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ON SOME CLASSES OF STATE-MORPHISM MV-ALGEBRAS

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Dedicated to Professor Andrej Pázman on the occasion of his 70th birthday

(*Communicated by Gejza Wimmer*)

ABSTRACT. Flaminio and Montagna recently introduced state MV-algebras as MV-algebras with an internal notion of a state. The present authors gave a stronger version of state MV-algebras, called state-morphism MV-algebras. We present some classes of state-morphism MV-algebras like local, simple, semisimple state-morphism MV-algebras, and state-morphism MV-algebras with retractive ideals. Finally, we describe state-morphism operators on m-free generated MV-algebras, $m < \infty$.

1. Introduction

States as averaging of the true-value were first introduced for MV-algebras by Mundici [Mun1]. This notion is not a genuine notion for algebraic structures, it is an external notion, however, in the realm of quantum structures, the mathematical background of quantum mechanics, it is a primary notion because it describes the measurement process in quantum physics (for theory of quantum structures and states on it we recommend the monograph [DvPu]). Therefore,

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states were known earlier for MV-algebras, as a special class of D-posets (equivalently effect algebras), see [KoCh].

In the last few years, the theory of states is studied by many experts in MV-algebras, see e.g. [RiMu], [Kro]. Kühr and Mundici studied states using an old notion of a coherent state by De Finnetti with motivation in Dutch book making, see [KuMu].

Flaminio and Montagna [FlMo] presented recently another approach to the state theory on MV-algebras: they add a new unary operation, σ, to the MV-algebra structure as an internal state. Such structures are called state MV-algebras (or MV-algebras with internal state). For a more detailed motivation of state MV-algebras and its relation to logic, see [FlMo].

The authors [DiDv] motivated by an important property that a state-morphism on an MV-algebra M is always an MV-homomorphism from M into the basic MV-algebra of the unit interval $[0,1]$, they introduced a stronger notion of state MV-algebras, called a state-morphism MV-algebra. That is, a couple (M,σ), where M is an MV-algebra and σ is an MV-homomorphism from M into itself such that $\sigma^2=\sigma$.

The basic properties of state-morphism MV-algebras, as well a complete description of subdirectly irreducible state-morphism MV-algebras were described in [DiDv].

In the present paper, we continue in the study of state-morphism MV-algebras. We describe some classes of state-morphism MV-algebras, like local, simple, semisimple state-morphism MV-algebras (Section 3), and state-morphism MV-algebras with retractive ideals (Section 4). Finally in Section 5, we describe state-morphism operators on m-free generated MV-algebras, $m<\infty$. The elements of state-morphism MV-algebras are given in the next section.

2. Elements of state-morphism MV-algebras

Let $M=(M;\oplus,\odot,^*,0,1)$ be an MV-algebra. That is, an algebra of type $\langle 2,2,1,0,0\rangle$ such that

(i) $\oplus$ is commutative and associative,

(ii) $0^*=1$,

(iii) $x\oplus 0=x$,

(iv) $x\oplus 1=1$,

(v) $x^{**}=x$,

(vi) $y \oplus (y \oplus x^*)^* = x \oplus (x \oplus y^*)^*$,

(vi) $x \odot y = (x^* \oplus y^*)^*$.

For more details on MV-algebras, see [Cha], [CDM].

We define a partial operation, $+$, in such a way that $x + y$ is defined in M iff $x \odot y = 0$, and in that a case we set $x + y = x \oplus y$. It is clear that $x \odot y = 0$ iff $x \leq y^*$. We recall that if A is a subset of an MV-algebra M, we set $A^* = \{x^* : x \in A\}$.

We recall that if (G, u) denotes an Abelian ℓ-group (= lattice ordered group) G with a fixed strong unit (= order unit), then $M = \Gamma(G, u) := [0, u]$ endowed with $x \oplus y = (x + y) \wedge u$, $x^* = u - x$, $x \odot y = (x + y - u) \vee 0$, and $0 = 0$, $1 = u$, is an MV-algebra, and thanks to the Mundici categorical representation of MV-algebras, [Mun], every MV-algebra is of the form $M = \Gamma(G, u)$ for some unital Abelian ℓ-group G with a fixed strong unit.

A *state* on M is a mapping $s : M \to [0, 1]$ such that

(i) $s(1) = 1$,

(ii) $s(x + y) = s(x) + s(y)$ whenever $x + y$ is defined in M.

A mapping $s : M \to [0, 1]$ is said to be a *state-morphism* if s is an MV-homomorphism from M into the standard MV-algebra of the real line $[0, 1] = \Gamma(\mathbb{R}, 1)$.

The set of all states on M, $\mathscr{S}(M)$, is convex, i.e. if s_1, s_2 are states on M, then $s = \lambda s_1 + (1 - \lambda)s_2$ is a state on M for any $\lambda \in [0, 1]$. A state s is *extremal* if from $s = \lambda s_1 + (1 - \lambda)s_2$ for $\lambda \in (0, 1)$ it follows that $s = s_1 = s_2$. We denote by $\mathscr{S}_\partial(M)$ the set of extremal states on M. Due to the Mundici categorical representation of MV-algebras via intervals in Abelian unital ℓ-groups, $\mathscr{S}(M)$ is non-void whenever $0 \neq 1$.

If we endow $\mathscr{S}(M)$ with the weak topology of states, i.e. a net $\{s_\alpha\}$ of states on M *converges weakly* to a state s if $s(a) = \lim\limits_{\alpha} s_\alpha(a)$ for any $a \in M$. Then $\mathscr{S}(M)$ becomes a compact Hausdorff topological space. In view of the Krein–Mil'man theorem, [Goo, Thm 5.17], every state on M is a weak limit of a net of convex combinations of extremal states.

We recall that by an *MV-ideal*, or shortly an ideal, we mean a non-void subset I of an MV-algebra M such that (i) $x, y \in I$, then $x \oplus y \in I$, and (ii) if $x \in M$, $y \in I$ and $x \leq y$, then $x \in I$.

According to [Mun1] or [DvPu, Thm 7.1.1], there is a one-to-one correspondence between extremal states, state-morphisms and maximal ideals: a state s is extremal iff s is a state-morphism iff $s(x \oplus y) = s(x) \oplus_{\mathbb{R}} s(y)$ (where $s \oplus_{\mathbb{R}} t := \min\{s + t, 1\}$) iff $s(x \vee y) = \max\{s(x), s(y)\}$ iff $\mathrm{Ker}(s) := \{x \in M : s(x) = 0\}$ is a maximal ideal of M. In addition, if I is a maximal ideal, then $s_I(x) := x/I$,

$x \in M$, is an extremal state, and there is a one-to-one correspondence between extremal states and maximal ideals given by $I \leftrightarrow s_I$.

