed values. The structural equation, say $X = G(Z, \xi)$, with $\xi = (\mu, \sigma)$ and $Z = (Z_1, \dots, Z_n)$, where $Z_i \sim N(0, 1)$ are mutually independent, is given by (2), and from that the following inequalities hold true

$$X_i - \frac{\delta}{2} \leq \mu + \sigma Z_i < X_i + \frac{\delta}{2}, \tag{16}$$

$i = 1, \dots, n$. Define the 'inverse' function $Q(x, z)$ as

$$Q(x, z) = \{(\mu, \sigma) : \; a_i \leq \mu + \sigma z_i < b_i, \; i = 1, \dots, n\} \tag{17}$$

where $a_i = x_i - \delta/2$ and $b_i = x_i + \delta/2$. For any fixed x and z, the outcome of the function $Q(x, z)$ results either in an empty set, or in a set which forms a polygon in a 2-dimensional parameter space of (μ, σ). The set $Q(x, z)$ is polygon if $z \in S(x)$, where

$$S(x) = \{(z_1, \dots, z_n) : \; \exists(\mu, \sigma) \text{ with } \; a_i \leq \mu + \sigma z_i < b_i, \; i = 1, 2, \dots, n\}. \tag{18}$$

If $Q(x, z)$ is a non-empty set (i.e. it is a polygon with infinite number of possible values (μ, σ)), it is necessary to choose a predetermined random mechanism $V(\cdot)$ for selection of one particular value (μ, σ). Although any $V(\cdot)$ with its support over the polygon $Q(x, z)$ is valid, here we suggest to use $V(\cdot)$ that is uniformly distributed over $Q(x, z)$. Originally, Hannig *et al.* (2007) suggested to use $V(\cdot)$ that is uniformly distributed over the vertices of the polygon given by $Q(x, z)$. However, we see some advantages in using $V(\cdot)$ that is uniformly distributed over $Q(x, z)$, as it gives a more informative picture about the values of the parameters (μ, σ) that could be attributed to the observed value x. For illustration, see the Figures 1–3.

From that, the joint fiducial distribution of the vector of parameters (μ, σ) (which depends on a particular choice of G, Q, and V) is given by

$$F_{(\mu,\sigma)}((\mu^\star, \sigma^\star)) = \Pr\left(V(Q(x, Z^\star)) \leq (\mu^\star, \sigma^\star) \mid Z^\star \in S(x)\right). \tag{19}$$

The explicit form of the fiducial distribution (19) is unknown. However, sampling from this distribution is directly applicable. In order to avoid the empty solutions during the sampling procedure, i.e. $Q(x, Z^\star) = \emptyset$, and/or the checking of the necessary condition, namely $Z^\star \in S(x)$, the MCMC (Markov Chain Monte Carlo) techniques are preferably suggested to be used. The detailed description of the sampling algorithm was given by Hannig *et al.* (2007). Here we summarize the important points of the algorithm:

(1) If the observed vector of measurements, say $x = (x_1, x_2, \dots, x_n)$, is consistent with the sampling mechanism, there will exist at least one parameter, say $\xi^{(0)} = \left(\mu^{(0)}, \sigma^{(0)}\right)$, such that $x = G\left(z, \xi^{(0)}\right)$ for some z, say $z^{(0)}$, i.e. $z^{(0)} \in S(x)$.

