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BANACH-STEINHAUS PROPERTIES OF STRICTLY $\mathscr{N}$-LOCALLY CONVEX SPACES BASED ON THE PRINCIPLE OF UNIFORM BOUNDEDNESS

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ABSTRACT. Using the principle of uniform boundedness in a strictly $\mathscr{N}$-locally convex spaces, we establish a Banach-Steinhaus-type result for sequentially continuous linear operators.

1. Introduction

Let (X, λ) and (Y, μ) be two locally convex spaces. An operator $T\colon (X, \lambda) \to (Y, \mu)$ is said to be sequentially continuous if $Tx_n \to_\mu Tx$ whenever $\{x_n\}$ is a sequence in X such that $x_n \to_\lambda x$; T is said to be (λ, μ)-bounded if T sends λ-bounded sets into μ-bounded sets. Clearly, continuous operators are sequentially continuous, and sequentially continuous operators are bounded but in general, converse implications fail. Let X'_λ, X^s_λ and X^b_λ denote the families of continuous linear functionals, sequentially continuous linear functionals and bounded linear functionals on (X, λ) respectively. In general, the inclusions $X'_\lambda \subset X^s_\lambda \subset X^b_\lambda$ are strict. Let $\mathscr{B}(X_\lambda)$, $\mathscr{C}(X_\lambda)$ and $\mathscr{C}_0(X_\lambda)$ denote the families of bounded sets in (X, λ), conditionally sequentially compact sets in (X, λ) and convergent sequences in (X, λ) to 0, respectively.

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Let σ be any family of bounded sets in (X, λ). By $\zeta_\sigma(\lambda)$ we denote the topology on X^b_λ generated by the family of semi-norms

$$P_C(T) = \sup_{x \in C} |Tx|, \qquad C \in \sigma.$$

Let $\beta(\lambda, \sigma)$ denote the topology of uniform convergence on $(X^s_\lambda, \zeta_\sigma(\lambda))$-bounded subsets of X^s_λ. A locally convex space (X, τ) is said to be strictly $\mathscr{N}$-locally convex space if there exist a complete norm $\|\cdot\|_X$ on X, a family σ of subsets of X and a locally convex topology λ on X such that

$$\mathscr{C}(X_\lambda) \subset \sigma \subset \mathscr{B}(X) \equiv \mathscr{B}(X, \|\cdot\|_X) \subset \mathscr{B}(X_\lambda) \qquad \text{and} \qquad \tau = \beta(\lambda, \sigma).$$

We can give many examples of such spaces. For example, if we take any Banach space E, then $(E, \|\cdot\|_E) = \big(E, \beta(\|\cdot\|_E, \mathscr{B}(E))\big)$ is a strictly $\mathscr{N}$-locally convex space. Assume now that $(X, \beta(\lambda, \sigma))$ and $(Y, \beta(\mu, \sigma_1))$ are two strictly $\mathscr{N}$-locally convex spaces respectively.

Denote by $\mathscr{B}_1(Y)$ the unit ball in $(Y, \|\cdot\|_Y)$. Let $\mathscr{L}^s(X_\lambda, Y_\mu)$ and $\mathscr{L}(X, Y)$ denote the families of sequentially continuous linear mappings from (X, λ) to (Y, μ) and continuous linear mappings from $(X, \|\cdot\|_X)$ to $(Y, \|\cdot\|_Y)$. Denote by X' the space of continuous linear functionals on $(X, \|\cdot\|_X)$. Put $\mathscr{F}_1 = \mathscr{B}\big(Y^s_\mu, \zeta_{\sigma_1}(\mu)\big)$.

The following proposition shows that if $(X, \beta(\lambda, \sigma))$ is a strictly $\mathscr{N}$-locally convex space, then $X^s_\lambda \subset X' \equiv (X, \|\cdot\|_X)'$ and every $\zeta_\sigma(\lambda)$-bounded subset of X^s_λ is bounded in $(X', \|\cdot\|_{X'})$.