If $a_1 = \cdots = a_n = a$, then $na := a_1 \oplus \cdots \oplus a_n$ and $a^n := a_1 \odot \cdots \odot a_n$. For any $a \in M$, we define $\mathrm{ord}(a)$ as the least integer m such that $ma = 1$, if it exists, otherwise, we set $\mathrm{ord}(a) = \infty$. We define also $a \ominus b := a \odot b^*$.

According to [FlMo], we say that a mapping $\sigma :\ M \to M$ is a *state operator* if, for all $x, y \in M$,

(i) $\sigma(1) = 1$,
(ii) $\sigma(x^*) = \sigma(x)^*$,
(iii) $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \odot y))$,
(iv) $\sigma(\sigma(x) \oplus \sigma(y)) = \sigma(x) \oplus \sigma(y)$.

A pair (M, σ) is said to be a *state MV-algebra*. The set of all state MV-algebras forms a variety.

For an MV-algebra M, let $\Sigma(M)$ be the set of all state operators on M. Then $\Sigma(M)$ is nonempty because it contains $\sigma = \mathrm{id}_M$.

The basic properties of a state operator were described in [FlMo, Lemma 3.2]:

(i) $\sigma(0) = 0$,
(ii) σ is monotone,
(iii) $\sigma(x \oplus y) \leq \sigma(x) \oplus \sigma(y)$, and if $x \odot y = 0$ then $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y)$,
(iv) $\sigma(\sigma(x)) = \sigma(x)$,
(v) $\sigma(M)$ is an MV-subalgebra of M.

It is easy to see that

(vi) $\sigma(M) = \{a \in M :\ a = \sigma(a)\}$.

In addition,

(vii) $\mathrm{ord}(x) < \infty \implies \sigma(x) \notin \mathrm{Rad}(M)$, where $\mathrm{Rad}(M)$ denotes the intersection of all maximal ideals of M.

In [DiDv], we have introduced a *state-morphism operator* on an MV-algebra M as an MV-homomorphism $\sigma :\ M \to M$ such that $\sigma^2 = \sigma$, and the couple (M, σ) is said to be a *state-morphism MV-algebra*, or more precisely, M with internal state. This notion was inspired by the above described basic property of extremal states which are only state-morphisms. For basic properties and notions on state-morphism MV-algebra, see [DiDv]. We recall [DiDv, Thm 4.1], that every state-morphism σ MV-algebra $M = \Gamma(G, u)$ can be uniquely extended to an ℓ-group homomorphism $\hat{\sigma} :\ G \to G$ such that $\hat{\sigma}^2 = \hat{\sigma}$ and $\hat{\sigma}(u) = u$, and vice-versa, the restriction to M of any ℓ-group homomorphism $h:\ G \to G$ such that $h(u) = u$ and $h^2 = h$ gives a state-morphism operator.

We recall, that (M, id_M) is always a state-morphism MV-algebra. Therefore, $\Sigma_\partial(M)$, the set of state-morphism operators is non-void.

Let (M, σ) be a state MV-algebra (a state-morphism MV-algebra). We say that a nonempty subset I of M is a *state-ideal* (a *state-morphism-ideal*) if I is an MV-ideal such that if $x \in I$, then $\sigma(x) \in I$.

We note that if σ is a state operator on M, then

$$\mathrm{Ker}(\sigma) = \{x \in M :\ \sigma(x) = 0\}$$

is an MV-ideal as well as a state-ideal.

We recall that if a is an element of M, then the MV-ideal of M generated by a is the set $I(a) = \{x \in M :\ x \le na \text{ for some } n \ge 1\}$, and the state-ideal (state-morphism-ideal) of M generated by a, is the set $I_\sigma(a) = \{x \in M :\ x \le n(a \oplus \sigma(a)) \text{ for some } n \ge 1\}$, [FlMo, Lem 4.2].

There is a one-to-one correspondence between congruences, $\sim$, and state ideals (state-morphism-ideals), I, given $x \sim_I y$ iff $x \odot y^*, y \odot x^* \in I$, and if $\sim$ is a congruence, then $I_\sim = \{x \in M :\ x \sim 0\}$ is a state-ideal of I and $\sim_{I_\sim} = \sim$.

Let $\mathscr{MI}_\sigma(M)$ be the set of all maximal state-ideals of M, and we set

$$\mathrm{Rad}_\sigma(M) = \bigcap \{I \in \mathscr{MI}_\sigma(M)\}.$$

According to [DiDv, Prop. 4.7],

$$\sigma(\mathrm{Rad}(M)) = \mathrm{Rad}(\sigma(M)) = \sigma(\mathrm{Rad}_\sigma(M)). \tag{2.1}$$

3. Classes of state-morphism MV-algebras

In the present section, we define some systems of state-morphism MV-algebras.

We recall that an MV-algebra M is

(i) *simple* if M has only two MV-ideals,

(ii) *semisimple* if the intersection of all maximal ideals is $\{0\}$, or equivalently M is MV-isomorphic to a system of fuzzy sets.

In what follows, we define the classes of state MV-algebras (M, σ), $\mathscr{SSMV}$ and $\mathscr{SSSMV}$, such that $\sigma(M)$ is a simple or semisimple MV-algebra, respectively.

Example 1. Let $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1,0))$ (called a *perfect MV-algebra*), where $\mathbb{Z} \overrightarrow{\times} G$ is the lexicographic product of the group of integers, $\mathbb{Z}$, with an Abelian ℓ-group ($(m_1, g_1) \geq (m_1, g_2)$ iff $m_1 > m_2$ or $m_1 = m_2$ and $g_1 \geq g_2$). Then clearly M satisfies the identity $2x^2 = (2x)^2$ because every element is of the form $x = (0, g)$ or $x = (1, -g)$ where $g \in G^+$. Hence, the operator

$$\sigma(x) = 2x^2, \qquad x \in M, \tag{3.1}$$

is a state-morphism operator on M and $\sigma(M) = B(M) = \{(0,0),(1,0)\}$ where $B(M) = \{x \in M : \ x \oplus x = x\}$, see also [DiLe, Thm 5.8]. Consequently, $(M, \sigma) \in \mathscr{S}\mathscr{S}\mathscr{M}\mathscr{V}$.

