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FIXED POINT OF CONTRACTIVE MAPPINGS IN GENERALIZED METRIC SPACES

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ABSTRACT. We prove a fixed point theorem for contractive mappings of Boyd and Wong type in generalized metric spaces, a concept recently introduced in [BRANCIARI, A.: *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publ. Math. Debrecen **57** (2000), 31–37].

1. Introduction

In 2000 Branciari [1] introduced a generalization of a metric space called *generalized metric space* by replacing the triangle inequality by a more general inequality. As such, every metric space is a generalized metric space but the converse is not true (see [1, Example 1]). We give here more examples which suggests that as a structure it may be weaker than a metric space. However the interesting point to note is that two very important fixed point theorems, namely Banach's fixed point theorem and Ciric's fixed point theorem have already been established in such a space ([1], [6]).

In this paper we continue in this direction and prove a fixed point theorem for contractive mappings of Boyd and Wong [2] (which is the most general one in the sense that it contains other types of contractive mappings ([5], [7]) as a subclass) in a generalized metric space.

2. Fixed point of contractive mappings

Let $\mathbb{R}^+$ denote the set of all non-negative real numbers and $\mathbb{N}$ the set of positive integers.

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Definition 1. (cf [1]) Let X be a set and $d\colon X^2 \to \mathbb{R}^+$ be a mapping such that for all $x, y \in X$ and for all distinct points $z_1, z_2, \dots, z_k \in X$ $(k \geq 2)$ each of them different from x and y, one has

(i) $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \leq d(x,z_1) + d(z_1,z_2) + \cdots + d(z_k,y)$

then we will say that (X,d) is a *generalized metric space* (or shortly g.m.s). Throughout this section a g.m.s will be denoted by (X,d) (or sometime by X only).

Any metric space is a g.m.s but the converse is not true ([1]).

In [1] it was claimed that as in a metric space, a topology can be generated in a g.m.s X with the help of the neighborhood basis given by $B = \{B(x,r) : x \in X,\ r \in \mathbb{R}^+ \setminus \{0\}\}$ where $B(x,r) = \{y \in X :\ d(x,y) < r\}$ is the open ball with centre x and radius r.

However consider the following example.

Example 1. Let us define $X = \left\{\frac{1}{n} :\ n = 1, 2, \dots\right\} \cup \{0\}$,

$$d\colon X \times X \to \mathbb{R}^+ : \quad d(x,y) = \begin{cases} 0 & \text{for } x = y \\ \frac{1}{n} & \text{for } \{x,y\} = \left\{0, \frac{1}{n}\right\}, x \neq y \\ 1 & \text{for } x \neq y,\ x, y \in X \setminus \{0\}. \end{cases}$$

Note that (X,d) satisfies axioms of generalized metric space, i.e. for all $x, y \in X$

(i) $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,y) \leq d(x,z_1) + d(z_1,z_2) + d(z_2,y)$, for distinct points x, y, z_1, z_2.

Observe that $B\left(\frac{1}{3}, \frac{1}{2}\right) \cap B\left(\frac{1}{4}, \frac{1}{2}\right) = \{0\}$ hence there is no $r > 0$ with $B(0,r) \subset B\left(\frac{1}{3}, \frac{1}{2}\right) \cap B\left(\frac{1}{4}, \frac{1}{2}\right)$. Therefore the family $\{B(x,r) :\ x \in X,\ r > 0\}$ is not a neighborhood basis for any topology on X.

In view of the above example, it seems more reasonable to construct the topology in a g.m.s X by taking the collection B as a subbasis. Further it can be observed from Example 1 that $\lim\limits_{n\to\infty} d(\frac{1}{2}, \frac{1}{n}) = 1$ whereas $d(\frac{1}{2}, 0) = \frac{1}{2} \neq 1$ which shows that d is not continuous in a sense presented in [1].

Consider now the following example:

Example 2. Let us define $Y = \left\{\frac{1}{n} :\ n = 1, 2, \dots\right\} \cup \{0, 2\}$,

$$d_1 : Y \times Y \to \mathbb{R}^+ : \quad d_1(x,y) = \begin{cases} 0 & \text{for } x = y \\ \frac{1}{n} & \text{for } x \in \{0,2\},\ y = \frac{1}{n} \\ \frac{1}{n} & \text{for } x = \frac{1}{n},\ y \in \{0,2\} \\ 1 & \text{otherwise.} \end{cases}$$

Note that (Y, d_1) is a g.m.s in which the points 0 and 2 do not have any disjoint open balls.