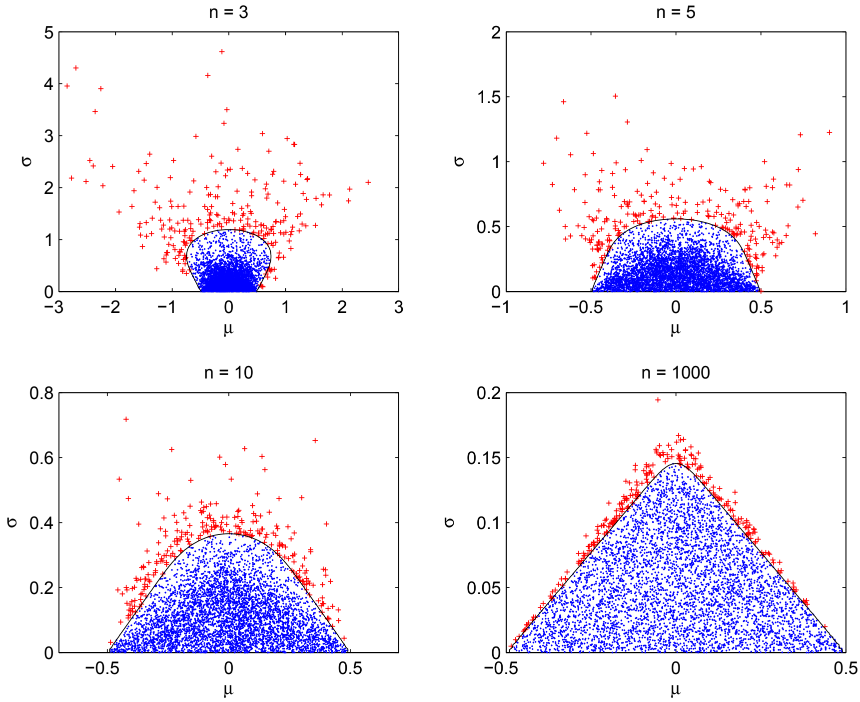

FIGURE 1. Random sample of size $M = 5000$ from the joint fiducial distribution of the parameters μ and σ based on n independent measurements with an instrument with resolution $\delta = 1$ ($n = 3$, 5, 10, and 1000). Here, all n observations resulted into single value, 0. The sampled values of the parameters (μ, σ) lying outside of the empirical 95% likelihood region discussed in Section 5 are marked by the sign $+$.

(2) For any $z \in S(x)$, say $z^{(k-1)}$, $Q\left(x, z^{(k-1)}\right)$ is a polygon containing an infinite number of possible values (μ, σ). Using the predetermined sampling mechanism $V(\cdot)$, generate a single value of the parameter vector from the polygon $Q\left(x, z^{(k-1)}\right)$, say $(\mu^{(k)}, \sigma^{(k)})$.

Here we suggest to use $V(\cdot)$ which samples a single point $(\mu^{(k)}, \sigma^{(k)})$ from the polygon $Q\left(x, z^{(k-1)}\right)$, in the parameter space (μ, σ), such that each point has equal chance of being selected[2] . In particular, the algorithm

[2]The reviewer correctly pointed out to the following problem: *'If the regions $Q(x, z)$ were parametrized in terms of μ and σ^2 instead then the regions would not be polygons. Furthermore, random sampling of points from the (μ, σ^2) region will not give the same fiducial distribution as random sampling from the (μ, σ) polygon. However, this issue is not a problem here, because the experimenter is free to define $V(\cdot)$ by some convenient rule, but it is worth bearing in mind.'*