More generally, if an MV-algebra M satisfies the identity $2x^2 = (2x)^2$, then by [DiLe, Thm 5.1, 5.11], M is a subdirect product of perfect MV-algebras. Then the operator σ defined by (3.1) is again a state-morphism operator, $\sigma(M) = B(M)$, and $(M, \sigma) \in \mathscr{S}\mathscr{S}\mathscr{S}\mathscr{M}\mathscr{V}$.

Proposition 3.1. *Let (M, σ) be a state-morphism MV-algebra. Then the following are equivalent:*

(1) $(M, \sigma) \in \mathscr{S}\mathscr{S}\mathscr{M}\mathscr{V}$.

(2) $\mathrm{Ker}(\sigma)$ *is a maximal MV-ideal of* M.

Proof. Let $x \notin \mathrm{Ker}(\sigma)$, then $\sigma(x) > 0$ and there is a positive integer n such that $n\sigma(x) = 1$. Hence $\sigma((x^*)^n) = 0$, that is $(x^*)^n \in \mathrm{Ker}(\sigma)$. So we have proved that $\mathrm{Ker}(\sigma)$ is a maximal MV-ideal of M.

Viceversa, assume that $\mathrm{Ker}(\sigma)$ is a maximal MV-ideal of M and $\sigma(x) > 0$. Then $\sigma(x) \notin \mathrm{Ker}(\sigma)$. But $\mathrm{Ker}(\sigma)$ is a maximal ideal of M, hence there exists an integer n such that $(\sigma(x)^*)^n \in \mathrm{Ker}(\sigma)$. Therefore $\sigma(n(\sigma(x))) = 1$. From (iv) of the definition of a state operator we get $n\sigma(x) = 1$. Hence $\mathrm{ord}(\sigma(x)) < \infty$ for every $\sigma(x) \neq 0$, i.e., $\sigma(M)$ is simple. $\square$

Proposition 3.2. *Let (M, σ) be a state MV-algebra. Then the following are equivalent:*

(1) $(M, \sigma) \in \mathscr{S}\mathscr{S}\mathscr{S}\mathscr{M}\mathscr{V}$.

(2) $\mathrm{Rad}(M) \subseteq \mathrm{Ker}(\sigma)$.

Proof. Assume $(M, \sigma) \in \mathscr{S}\mathscr{S}\mathscr{S}\mathscr{M}\mathscr{V}$, then $\mathrm{Rad}(\sigma(M)) = \{0\}$. But $\sigma(\mathrm{Rad}(M)) = \mathrm{Rad}(\sigma(M))$. Hence $\sigma(\mathrm{Rad}(M)) = \{0\}$, that is $\mathrm{Rad}(M) \subseteq \mathrm{Ker}(\sigma)$.

Now assume that $\mathrm{Rad}(M) \subseteq \mathrm{Ker}(\sigma)$. Then $\sigma(\mathrm{Rad}(M)) = \{0\}$. By (2.1), $\sigma(\mathrm{Rad}(M)) = \mathrm{Rad}(\sigma(M))$, then we get $\mathrm{Rad}(\sigma(M)) = \{0\}$, i.e., $(M, \sigma) \in \mathscr{S}\mathscr{S}\mathscr{S}\mathscr{M}\mathscr{V}$. $\square$

We recall that perfect MV-algebras were defined in Example 1, and an MV-algebra M is perfect iff, for each $x \in M$, either $x \in \mathrm{Rad}(M)$ or $x^* \in \mathrm{Rad}(M)$, [DiLe].

Proposition 3.3. *Let (M, σ) be a state MV-algebra. Then the following are equivalent*

(1) *M is perfect;*
(2) *for every $x \in M$, $(\sigma(x) \in \mathrm{Rad}(M) \implies x \in \mathrm{Rad}(M))$ and $\sigma(M)$ is perfect.*

P r o o f. Assume M is perfect. For $x \in M$, let $\sigma(x) \in \mathrm{Rad}(M)$ and $x \in \mathrm{Rad}(M)^*$. Hence $x^* \oplus \sigma(x) \in \mathrm{Rad}(M)$ and $\sigma(x) \leq x$. Furthermore, we get:

$$\begin{aligned}\sigma(x^* \oplus \sigma(x)) &= \sigma(x^*) \oplus \sigma(\sigma(x) \odot (x \oplus \sigma(x^*)))\\ &= \sigma(x^*) \oplus \sigma(\sigma(x) \wedge x) = \sigma(x^*) \oplus \sigma(\sigma(x))\\ &= \sigma(x^*) \oplus \sigma(x) = 1.\end{aligned}$$

Hence for every $y \in \mathrm{Rad}(M)^*$, $\sigma(y) = 1$, in fact we have $x^* \oplus \sigma(x) \leq y$ and $1 = \sigma(x^* \oplus \sigma(x)) \leq \sigma(y)$. Therefore, for every $z \in \mathrm{Rad}(M)$, $\sigma(z) = 0$, in contrast with $\sigma(x^* \oplus \sigma(x)) = 1$. Thus, assuming that $\sigma(x) \in \mathrm{Rad}(M)$ necessarily $x \in \mathrm{Rad}(M)$. Of course $\sigma(M)$ is perfect, being a subalgebra of a perfect algebra. We proved that (1) $\implies$ (2).

To prove that (2) $\implies$ (1), assume $\sigma(M)$ is perfect. Let $x \in M$. Then, if $\sigma(x) \in \mathrm{Rad}(M)$, by the hypothesis we get $x \in \mathrm{Rad}(M)$. If $\sigma(x) \in \mathrm{Rad}(M)^*$, then $\sigma(x^*) \in \mathrm{Rad}(M)$ and again, by the hypothesis, $x^* \in \mathrm{Rad}(M)$ and $x \in \mathrm{Rad}(M)^*$. From that, easily can be seen that M has to be perfect. $\square$

The Proposition 3.3 pushes us to give the following definition, which generalizes the notion of a faithful state operator. Indeed, let (M, σ) be a state MV-algebra; we say that σ is *radical-faithful* if, for every $x \in M$, $\sigma(x) \in \mathrm{Rad}(M)$ implies $x \in \mathrm{Rad}(M)$. So we can paraphrase the Proposition 3.3 saying that every state operator on a perfect MV-algebra is radical-faithful.