All this points out to the fact that a g.m.s (which is not a metric space) may sometimes be perceived as a much weaker structure than a metric space due to the weakening of the triangle inequality. The results of [1] and [6], in a sense, prove the existence of fixed points of contraction mappings or quasi-contraction mappings in more general spaces. Here we continue in this line and prove the existence of fixed points for contractive mappings of Boyd and Wong [2] which plays a prominent role in the literature (see [2], [8], [9]) and is the most general type of contractive map (see [5], [7]). It should be noted that throughout we use only the generalized metric axioms (Definition 1) in our investigations.

Before we prove main results, we recall the following definitions.

Definition 2. (cf [1]) A sequence $\{x_n\}$ is said to be *g.m.s convergent to a point* x *in* (X, d) if $d(x_n, x) \to 0$ as $n \to \infty$.

Definition 3. (cf [1]) Let (X, d) be a g.m.s. A sequence $\{x_n\}$, $n \in \mathbb{N}$, in X is said to be a *g.m.s Cauchy sequence* if for any $\varepsilon > 0$ there exists a natural number $n_\varepsilon \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, $n \geq n_\varepsilon$, one has $d(x_n, x_{n+m}) < \varepsilon$. (X, d) is called complete g.m.s if every g.m.s Cauchy sequence is convergent in X.

Note 1. The convergence of a sequence as defined in Definition 2 (see also [1], [6]) is actually weaker than the topological convergence (in Example 1, $\{\frac{1}{n}\}$ converges to 0, but 0 is isolated in the topology generated by d). So instead of following ([1]), it seems more appropriate to rename the convergence as g.m.s convergence.

Definition 4. (cf [5]) A mapping $T\colon X \to X$ is said to be *contractive* if for any two distinct points $x, y \in X$, $d(Tx, Ty) < d(x, y)$.

We now prove the following fixed point theorem for Boyd and Wong's contractive mappings ([2]).

Theorem 1. *Let X be a complete g.m.s and let $T\colon X \to X$ satisfy*

$$d(Tx, Ty) \leq \psi(d(x, y)) \tag{2.1}$$

where $\psi\colon \overline{P} \to [0, \infty)$ is upper semi-continuous from right on $\overline{P}$ (the closure of the range of d) and satisfies $\psi(t) < t$ for all $t \in \overline{P} \setminus \{0\}$. Then T has a unique fixed point x_0 and $T^n x \to x_0$ for each $x \in X$.

Proof. Given $x \in X$, define

$$c_n = d(T^n x, T^{n-1} x). \tag{2.2}$$

Then by (2.1), the sequence $\{c_n\}$ is decreasing and hence has a limit c (say). We shall show that $c = 0$.

If $c > 0$, we have

$$c_{n+1} \leq \psi(c_n) \tag{2.3}$$

so that

$$c \leq \limsup_{t \to c^+} \psi(t) \leq \psi(c) \tag{2.4}$$

which is a contradiction.

For each $x \in X$ consider the sequence $\{T^n x\}$. First assume that it is eventually constant, so there is some $n \in \mathbb{N}$ such that $T^m x = T^n x = y$ for each $m > n$. Then $T^{m-n}(T^n x) = T^n x$, so denoting $k = m - n$, we have $T^k y = y$ for all $k \in \mathbb{N}$. It follows that $d(y, Ty) = d(T^k y, T^{k+1} y) = c_k$ for all k, and since $c_k \to 0$, $d(y, Ty) = 0$, so $y = Ty$. Thus y is a fixed point of T.

If $\{T^n x\}$ is not eventually constant, then it has a subsequence with pairwise distinct terms. Without loss of generality, assume that $\{T^n x\}$ is this subsequence. We will show that $\{T^n x\}$ is a g.m.s Cauchy sequence. By contradiction, suppose that there is an $\varepsilon > 0$ and sequences $\{m(k)\}$, $\{n(k)\}$ of positive integers with $m(k) > n(k) \geq k$ such that

$$d(T^{m(k)} x, T^{n(k)} x) \geq \varepsilon \qquad \text{for all} \quad k \in \mathbb{N}. \tag{2.5}$$

Since this is true for all $k \in \mathbb{N}$, we can conclude that for every $k \in \mathbb{N}$ there will exist $n(k) \geq k$ and an infinite number of $m(k) > n(k)$ for which

$$d(T^{m(k)} x, T^{n(k)} x) \geq \frac{\varepsilon}{3}. \tag{2.6}$$

For otherwise let $m_1(k) > n(k)$ be the highest positive integer for which (2.6) holds. Since $c_k \to 0$ as $k \to \infty$ we can find $m_2 \in \mathbb{N}$ such that

$$c_k = d(T^k x, T^{k-1} x) < \frac{\varepsilon}{3} \qquad \text{for all} \quad k \geq m_2. \tag{2.7}$$

Now if $m_0 = \max\{m_1, m_2\}$ then for any $i, j > m_0$,

$$\begin{aligned} d(T^i x, T^j x) &= d(T^i x, T^{i+1} x) \\ &< \frac{\varepsilon}{3} < \varepsilon \qquad \text{if} \quad j = i+1, \qquad \text{or,} \\ d(T^i x, T^j x) &\leq d(T^i x, T^{i+1} x) + d(T^{i+1} x, T^n x) + d(T^n x, T^j x) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \qquad \text{if} \quad j > i+1 \end{aligned}$$

which contradicts (2.5).