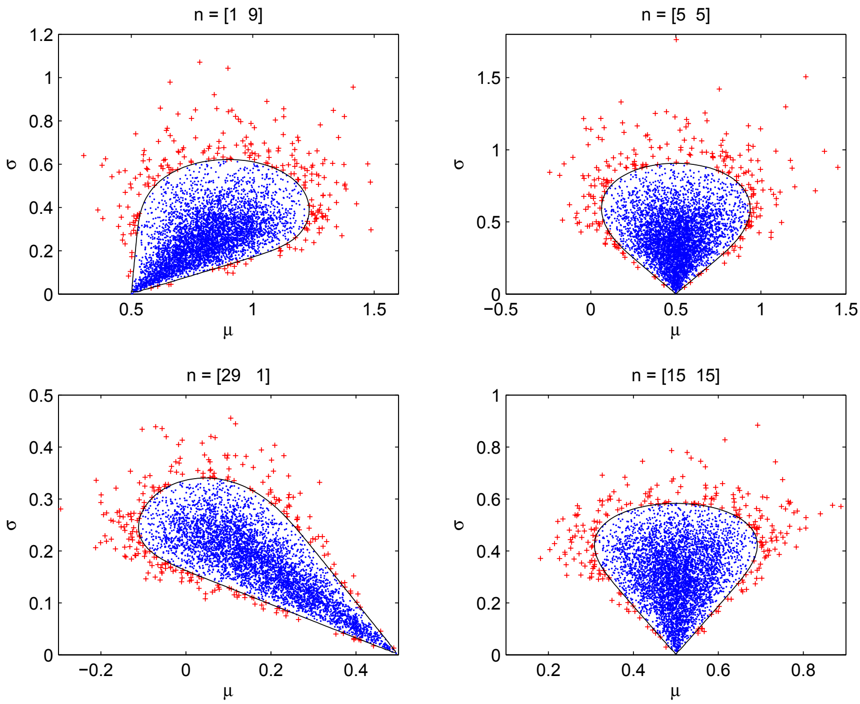

FIGURE 2. Random sample of size $M = 5000$ from the joint fiducial distribution of the parameters μ and σ based on n independent measurements with an instrument with resolution $\delta = 1$ ($n = 10$ and 30). Here, all observations resulted into two single values, 0 and 1, (with $n = n_1 + n_2$, where $[n_1, n_2] = [1, 9]$, $[5, 5]$, $[29, 1]$, and $[15, 15]$). The sampled values of the parameters (μ, σ) lying outside of the empirical 95% likelihood region discussed in Section 5 are marked by the sign $+$.

generates randomly a candidate point from the smallest square containing the polygon and then the acceptance-rejection method is applied.

(3) Given $(\mu^{(k)}, \sigma^{(k)})$, randomly select a new candidate vector of errors z, $z = (z_1, \ldots, z_n)$, i.e. such that $z \in S(x)$. Then, for given $z \in S(x)$, update z to $z^{(k)} = \left(z_1^{(k)}, \ldots, z_n^{(k)}\right)$ by a random realization of

$$z_i^{(k)} = \frac{1}{n} U^\star + \frac{\sqrt{W^\star}}{\sum_{i=1}^{n} (z_i - \bar{z})^2} (z_i - \bar{z}), \qquad i = 1, \ldots, n, \tag{20}$$