We recall that an MV-algebra is *local* if it has a unique maximal ideal. So we have:

Proposition 3.4. *Let (M, σ) be a state-morphism MV-algebra with radical-faithful σ. Then the following are equivalent:*

(1) *M is a local MV-algebra.*
(2) *$\sigma(M)$ is a local MV-algebra.*

P r o o f. The implication (1) $\implies$ (2) is trivial. Now we are going to prove that (2) $\implies$ (1). Let $\sigma(M)$ be local. We shall prove that $\mathrm{Rad}(M)$ is a prime ideal, so M is shown to be local. Take $x, y \in M$ such that $x \wedge y \in \mathrm{Rad}(M)$. Then

$$\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y) \in \sigma(\mathrm{Rad}(M)) = \mathrm{Rad}(\sigma(M)).$$

Since $\sigma(M)$ is local, then $\mathrm{Rad}(\sigma(M))$ is a prime ideal of $\sigma(M)$, this implies that either $\sigma(x) \in \mathrm{Rad}(\sigma(M)) \subseteq \mathrm{Rad}(M)$, or $\sigma(y) \in \mathrm{Rad}(\sigma(M)) \subseteq \mathrm{Rad}(M)$. Thus $\sigma(x) \in \mathrm{Rad}(M)$ or $\sigma(y) \in \mathrm{Rad}(M)$. Since σ is radical-faithful, we get $x \in \mathrm{Rad}(M)$ or $y \in \mathrm{Rad}(M)$, so it is now proved that $\mathrm{Rad}(M)$ is prime. □

Proposition 3.5. *Let (M, σ) be a state-morphism MV-algebra with faithful σ. Then the following are equivalent:*

(1) $(M, \sigma) \in \mathscr{S}\mathscr{S}\mathscr{M}\mathscr{V}$;

(2) *M is a local MV-algebra and* $\mathrm{Ker}(\sigma) = \mathrm{Rad}(M)$.

P r o o f.

(1) $\implies$ (2). Assume $(M, \sigma) \in \mathscr{S}\mathscr{S}\mathscr{M}\mathscr{V}$. Then $\sigma(M)$ is simple and it is local. By Proposition 3.4, M is local. Since M is local, then $\mathrm{Rad}(M)$ is a unique maximal ideal of M and $\mathrm{Rad}(\sigma(M)) = \{0\}$. Hence, $\mathrm{Rad}(M) \subseteq \mathrm{Ker}(\sigma)$. But $\mathrm{Ker}(\sigma)$ is an ideal of M, hence $\mathrm{Ker}(\sigma) \subseteq \mathrm{Rad}(M)$. So we shown that $\mathrm{Rad}(M) = \mathrm{Ker}(\sigma)$.

(2) $\implies$ (1). Since M is local, $\sigma(M)$ is so. Moreover, from $\mathrm{Rad}(M) = \mathrm{Ker}(\sigma)$ we get $\mathrm{Ker}(\sigma)$ is a maximal ideal of M, thus by Proposition 3.1, $(M, \sigma) \in \mathscr{S}\mathscr{S}\mathscr{M}\mathscr{V}$. □

We can ask whether choosing a subalgebra B of M it happens that $\sigma(B)$ is a subalgebra of M. In the next lemma we provide a sufficient condition for a positive answer to the question:

Let B be a subalgebra of an MV-algebra M and let σ be a state operator on M. If $\sigma(B) \subseteq B$, then by the basic properties of σ, $\sigma(B)$ is a subalgebra of B.

Let M be an MV-algebra and X a non-empty subset of M. Then $\mathrm{Alg}(X)$ denotes the subalgebra of M generated by X.

Proposition 3.6. *Let (M, σ) be a state MV-algebra. Then* $\sigma(\mathrm{Alg}(\mathrm{Rad}(M)))$ *is a subalgebra of* $\mathrm{Alg}(\mathrm{Rad}(M))$.

P r o o f. By the equality (2.1), $\sigma(\mathrm{Rad}(M)) \subseteq \mathrm{Rad}(M)$. Furthermore, we have $\sigma((\mathrm{Rad}(M))^*) \subseteq (\mathrm{Rad}(M))^*$, in fact for every $x \in \mathrm{Rad}(M)$, $\sigma(x^*) = \sigma(x)^* \in (\mathrm{Rad}(M))^*$. Since $\mathrm{Alg}(\mathrm{Rad}(M)) = \mathrm{Rad}(M) \cup \mathrm{Rad}(M)^*$, hence $\sigma(\mathrm{Alg}(\mathrm{Rad}(M))) \subseteq \mathrm{Alg}(\mathrm{Rad}(M))$. By the remark just before the proposition, we get the claim of Proposition true. □

As a comment we can say that in any state MV-algebra M, the subalgebra $\mathrm{Alg}(\mathrm{Rad}(M)) = \mathrm{Rad}(M) \cup \mathrm{Rad}(M)^*$ is the greatest subalgebra of M that is perfect (see Example 2), [DiLe], and is called the *perfect skeleton.* Therefore, the perfect skeleton is stable with respect to the state operator.

4. State-morphism MV-algebras and retractive ideals

We recall that an MV-algebra A is said to be a *retract* of the MV-algebra B (with respect to h and ε) if there are MV-homomorphisms $\varepsilon\colon A \to B$ and $h\colon B \to A$ such that $h \circ \varepsilon = \mathrm{id}_A$. Also we say that a homomorphism $\rho\colon A \to B$ is *retractive* (with respect to δ) provided that there is an MV-homomorphism $\delta\colon B \to A$ such that $\rho \circ \delta = \mathrm{id}_B$. A congruence relation θ on A is called *retractive* when the canonical projection $\pi_\theta\colon A \to A/\theta$ is retractive. An ideal I of A is called *retractive* if the associated congruence is retractive.

Let M be an MV-algebra and I a retractive ideal of M, then we call an *ideal retraction* of M any pair (I, φ), where φ is a homomorphism from M/θ_I to M such that $\pi_{\theta_I} \circ \varphi = \mathrm{id}_{M/\theta_I}$.

Lemma 4.1. *Let M be an MV-algebra and $\sigma : M \to M$ an endomorphism. Then the following statements are equivalent:*

(1) *(M, σ) is a state-morphism MV-algebra.*

(2) *σ is a retractive homomorphism from M to $\sigma(M)$ with respect to the identity map from $\sigma(M)$ to M.*

Proof.