Now in the view of (2.6) we can choose $m(k)$ as the least positive integer greater than $n(k) + 2$ for which

$$d_k = d(T^{m(k)} x, T^{n(k)} x) \geq \frac{\varepsilon}{3} \qquad \text{for all} \quad k \in \mathbb{N}. \tag{2.8}$$

Assume that $k \geq m_2$. Now if

(i) $m \geq n+5$ then clearly

$$\begin{aligned} d_k = d(T^m x, T^n x) &\leq d(T^m x, T^{m-1}x) + d(T^{m-1}x, T^{m-2}x) + d(T^{m-2}x, T^n x) \\ &< c_m + c_{m-1} + \frac{\varepsilon}{3} \qquad \text{(in view of (2.8))} \\ &< 2c_k + \frac{\varepsilon}{3}. \end{aligned}$$

(ii) If $m = n+3$ then, by (2.7), $d(T^{m-2}x, T^n x) < \frac{\varepsilon}{3}$, so

$$\begin{aligned} d_k = d(T^m x, T^n x) &\leq d(T^m x, T^{m-1}x) + d(T^{m-1}x, T^{m-2}x) + d(T^{m-2}x, T^n x) \\ &< 2c_k + \frac{\varepsilon}{3} \qquad \text{(as above)}. \end{aligned}$$

(iii) If $m = n+4$ then

$$\begin{aligned} d_k &= d(T^m x, T^n x) \\ &\leq d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) + d(T^{n+2}x, T^{n+3}x) + d(T^{n+3}x, T^{n+4}x) \\ &< 3c_k + \frac{\varepsilon}{3}. \end{aligned}$$

Hence $d_k \to \frac{\varepsilon}{3}^+$ as $k \to \infty$.

Again

$$\begin{aligned} d_k = d(T^m x, T^n x) &\leq d(T^m x, T^{m+1}x) + d(T^{m+1}x, T^{n+1}x) + d(T^{n+1}x, T^n x) \\ &= c_{m+1} + d(T^{m+1}x, T^{n+1}x) + c_{n+1} \\ &\leq 2c_k + \psi(d(T^m x, T^n x)) \\ &= 2c_k + \psi(d_k). \end{aligned} \tag{2.9}$$

Thus as $k \to \infty$, from (2.9), we obtain $\frac{\varepsilon}{3} \leq \psi(\frac{\varepsilon}{3})$ which contradicts the given condition since $\varepsilon > 0$. Therefore in this case $\{T^n x\}$ is g.m.s Cauchy and as X is complete, $\{T^n x\}$ converges to a point x_0 in X.

We shall show that $Tx_0 = x_0$. We divide the proof into two parts. First let $T^n x$ be different from both x_0 and Tx_0 for any $n \in \mathbb{N}$. Then

$$\begin{aligned} d(x_0, Tx_0) &\leq d(Tx_0, T^n x) + d(T^n x, T^{n+1}x) + d(T^{n+1}x, Tx_0) \\ &\leq d(x_0, T^n x) + c_{n+1} + \psi(d(x_0, T^n x_0)) \\ &< d(x_0, T^n x) + c_{n+1} + d(x_0, T^n x_0) \\ &\to 0 \qquad \text{as} \quad n \to \infty \end{aligned}$$

which implies $Tx_0 = x_0$.

Next assume that $T^k x = x_0$ or $T^k x = Tx_0$ for some $k \in \mathbb{N}$. Obviously then $x_0 \neq x$ and one can easily show that $\{T^n x_0\}$ is a sequence with the following properties

(i) $\lim\limits_{n \to \infty} d(T^n x_0, x_0) = 0$

(ii) $T^n x_0 \neq x_0$ for any $n \in \mathbb{N}$

(iii) $T^p x_0 \neq T^r x_0$ for $p, r \in \mathbb{N}$, $p \neq r$.

Hence proceeding as above it immediately follows that x_0 is a fixed point of T.

That the fixed point of T is unique easily follows from the definition of T. $\square$

Remark 1. As in [2] we note that if we take $\psi(t) = \alpha(t)t$ where α is a decreasing function and $\alpha(t) < 1$ for $t > 0$ then we can obtain the Rakotch's fixed point theorem [7] for contractive mappings $T\colon X \to X$ satisfying the condition

$$d(Tx, Ty) \leq \alpha(d(x,y))d(x,y) \qquad \text{for all} \quad x, y \in X$$

where α is a mapping as defined above.

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