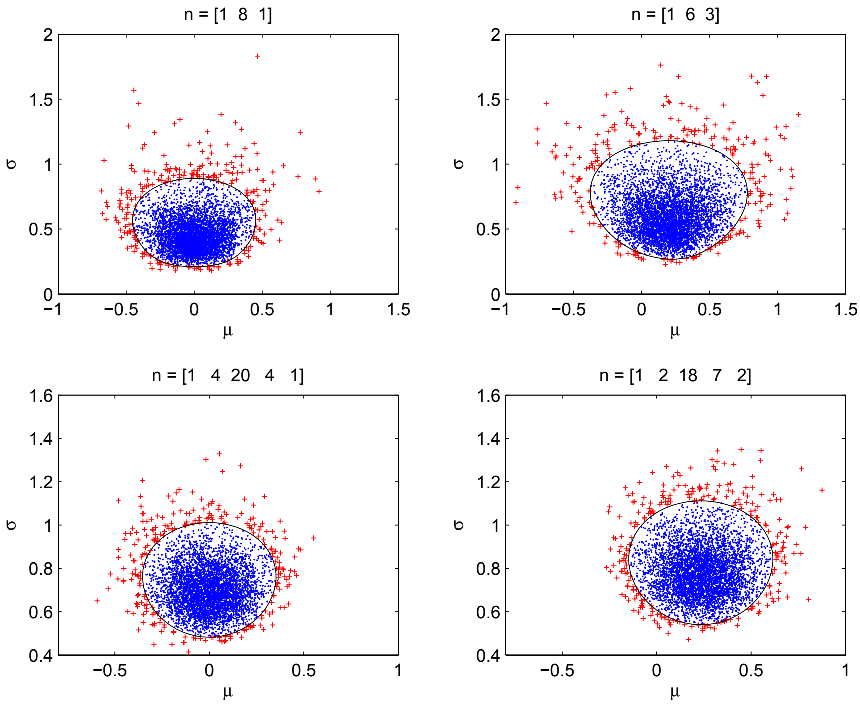

Figure 3. Random sample of size $M = 5000$ from the joint fiducial distribution of the parameters μ and σ based on n independent measurements with an instrument with resolution $\delta = 1$ ($n = 10$ and 30). Here, all observations resulted into small number of distinct values, namely $[-1, 0, 1]$ with multiplicities $n = [1, 8, 1]$, and $n = [1, 6, 4]$, and $[-2, -1, 0, 1, 2]$ with multiplicities $n = [1, 4, 20, 4, 1]$, and $n = [1, 2, 18, 7, 2]$. The sampled values of the parameters (μ, σ) lying outside of the empirical 95% likelihood region discussed in Section 5 are marked by the sign $+$.

where $\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$. $U^\star \sim N(0,1)$ and $W^\star \sim \chi^2_{(n-1)}$ are independent random variables, common for all $i = 1, \dots, n$. Notice that this mechanism ensures that $z^{(k)} \in S(x)$.

Continue with step 2 until the desired number of fiducial samples is reached.

The MATLAB algorithm for sampling from the fiducial distribution based on quantized observations is available upon request from the authors. A version of the algorithm is also available at `http://www.mathworks.com/matlabcentral/fileexchange/` (look for the file `DigitFD`).

Now, we will discuss possibilities how to construct the appropriate interval estimates which could serve as an approximate $(1-\alpha)\times 100\%$-confidence intervals for the parameter μ, based on a sample from fiducial distribution (19).

The standard approach for constructing the interval estimate for the parameter μ is based on the marginal fiducial distribution of the parameter μ. However, as we now explain, the simple two-sided $(1-\alpha)\times 100\%$ interval estimator, based on the fiducial distribution (19), derived with sampling mechanism $V(\cdot)$ that is uniformly distributed over the polygon $Q(x,z)$, could be inadequate, especially in cases when the number of distinct observed results is equal either to 1 or 2, see Figure 1 and Figure 2.

In particular, consider the situation with one observed value, i.e. $x_{(n)} = x_{(1)}$, and with large number of observations n, say $n = 1000$. In such cases, for any significance level α, there exist a parameter μ, close to the border $a = x_{(1)} - \delta/2$ or $b = x_{(n)} + \delta/2$, and sufficiently small parameter σ, that a random sample of size n will result in all observations equal to one unique value, and the two-sided $(1-\alpha)\times 100\%$ interval estimate based on the marginal fiducial distribution for μ will not cover the true value μ.

We suggest using an interval estimate for μ based on the marginal fiducial distribution of μ, but with an adjustment for the case with $x_{(1)} - x_{(n)} \leq \delta$: Let $\mu_{(\alpha/2)}$ and $\mu_{(1-\alpha/2)}$ denote the quantiles of the marginal fiducial distribution for μ. Then, the adjusted interval estimator, suggested as an approximate $(1-\alpha)\times 100\%$ confidence interval for μ, is given by

$$CI_{\mu}^{FD}(1-\alpha) := \begin{cases} \left[\mu^{\star}_{(\alpha/2)}, \mu^{\star}_{(1-\alpha/2)}\right], & \text{if } x_{(n)} - x_{(1)} \leq \delta, \\ \left[\mu_{(\alpha/2)}, \mu_{(1-\alpha/2)}\right], & \text{otherwise,} \end{cases} \tag{21}$$

where $\left[\mu^{\star}_{(\alpha/2)}, \mu^{\star}_{(1-\alpha/2)}\right] = \left[\min\{\mu_{(\alpha/2)}, x_{(n)} - \delta/2\}, \max\left\{\mu_{(1-\alpha/2)}, x_{(1)} + \delta/2\right\}\right]$.