(1) $\Longrightarrow$ (2). Then $\sigma^2 = \sigma$. We have to show that there is an MV-homomorphism $\delta\colon \sigma(M) \to M$ such that $\sigma \circ \delta = \mathrm{id}_{\sigma(M)}$. Let δ be defined as the identity map from $\sigma(M)$ to M, i.e., $\delta(\sigma(a)) = \sigma(a)$, for $a \in M$. Hence we get

$$\sigma(\delta(\sigma(a))) = \sigma(\sigma(a)) = \sigma(a).$$

(2) $\Longrightarrow$ (1). Let φ denote the identity map from $\sigma(M)$ to M. Then we only have to verify that $\sigma^2 = \sigma$. Indeed we have, for $a \in M$:

$$\sigma(\sigma(a)) = \sigma(\varphi(\sigma(a))) = \sigma(a).$$

$\square$

Proposition 4.2. *Let (I, φ) be an ideal retraction of an MV-algebra M. Then there is a state-morphism operator σ on M such that* $\mathrm{Ker}(\sigma) = I$*.*

P r o o f. Let (I,φ) be an ideal retraction of M, θ_I the associated congruence relation to I and $\pi_{\theta_I}\colon M \to M/\theta_I$ the canonical projection. Thus π_{θ_I} is a retractive homomorphism and $\pi_{\theta_I}\circ\varphi = \mathrm{id}_{M/I}$. Let us define the mapping $\sigma\colon M\to M$ as follows:

$$\sigma(a) = \varphi(\pi_{\theta_I}(a)), \quad a\in M.$$

Now we are going to show that σ is an MV-endomorphism such that $\sigma^2=\sigma$. Indeed we have:

$$\sigma(0) = \varphi(\pi_{\theta_I}(0)) = \varphi(0/I) = 0.$$

For every $x,y\in M$,

$$\sigma(x\oplus y) = \varphi(\pi_{\theta_I}(x\oplus y)) = \varphi(x/I\oplus y/I) = \varphi(\pi_{\theta_I}(x))\oplus\varphi(\pi_{\theta_I}(y)) = \sigma(x)\oplus\sigma(y);$$

$$\sigma(x^*) = \varphi(\pi_{\theta_I}(x^*)) = \varphi((x^*)/I) = (\varphi(x/I))^* = \varphi(\pi_{\theta_I}(x))^* = (\sigma(x))^*;$$

furthermore, since $\pi_\theta\circ\varphi = \mathrm{id}_{M/I}$, for every $x\in M$ we have:

$$\sigma(\sigma(x)) = \sigma(\varphi(\theta_I(x))) = \varphi(\theta_I(\varphi(\theta_I(x)))) = \varphi(\theta_I(x)) = \sigma(x).$$

By Lemma 4.1, we get that (M,σ) is a state-morphism MV-algebra. Now we only have to prove that $\mathrm{Ker}(\varphi\circ\pi_{\theta_I}) = I$. Let $a\in I$, then

$$\varphi(\pi_{\theta_I}(a)) = \varphi(\pi_{\theta_I}(0)) = 0$$

hence $I\subseteq \mathrm{Ker}(\varphi\circ\pi_{\theta_I})$. Assume now that $a\in\mathrm{Ker}(\varphi\circ\pi_{\theta_I})$. Then

$$\varphi(\pi_{\theta_I}(a)) = 0$$

so

$$\pi_{\theta_I}(\varphi(\pi_{\theta_I}(a))) = \pi_{\theta_I}(0)$$

hence we get $\pi_{\theta_I}(a) = \pi_{\theta_I}(0)$, $a\in I$ and then $\mathrm{Ker}(\varphi\circ\pi_{\theta_I})\subseteq I$. We can then conclude that

$$\mathrm{Ker}(\varphi\circ\pi_{\theta_I}) = I.$$

□

Proposition 4.3. *Let (M,σ) be a state-morphism MV-algebra. Then the pair $(\mathrm{Ker}(\sigma),\iota)$ is an ideal retraction of M, where $\iota : M/\mathrm{Ker}(\sigma)\to M$ is defined by $\iota(x/\mathrm{Ker}(\sigma)) = \sigma(x)$, $x\in M$.*

P r o o f. Let (M,σ) be a state-morphism MV-algebra with an MV-reduct M, then, by Lemma 4.1, σ is a retractive homomorphism from M to $\sigma(M)$ with respect to the mapping $\iota\colon\sigma(M)\to M$, and $\mathrm{Ker}(\sigma)$ is a retractive ideal of M. □

Corollary 4.4. *Let (M,σ) be a state-morphism MV-algebra. Then $\sigma(M)$ is a retract of M.*

P r o o f. Assume (M, σ) be a state-morphism MV-algebra, then σ is an MV-homomorphism and $\sigma(M)$ is an MV-subalgebra of M, hence we have:

$$\begin{tikzcd} \sigma(M) \arrow[rr, "i"] \arrow[dr, "\mathrm{id}_M"'] & & M \arrow[dl, "\sigma"] \\ & \sigma(M) & \end{tikzcd}$$

where i is the inclusion map and $\mathrm{id}_{\sigma(M)}$ is the identity map of $\sigma(M)$. Indeed we get $i(\sigma(x)) = \sigma(x) \in M$ and $\sigma(\sigma(x)) = \sigma(x)$, for every $x \in M$. Thus $\sigma(M)$ is a retract of M, with respect to the MV-homomorphisms i and σ. $\square$

We call a *proper ideal retraction* of an MV-algebra M any ideal retraction (I, φ) of M such that the canonical projection of the congruence associated to I, $\varphi \circ \pi_{\theta_I}$, is an endomorphism of M such that $(\varphi \circ \pi_{\theta_I}) \circ (\varphi \circ \pi_{\theta_I}) = (\varphi \circ \pi_{\theta_I})$.

Let M be a given MV-algebra. We define $\mathfrak{PIR}(M)$, the set of all proper ideal retractions of M, and let $\mathscr{S}\mathrm{mor}(M)$ be the set of all state-morphism MV-algebras (M, σ),

So, we can define a mapping $\Xi : \ \mathfrak{PIR}(M) \to \mathscr{S}\mathrm{mor}(M)$ setting $\Xi(I, \varphi) = (M, \varphi \circ \pi_{\theta_I})$, and a mapping $\Psi \colon \mathscr{S}\mathrm{mor}(M) \to \mathfrak{PIR}(M)$ setting

$$\Psi((M, \sigma)) = (\mathrm{Ker}(\sigma), \iota).$$

Proposition 4.5. *Let M be an MV-algebra. Then there is a one-to-one correspondence between proper ideal retractions of M and state-morphism MV-algebras (M, σ).*

P r o o f. To prove the proposition we shall show that the mapping Ξ defined above is bijective. Let $(I, \varphi), (J, \psi)$ be proper ideal retractions of M. Assume that

$$\Xi((I, \varphi)) = \Xi((J, \psi)),$$

that is,

$$(M, \varphi \circ \pi_{\theta_I}) = (M, \psi \circ \pi_{\theta_J}).$$

Then

$$(\varphi \circ \pi_{\theta_I})(x) = (\psi \circ \pi_{\theta_J})(x) \qquad \text{for every} \quad x \in M,$$

that is,

$$x/I = x/J \qquad \text{for every} \quad x \in M.$$

Hence $I = J$ and $\varphi = \psi$. Then we get $(I, \varphi) = (J, \psi)$. So we proved that Ξ is injective.