4. Simulations

We have performed a simulation study, based on $N = 10000$ independent Monte Carlo trials per each considered experimental design, in order to compare the basic statistical properties (the coverage probabilities and the expected lengths) of the considered interval estimators CI_{μ}^{ML}, CI_{μ}^{St}, CI_{μ}^{W}, and CI_{μ}^{FD} for the parameter μ, defined by (9), (10), (12), and (21), respectively.

In this study, we have considered experimental designs with small sample sizes n, $n \in \{5, 10, 30\}$, and relatively small standard errors σ of the inherent error (with respect to the known resolution $\delta > 0$), $\sigma \in \{0.1\delta, 0.3\delta, 0.5\delta, 1.0\delta\}$. Further, we have considered different values of the true parameter μ within the unit of the measurement scale, $\mu \in \{0.0\delta, 0.1\delta, 0.2\delta, 0.3\delta, 0.4\delta, 0.5\delta\}$. Without loss of generality we have considered designs with resolution $\delta = 1$, only.

TABLE 1. Approximate coverage probabilities and approximate average lengths of the interval estimators for μ corresponding to sample size $n = 5$.

σ	μ	Fiducial		Willink		Student		ML	
0.1	0.0	1.0000	(1.00)	1.0000	(1.60)	1.0000	(0.00)	1.0000	(0.00)
	0.1	1.0000	(1.00)	1.0000	(1.60)	0.0001	(0.00)	0.0000	(0.00)
	0.2	1.0000	(1.00)	1.0000	(1.60)	0.0064	(0.01)	0.0000	(0.00)
	0.3	1.0000	(1.01)	1.0000	(1.55)	0.1103	(0.12)	0.0000	(0.00)
	0.4	1.0000	(1.10)	1.0000	(1.36)	0.5834	(0.69)	0.5533	(0.10)
	0.5	1.0000	(1.22)	1.0000	(1.30)	0.9384	(1.20)	0.6307	(0.15)
0.3	0.0	1.0000	(1.03)	1.0000	(1.62)	1.0000	(0.24)	0.8039	(0.01)
	0.1	1.0000	(1.04)	1.0000	(1.51)	0.2777	(0.32)	0.0090	(0.01)
	0.2	0.9986	(1.07)	0.9986	(1.41)	0.4592	(0.54)	0.0050	(0.01)
	0.3	0.9997	(1.13)	1.0000	(1.33)	0.6910	(0.84)	0.0018	(0.00)
	0.4	0.9955	(1.19)	1.0000	(1.30)	0.8750	(1.10)	0.6515	(0.13)
	0.5	0.9999	(1.22)	1.0000	(1.30)	0.9332	(1.19)	0.6279	(0.15)
0.5	0.0	0.9954	(1.39)	0.9970	(1.85)	0.9954	(1.25)	0.4656	(0.38)
	0.1	0.9857	(1.37)	0.9922	(1.69)	0.8446	(1.25)	0.2944	(0.36)
	0.2	0.9773	(1.37)	0.9805	(1.57)	0.8688	(1.28)	0.2697	(0.34)
	0.3	0.9880	(1.38)	0.9998	(1.52)	0.9088	(1.33)	0.2350	(0.30)
	0.4	0.9718	(1.36)	0.9982	(1.47)	0.9235	(1.34)	0.6424	(0.35)
	0.5	0.9860	(1.37)	0.9919	(1.44)	0.9376	(1.35)	0.6871	(0.37)
1.0	0.0	0.9623	(2.37)	0.9697	(2.55)	0.9623	(2.37)	0.7475	(1.28)
	0.1	0.9438	(2.34)	0.9628	(2.50)	0.9356	(2.34)	0.7239	(1.26)
	0.2	0.9583	(2.36)	0.9771	(2.50)	0.9528	(2.37)	0.7419	(1.29)
	0.3	0.9561	(2.38)	0.9838	(2.50)	0.9509	(2.39)	0.7476	(1.30)
	0.4	0.9451	(2.34)	0.9823	(2.46)	0.9385	(2.35)	0.7965	(1.28)
	0.5	0.9519	(2.37)	0.9764	(2.45)	0.9447	(2.38)	0.8130	(1.29)

For each experimental design, with particular values of n, μ and σ, we have generated $N = 10000$ realizations of the quantized observations. For each generated vector of measurements $(x_1, x_2, \ldots, x_n)$ we have calculated the interval estimates for the parameter μ at the nominal (significance) level $\alpha = 0.05$ (i.e. intervals suggested as an approximate $(1-\alpha) \times 100\%$ confidence intervals): $CI_\mu^{ML}(1-\alpha)$, $CI_\mu^{St}(1-\alpha)$, $CI_\mu^{W}(1-\alpha)$, and $CI_\mu^{FD}(1-\alpha)$, which was calculated from an independent sample of size $M = 10000$ from the fiducial distribution given by (19).

Table 2. Approximate coverage probabilities and approximate average lengths of the interval estimators for μ corresponding to sample size $n = 10$.

σ	μ	Fiducial		Willink		Student		ML	
0.1	0.0	1.0000	(1.00)	1.0000	(1.31)	1.0000	(0.00)	1.0000	(0.00)
	0.1	1.0000	(1.00)	1.0000	(1.31)	0.0002	(0.00)	0.0000	(0.00)
	0.2	1.0000	(0.99)	1.0000	(1.30)	0.0144	(0.01)	0.0000	(0.00)
	0.3	1.0000	(0.93)	1.0000	(1.25)	0.2058	(0.10)	0.0000	(0.00)
	0.4	0.9999	(0.70)	0.9999	(0.95)	0.4865	(0.31)	0.8099	(0.09)
	0.5	1.0000	(0.65)	0.9920	(0.73)	0.9808	(0.71)	0.8937	(0.15)
0.3	0.0	1.0000	(0.87)	1.0000	(1.28)	1.0000	(0.19)	0.6564	(0.02)
	0.1	0.9976	(0.83)	1.0000	(1.17)	0.4754	(0.24)	0.0304	(0.02)
	0.2	0.9907	(0.73)	0.9998	(1.03)	0.7069	(0.38)	0.0166	(0.01)
	0.3	0.9982	(0.67)	0.9989	(0.87)	0.9052	(0.55)	0.0062	(0.00)
	0.4	0.9948	(0.65)	0.9949	(0.77)	0.9008	(0.63)	0.7496	(0.10)
	0.5	1.0000	(0.65)	0.9897	(0.72)	0.9776	(0.70)	0.8880	(0.15)
0.5	0.0	0.9625	(0.80)	0.9817	(1.04)	0.9625	(0.76)	0.6902	(0.50)
	0.1	0.9353	(0.78)	0.9800	(0.94)	0.9478	(0.76)	0.6275	(0.47)
	0.2	0.9547	(0.78)	0.9816	(0.88)	0.9398	(0.77)	0.5543	(0.43)
	0.3	0.9713	(0.77)	0.9798	(0.84)	0.9648	(0.79)	0.4464	(0.35)
	0.4	0.9670	(0.76)	0.9698	(0.82)	0.9472	(0.79)	0.6977	(0.35)
	0.5	0.9786	(0.77)	0.9852	(0.82)	0.9787	(0.81)	0.9057	(0.38)
1.0	0.0	0.9497	(1.39)	0.9576	(1.41)	0.9524	(1.40)	0.8921	(1.09)
	0.1	0.9461	(1.39)	0.9578	(1.41)	0.9529	(1.40)	0.8895	(1.09)
	0.2	0.9454	(1.38)	0.9538	(1.40)	0.9497	(1.39)	0.8886	(1.08)
	0.3	0.9524	(1.39)	0.9536	(1.40)	0.9517	(1.40)	0.8913	(1.09)
	0.4	0.9539	(1.39)	0.9565	(1.40)	0.9549	(1.40)	0.9025	(1.10)
	0.5	0.9527	(1.40)	0.9556	(1.41)	0.9447	(1.40)	0.9014	(1.09)

Fortunately, because of the discrete nature of the random vector $(X_1, X_2, \ldots, X_n)$, the observations $(x_1, x_2, \ldots, x_n)$ resulted in relatively small number of distinct values, especially if σ was small. This helped to reduce remarkably the computational demands, required by the used algorithm for sampling from the fiducial distribution.