To show the surjectivity of Ξ take a state-morphism MV-algebra (M, σ). Then by Proposition 3.3 $(\mathrm{Ker}(\sigma), \iota)$ is a proper ideal retraction of M such that

$$\Xi((\mathrm{Ker}(\sigma), \iota)) = (M, \iota \circ \pi_{\theta_{\mathrm{Ker}(\sigma)}}) = (M, \sigma).$$

The proposition is now proved. □

Notice that a retractive extremal state over an MV-algebra M determines a retractive maximal ideal of M, via its kernel. So it is worth to provide information about maximal retractive ideals of an MV-algebra. The following proposition aims to do that. We recall that given an MV-algebra M there is the greatest local subalgebra of M, here denoted by $\mathscr{L}(M)$, see [DEG].

Proposition 4.6. *Let M be an MV-algebra and J a maximal ideal of M. Then the following are equivalent:*

(1) *J is retractive;*
(2) *$M/J \cong \mathscr{L}(M)/J$ and $\mathscr{L}(M)$ has the radical retractive.*

Proof. Let J be a retractive maximal ideal of the MV-algebra M and π_J be the canonical projection of M to M/J. Then there exists an MV-homomorphism $\delta\colon M/J \to M$ such that

$$\pi_J \circ \delta = \mathrm{id}_{M/J}\,. \tag{4.1}$$

Notice that $\delta(M/J)$ is a simple subalgebra of M, then it is a local subalgebra of M, therefore

$$\delta(M/J) \subseteq \mathscr{L}(M).$$

Hence $\pi_J(\delta(M/J)) \subseteq \pi_J(\mathscr{L}(M))$ and, by (4.1),

$$M/J \subseteq \mathscr{L}(M)/J.$$

Moreover, we have $\mathscr{L}(M) \subseteq M$, and then $\mathscr{L}(M)/J \subseteq M/J$. Hence we proved that

$$M/J = \mathscr{L}(M)/J.$$

Now we are going to prove that $\mathscr{L}(M)$ has the radical as a retractive ideal. Indeed, we notice that for $t, l \in \mathscr{L}(M)$ is:

$$t/\mathrm{Rad}(\mathscr{L}(M)) = l/\mathrm{Rad}(\mathscr{L}(M))$$
$$\Longleftrightarrow$$
$$(t \odot l^* \oplus t^* \odot l) \in \mathrm{Rad}(\mathscr{L}(M))$$
$$\Longleftrightarrow$$
$$(t/\mathrm{Rad}(M) = l/\mathrm{Rad}(M))$$
$$\Longleftrightarrow$$
$$(t \odot l^* \oplus t^* \odot l) \in \mathrm{Rad}(M)$$
$$\Longrightarrow$$
$$(t \odot l^* \oplus t^* \odot l) \in J$$
$$\Longleftrightarrow$$
$$t/J = l/J.$$

Then we define a mapping $\widehat{\delta} : \mathscr{L}(M)/\mathrm{Rad}(\mathscr{L}(M)) \to \mathscr{L}(M)$ as follows:
for any $l \in \mathscr{L}(M)$,

$$\widehat{\delta}(l/\mathrm{Rad}(\mathscr{L}(M))) = \delta(l/J) \in \mathscr{L}(M).$$

Indeed, $\delta(M/J) \subseteq \mathscr{L}(M)$, being $\delta(M/J)$ simple and therefore local.

Let us show that $\pi_{\mathrm{Rad}(\mathscr{L}(M))} \circ \widehat{\delta} = \mathrm{id}_{\mathrm{Rad}(\mathscr{L}(M))}$. In fact we have:

$$\begin{aligned}\pi_{\mathrm{Rad}(\mathscr{L}(M))} \circ \widehat{\delta}(l/\mathrm{Rad}(\mathscr{L}(M))) &= \pi_{\mathrm{Rad}(\mathscr{L}(M))}(\delta(l/J)) \\ &= \pi_J(\delta(l/J)) = l/J = l/\mathrm{Rad}(\mathscr{L}(M)).\end{aligned}$$

Vice versa, let $M/J \cong \mathscr{L}(M)/J$ and let $\mathscr{L}(M)$ have the radical as a retractive ideal. Then there exists an MV-homomorphism $\lambda\colon \mathscr{L}(M)/\mathrm{Rad}(\mathscr{L}(M)) \to \mathscr{L}(M)$ such that

$$\pi_{\mathrm{Rad}(\mathscr{L}(M))} \circ \lambda = \mathrm{id}_{\mathscr{L}(M)/\mathrm{Rad}(\mathscr{L}(M))}.$$

Let us define a mapping $\gamma\colon M/J \to M$, as follows:
for every $m \in M$ there is an element $l \in \mathscr{L}(M)$ such that $m/J = l/J$, then we set

$$\gamma(m/J) = \lambda(l/\mathrm{Rad}(\mathscr{L}(M))) \in \mathscr{L}(M) \subseteq M.$$

Hence we get

$$\begin{aligned}\pi_J(\gamma(m/J)) &= \pi_J(\lambda(l/\mathrm{Rad}(\mathscr{L}(M))))\\ &= \pi_{\mathrm{Rad}(\mathscr{L}(M))}\lambda(l/\mathrm{Rad}(\mathscr{L}(M)))\\ &= l/\,\mathrm{Rad}(\mathscr{L}(M)) = m/J.\end{aligned}$$

□

As it was already mentioned in Section 2, there is a one-to-one correspondence between extremal states on an MV-algebra M and maximal ideals of M, and furthermore that the set of extremal states on M, $\mathscr{S}_\partial(M)$, is compact. It makes sense to explore suitable subsets of $\mathscr{S}_\partial(M)$ that are someway linked with sets of state-morphism operators on M, via a selection of classes of maximal ideals of M.

Let $\mathrm{Max}(M)$ denote the set of maximal ideals of M. We say that an extremal state s on M is *retractive* if s is retractive as an MV-homomorphism from M to $s(M)$.