The correctness of the considered interval estimators should be evaluated with respect to the criterion $\Pr(\mu \in CI_\mu(1-\alpha)) \geq 1-\alpha$, where μ stands for the true value of the parameter. The results of the simulation study (the empirical coverage probabilities and the average lengths of the interval estimators) are

TABLE 3. Approximate coverage probabilities and approximate average lengths of the interval estimators for μ corresponding to sample size $n = 30$.

σ	μ	Fiducial		Willink		Student		ML	
0.1	0.0	1.0000	(1.00)	1.0000	(1.18)	1.0000	(0.00)	0.9998	(0.00)
	0.1	1.0000	(1.00)	1.0000	(1.18)	0.0015	(0.00)	0.0000	(0.00)
	0.2	1.0000	(0.98)	1.0000	(1.18)	0.0000	(0.00)	0.0000	(0.00)
	0.3	1.0000	(0.78)	1.0000	(1.13)	0.0001	(0.00)	0.0000	(0.00)
	0.4	1.0000	(0.44)	1.0000	(0.81)	0.0913	(0.03)	0.7097	(0.04)
	0.5	1.0000	(0.31)	0.9757	(0.37)	0.9548	(0.36)	0.9026	(0.09)
0.3	0.0	0.9708	(0.60)	1.0000	(0.95)	0.9950	(0.13)	0.4898	(0.08)
	0.1	0.9540	(0.54)	0.9306	(0.91)	0.7452	(0.14)	0.1711	(0.06)
	0.2	0.9881	(0.47)	0.9773	(0.86)	0.6912	(0.18)	0.0628	(0.02)
	0.3	0.9943	(0.39)	0.9912	(0.68)	0.7784	(0.25)	0.0138	(0.00)
	0.4	0.9885	(0.32)	0.9963	(0.45)	0.8559	(0.31)	0.3870	(0.03)
	0.5	1.0000	(0.31)	0.9802	(0.37)	0.9591	(0.36)	0.9058	(0.09)
0.5	0.0	0.9512	(0.41)	0.9646	(0.42)	0.9561	(0.40)	0.9264	(0.38)
	0.1	0.9431	(0.41)	0.9562	(0.41)	0.9534	(0.40)	0.9300	(0.38)
	0.2	0.9402	(0.41)	0.9516	(0.42)	0.9513	(0.41)	0.8864	(0.37)
	0.3	0.9421	(0.40)	0.9522	(0.42)	0.9512	(0.41)	0.8074	(0.34)
	0.4	0.9454	(0.39)	0.9525	(0.42)	0.9479	(0.41)	0.7525	(0.31)
	0.5	0.9580	(0.39)	0.9528	(0.42)	0.9462	(0.41)	0.9290	(0.32)
1.0	0.0	0.9508	(0.73)	0.9513	(0.74)	0.9513	(0.74)	0.9359	(0.68)
	0.1	0.9461	(0.73)	0.9473	(0.73)	0.9473	(0.73)	0.9324	(0.68)
	0.2	0.9470	(0.73)	0.9477	(0.73)	0.9477	(0.73)	0.9323	(0.68)
	0.3	0.9470	(0.73)	0.9451	(0.73)	0.9451	(0.73)	0.9338	(0.68)
	0.4	0.9478	(0.73)	0.9472	(0.73)	0.9472	(0.73)	0.9325	(0.68)
	0.5	0.9522	(0.73)	0.9507	(0.73)	0.9507	(0.73)	0.9376	(0.68)

presented in Tables 1–3. Note that based on $N = 10000$ independent Monte Carlo trials the standard error of an estimate of a true proportion close to 0.95 is ≈ 0.0022. Due to limited space, we did not quote the standard errors of the mean interval widths. However, the values are available upon request from the authors.

As could be seen from the presented results, the approximate confidence interval CI_μ^{ML}, although known to be asymptotically correct, is inadequate for the considered sample sizes. It has been observed in subsequent studies that if σ is small, the sufficient sample size n can be prohibitively large, if the goal is to reach (at least approximately) the asymptotic behavior of this interval estimator. The interval estimator CI_μ^{St} is correct, even for small sample sizes n, if σ is sufficiently large (i.e. if it is close to the resolution δ or greater). The interval estimator CI_μ^{W} was correct for almost all considered situations, although with slightly greater average lengths, if compared with the interval estimator based on the fiducial distribution, CI_μ^{FD}, which was similarly correct for almost all considered situations.

In [14], the authors have presented only simulation results corresponding to $n = 10$, some selected values of (σ, δ) (typically with large ratio of σ/δ), and only for one value of the true parameter, $\mu = 10$. Our simulation study is in good agreement with those results, however, the experimental designs considered in this paper are largely extended. Moreover, we have used a different version of the fiducial distribution and a more efficient numerical algorithm for sampling from the joint fiducial distribution of the parameters (μ, σ).

5. Discussion and conclusions

We have considered the problem of making statistical inference about the mean of a normal distribution based on random sample of quantized observations. By simulation we have compared the basic properties of four interval estimators (suggested as approximate confidence intervals) for the parameter μ based on quantized observations with small to moderate sample sizes and relatively small standard errors σ of the inherent errors.

As shown, the ML based interval estimator CI_μ^{ML} is inadequate for considered situations. The best performance have been recorded for the interval estimator CI_μ^{FD} based on the fiducial distribution, followed by the interval estimator CI_μ^{W}, proposed by Willink (2007). The interval estimator CI_μ^{St}, which is known to be exact for the continuous observations, is correct also for small sample sizes n, if σ is close to the resolution δ or greater. To conclude, due to its simplicity, the approximate confidence interval CI_μ^{W} could be suggested for practical purposes.

This study extends the results of Hannig *et al.* (2007). Here we have used a different version of the algorithm for sampling from the joint fiducial distribution of the parameters (μ, σ). We suggest to use a random mechanism $V(\cdot)$ that samples uniformly over the polygon of possible parameter values $Q(x, z)$ for each fixed vector x and z. The current version of the numerical algorithm for sampling from the joint fiducial distribution of (μ, σ) is available upon request from the authors.

Moreover, the joint fiducial distribution (19) gives more detailed information about the unknown parameters μ and σ, which could be used for more complex statistical inferences. We would like to stress the possibility to construct the simultaneous regions for the parameters (μ, σ), here denoted as the joint $(1 - \alpha) \times 100\%$ likelihood region, which could serve as an approximate simultaneous $(1 - \alpha) \times 100\%$-confidence region for the parameters (μ, σ).

In particular, given the observed quantized observations, say $(x_1, \ldots, x_n)$, we suggest to construct a joint empirical $(1 - \alpha) \times 100\%$ likelihood region for the parameters (μ, σ) based on ordering the sampled values from the the fiducial distribution (19) with respect to their log-likelihood value. For illustration, see the examples given in Figures 1-3. The empirical 95% likelihood region have been determined as a region of sampled parameter values (μ, σ) for which their log-likelihood value $l(\mu, \sigma \mid x_1, \ldots, x_n)$, given by (4), was greater then the empirical estimate of the log-likelihood α-quantile, say $l_{(\alpha)}(\mu, \sigma \mid x_1, \ldots, x_n)$, with $\alpha = 0.05$. Analysis of the basic statistical properties of the empirical likelihood regions is subject of further research.

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