We denote the set of retractive extremal states on M by $\mathscr{S}_{\mathrm{Retr}\,\partial}(M)$ and by $\Sigma_{\mathrm{Retr}\,\partial}(M)$ the set of state-morphism operators σ on M with $\mathrm{Ker}(\sigma) \in \mathrm{Max}(M)$ and a retractive ideal of M.

Proposition 4.7. *Let M be an MV-algebra. Then there is a bijective mapping from $\mathscr{S}_{\mathrm{Retr}\,\partial}(M)$ onto $\Sigma_{\mathrm{Retr}\,\partial}(M)$.*

Proof. Let s be a state-morphism on M. So, by definition of a retractive extremal state there is an MV-homomorphism $\delta_s : s(M) \to M$ such that $s \circ \delta_s = \mathrm{id}_{s(M)}$. Then we can define a mapping $\sigma : M \to M$ setting $\sigma(a) = \delta_s(s(a))$ for every $a \in M$. Easily it can be seen that σ is an MV-endomorphism of M. Let us show that $\sigma \circ \sigma = \sigma$. Indeed, for every $a \in M$, we have:

$$\sigma(\sigma(a)) = \sigma(\delta_s(s(a))) = \delta_s(s(\delta_s(s(a)))) = \delta_s(s(a)) = \sigma(a).$$

Hence, by Lemma 4.1, we get that σ is a state-morphism operator on M. Now we are going to prove that $\mathrm{Ker}(\sigma) \in \mathrm{Max}(M)$ and that $\mathrm{Ker}(\sigma)$ is a retractive ideal of M. A direct verification can show that:

$$\mathrm{Ker}(s) = \mathrm{Ker}(\sigma).$$

Since $\mathrm{Ker}(s)$ is a maximal ideal of M, so is $\mathrm{Ker}(\sigma)$. By Proposition 3.1, $(M, \sigma) \in \mathscr{SSMV}$. Since $\mathrm{Ker}(\sigma) = \mathrm{Ker}(s) \in \mathrm{Max}(M)$, by Proposition 4.3, $\mathrm{Ker}(\sigma)$ is a retractive ideal of M.

Let now $\psi\colon \mathscr{S}_{\mathrm{Retr}\,\partial}(M) \to \Sigma_{\mathrm{Retr}\,\partial}(M)$ be defined by $\psi(s) = \sigma$. Suppose that $\psi(s_1) = \psi(s_2)$. Then by the above, $\mathrm{Ker}(s_1) = \mathrm{Ker}(s_2)$ which means $s_1 = s_2$.

Choose $\sigma \in \Sigma_{\mathrm{Retr}\,\partial}(M)$. Because $\mathrm{Ker}(\sigma)$ is maximal, there is a unique extremal state, s, on M such that $\mathrm{Ker}(s) = \mathrm{Ker}(\sigma)$. Because $\mathrm{Ker}(\sigma)$ is a retractive ideal, s is a retractive extremal state, and $\psi(s) = \sigma$. $\square$

5. State-morphism operators on $\mathrm{Free}(m)$

Let A be an MV-algebra of functions from a set X^m, where $m \geq 1$ is an integer, to the real interval $[0,1]$. We say that a state-morphism operator σ on A satisfies the *m-projective* property if and only if there exist m elements from A, $\{t_1, \dots, t_m\}$, such that:

$$\sigma(t_i)(\sigma(t_1)(x_1, \dots, x_n), \dots, \sigma(t_m)(x_1, \dots, x_n)) = \sigma(t_i)(x_1, \dots, x_n)$$

for all $(x_1, \dots, x_n) \in X^m$.

Let $F(m)$ denote the m-generated free MV-algebra. It is well known that $F(m)$ is isomorphic to the MV-algebra of McNaughton functions from $[0,1]^m$ to $[0,1]$, [CDM]. $F(m)$ is also closed under the functional composition. Let f_0 and f_1 denote the zero constant function, which is the zero element of $F(m)$ and the 1-constant function, which is the unit of $F(m)$, respectively. Let $g_1, \dots, g_m$ be the set of its free generators $g_i(x_1, \dots, x_m) = x_i$. Let $\mathrm{Alg}(f_1, \dots, f_m)$ denote the subalgebra of $F(m)$ generated by $f_1, \dots, f_m \in F(m)$. Let $\mathfrak{F} = \{f_1, \dots, f_m\}$ be a subset of non-constant elements of $F(m)$. The system $\mathfrak{F}$ is called a *projective system* iff

$$f_i(f_1(x_1, \dots, x_m), \dots, f_m(x_1, \dots, x_m)) = f_i(x_1, \dots, x_m),$$

for all $(x_1, \dots, x_m) \in [0,1]^m$.

Proposition 5.1. *Let σ be a state-morphism operator on $F(m)$ such that σ satisfies the m-projective property. Then $\sigma(F(m))$ is a projective subalgebra generated by $\sigma(g_1), \dots, \sigma(g_m)$.*

Proof. It is clear that $\mathrm{Alg}(\{\sigma(g_1), \dots, \sigma(g_m)\}) \subseteq \sigma(F(m))$. Let $f \in F(m)$, then there exists an MV-polynomial $P_f(\mathbf{x})$ such that for every $\mathbf{x} \in [0,1]^m$, $P_f(\mathbf{x}) = f(\mathbf{x})$. Then, for every $\mathbf{x} \in [0,1]^m$,

$$P_f(g_1(\mathbf{x}), \dots, g_m(\mathbf{x})) = f(\mathbf{x}).$$

That is $\sigma(P_f(g_1, \dots, g_m)) = \sigma(f)$. Since σ is an MV-endomorphism, then we get $P_f(\sigma(g_1, \dots, g_m)) = \sigma(f)$. Thus, we proved that

$$\mathrm{Alg}(\sigma(g_1), \dots, \sigma(g_1)) = \sigma(F(m)).$$

Let us prove that $\{\sigma(g_1), \ldots, \sigma(g_m)\}$ is a projective system of generators of $\sigma(F(m))$. Indeed, this follows from the m-projective property of σ. Hence, by [DiGr, Thm 9], $\sigma(F(m))$ is an m-generated projective subalgebra of $F(m)$ which is generated by $\sigma(g_1), \ldots, \sigma(g_m)$. □

Proposition 5.2. *Let A be a projective m-generated subalgebra of $F(m)$, with a projective system of generators $\mathfrak{F} = \{t_1, \ldots, t_m\}$. Then there exists a state-morphism operator, σ, on $F(m)$ such that $A = \sigma(F(m))$ and σ satisfies the m-projective property.*

Proof. Let A be a projective m-generated subalgebra of $F(m)$, with a projective system of generators $\mathfrak{F} = \{t_1, \ldots, t_m\}$. Then, by [DiGr, Cor. 27], A coincides with

$$\{f(t_1(x_1, \ldots, x_m), \ldots, t_1(x_1, \ldots, x_m)) : \ f \in F(m), (x_1, \ldots, x_m) \in [0,1]^m\}.$$

Let us define a mapping $\sigma : F(m) \to F(m)$ by setting

$$\sigma(f) = f(t_1, \ldots, t_m).$$

Hence we get, for every $i = 1, \ldots, m$,

$$\sigma(g_i) = g_i(t_1, \ldots, t_m) = t_i,$$

and then A is generated by $\sigma(g)$, furthermore, for every $f \in F(m)$,

$$\sigma(\sigma(f)) = \sigma(f(t_1, \ldots, t_m)) = (f(t_1, \ldots, t_m))(t_1, \ldots, t_m).$$

Hence, for every $\mathbf{x} \in [0,1]^m$,

$$\begin{aligned}\sigma(\sigma(f))(\mathbf{x}) &= (f(t_1, \ldots, t_m))(t_1(\mathbf{x}), \ldots, t_m(\mathbf{x})) \\ &= f(t_1(t_1(\mathbf{x}), \ldots, t_m(\mathbf{x})), \ldots, t_m(t_1(\mathbf{x}), \ldots, t_m(\mathbf{x}))) \\ &= f(t_1(\mathbf{x}), \ldots, t_m(\mathbf{x})) = \sigma(f)(\mathbf{x}).\end{aligned}$$

That is $\sigma(\sigma(f)) = \sigma(f)$. It is plain to check that σ is an MV-endomorphism.

Then, by Lemma 4.1, σ is a state-morphism operator on $F(m)$, $\sigma(F(m)) = A$, $\sigma(F(m))$ is generated by $\sigma(g_1), \ldots, \sigma(g_m)$ and σ satisfies the m-projective property. □

Theorem 5.3. *There is a one-to-one correspondence between m-generated projective subalgebras of $F(m)$ and state-morphism operators on $F(m)$ satisfying the m-projective property.*

Proof. It follows from Propositions 5.1–5.2. □

As an example, we can say in more details if we study the one-generated free MV-algebra $F(1)$. The identity map $g = \mathrm{id}_{[0,1]}$ of $F(1)$ is a free generator of $F(1)$. To any 1-variable McNaughton function f there is associated a partition, $0 = a_0 < a_1 < \cdots < a_n = 1$, of the unit interval $[0,1]$ in such a way that the points $\{(a_0, f(a_0)), (a_1, f(a_1)), \dots, (a_n, f(a_n))\}$ are the knots of f and the function f is linear on each interval $[a_{i-1}, a_i]$, with $i = 1, \dots, n$.

From Propositions 5.1–5.2 we have that there is a one-to-one correspondence between one-generated projective subalgebras of $F(1)$ and state-morphism operators on $F(1)$.

Thus all state-morphism operators σ on $F(1)$ are obtained as follows:

$$\sigma(f) = f \circ t$$

with $f, t \in F(1)$ and t such that $t \circ t = t$. By [DiGr, Thm 23], t is characterized by one of the following conditions:

(1) $\max\{t(x), x \in [0,1]\} = t(a_1)$ and for a nonzero function t and for every $x \in [0, a_1]$, $t(x) = x$;

(2) $\min\{t(x), x \in [0,1]\} = t(a_{n-1})$ and for a non-unit function t and for every $x \in [a_{n-1}, a_n]$, $t(x) = x$.

Examples of such generators are: $t = g \wedge g^*$, $t = g \wedge (g^2)^*$, $t = g \vee g^*$, $t = g \vee 2(g)^*$.

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REFERENCES

[Cha] CHANG, C. C.: *Algebraic analysis of many valued logics*, Trans. Amer. Math. Soc. **88** (1958), 467–490.

[CDM] CIGNOLI, R.—D'OTTAVIANO, I. M. L.—MUNDICI, D.: *Algebraic Foundations of Many-valued Reasoning*, Kluwer Academic Publ., Dordrecht, 2000.

[DiDv] DI NOLA, A.—DVUREČENSKIJ, A.: *State-morphism MV-algebras*, Ann. Pure Appl. Logic **161** (2009), 161–173.

[DEG] DI NOLA, A.—ESPOSITO, I.—GERLA, B.: *Local algebras in the representation of MV-algebras*, Algebra Universalis **56** (2007), 133–164.

[DiGr] DI NOLA, A.—GRIGOLIA, R.: *On projective MV-algebras* Ann. Pure Appl. Logic (Submitted).

[DiLe] DI NOLA, A.—LETTIERI, A.: *Perfect MV-algebras are categorical equivalent to abelian ℓ-groups*, Studia Logica **53** (1994), 417–432.

[DvPu] DVUREČENSKIJ, A.—PULMANNOVÁ, S.: *New Trends in Quantum Structures*, Kluwer Acad. Publ./Ister Science, Dordrecht/Bratislava, 2000.

[FlMo] FLAMINIO, T.—MONTAGNA, F.: *An algebraic approach to states on MV-algebras.* In: Fuzzy Logic 2, Proc. of the 5th EUSFLAT Conf., Sept. 11-14, 2007, Ostrava, Vol. II (V. Novák, ed.), pp. 201–206.

[Goo] GOODEARL, K. R.: *Partially Ordered Abelian Groups with Interpolation.* Math. Surveys Monogr. 20, Amer. Math. Soc., Providence, RI, 1986.

[KoCh] KÔPKA, F.—CHOVANEC, F.: *D-posets*, Math. Slovaca **44** (1994), 21–34.

[Kro] KROUPA, T.: *Every state on semisimple MV-algebra is integral*, Fuzzy Sets and Systems **157** (2006), 2771–2782.

[KuMu] KÜHR, J.—MUNDICI, D.: *De Finetti theorem and Borel states in* $[0,1]$*-valued algebraic logic*, Internat. J. Approx. Reason. **46** (2007), 605–616.

[Mun] MUNDICI, D.: *Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus*, J. Funct. Anal. **65** (1986), 15–63.

[Mun1] MUNDICI, D.: *Averaging the truth-value in Łukasiewicz logic*, Studia Logica **55** (1995), 113–127.

[RiMu] RIEČAN, B.—MUNDICI, D.: *Probability on MV-algebras.* In: Handbook of Measure Theory, Vol. II (E. Pap, ed.), Elsevier Science, Amsterdam, 2002, pp. 869–909.

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