Proposition 1. *We have $X^s_\lambda \subset X' \equiv (X, \|\cdot\|_X)'$. Moreover, $\mathscr{B}(X^s_\lambda, \zeta_\sigma(\lambda)) \subset \mathscr{B}(X', \|\cdot\|_{X'})$*

Proof. See [7]. □

It follows immediately from Proposition 1 that if $(Y, \beta(\mu, \sigma_1))$ is a strictly $\mathscr{N}$-locally convex space, then

$$\forall A \in \mathscr{F}_1 \;\; \forall y \in Y \;\; \mathscr{P}_A(y) \equiv \sup_{f \in A} |f(y)| < +\infty.$$

Thus, the topology $\beta(\mu, \sigma_1)$ is generated by the family $\mathscr{P}_A$, $A \in \mathscr{F}_1$ of semi-norms on Y. It follows also from the definition of strictly $\mathscr{N}$-locally convex space and Proposition 1 that the topologies μ and $\beta(\mu, \sigma_1)$ are coarser than the norm $\|\cdot\|_Y$. Let $T_n \colon X \to Y$ be a sequence of linear operators. $\{T_n\}$ is said to be (σ_1, μ)-bounded if

$$\forall A \in \mathscr{F}_1 \;\; \exists C_1 > 0 \;\; \forall x \in \mathscr{B}_1(X) \;\; \forall f \in A \;\; \forall n \in \mathbb{N} \;\; |f(T_n x)| \le C_1.$$

Let us recall now the principle of uniform boundedness in a strictly $\mathscr{N}$-locally convex space established in [7].

Theorem 2. *Let $(X, \beta(\lambda, \sigma))$, $(Y, \beta(\mu, \sigma_1))$ be two strictly $\mathscr{N}$-locally convex spaces and let $T_n\colon X \to Y$ be a sequence of (λ, μ)-sequentially continuous linear operators, $n \in \mathbb{N}$, such that the sequence $\{T_n x\}$ is bounded in $(Y, \beta(\mu, \sigma_1))$ for each $x \in X$. Then $\{T_n\}$ is (σ_1, μ)-bounded.*

Using this principle, we will establish a Banach-Steinhaus type result in strictly $\mathscr{N}$-locally convex spaces for sequentially continuous linear operators.

2. Banach-Steinhaus properties of strictly $\mathscr{N}$-locally convex spaces

Assume for the moment that X, Y are two Banach spaces and assume also that there exists a sequence $T_n\colon X \to Y$ of continuous linear operators such that for some $x \in X$, the sequence $\{T_n x\}$ do not converge in $(Y, \|\cdot\|_Y)$. Then, we cannot apply the Banach-Steinhaus theorem in Banach spaces. But, as we have seen before, if there exist a locally convex topology μ coarser than the norm $\|\cdot\|_Y$ and a family σ_1 of subsets of Y such that

$$\mathscr{C}(Y_\mu) \subset \sigma_1 \subset \mathscr{B}(Y, \|\cdot\|_Y) \subset \mathscr{B}(Y_\mu),$$

then μ and $\beta(\mu, \sigma_1)$ are coarser than the norm $\|\cdot\|_Y$. Consequently, the sequence $\{T_n x\}$ can converge to Tx in (Y, μ) or in $(Y, \beta(\mu, \sigma_1))$ for every $x \in X$ even if it is not converge in $(Y, \|\cdot\|_Y)$. Thus, we can apply the Banach-Steinhaus theorem in the strictly $\mathscr{N}$-locally convex space $(Y, \beta(\mu, \sigma_1))$. We prove under the assumptions: $T_n\colon X \to Y$ is a sequence of (λ, μ)-sequentially continuous linear operators and $\{T_n x\}$ converges to Tx in $(Y, \beta(\mu, \sigma_1))$ for each $x \in X$ that the limit operator T is $(\beta(\lambda, \sigma), \beta(\mu, \sigma_1))$-sequentially continuous. It is Banach-Steinhaus theorem in strictly $\mathscr{N}$-locally convex spaces.

Theorem 3. *Let $(X, \beta(\lambda, \sigma))$, $(Y, \beta(\mu, \sigma_1))$ be two strictly $\mathscr{N}$-locally convex spaces and let $T_n\colon X \to Y$ be a sequence of (λ, μ)-sequentially continuous linear operators, $n \in \mathbb{N}$, such that the sequence $\{T_n x\}$ converges to Tx in $(Y, \beta(\mu, \sigma_1))$ for each $x \in X$. Then the limit operator T is $(\beta(\lambda, \sigma), \lambda(\mu, \sigma_1))$-sequentially continuous.*

P r o o f. Since $\{T_n x\}$ converges to Tx in $(Y, \beta(\mu, \sigma_1))$ for every $x \in X$, then the sequence $\{T_n x\}$ is bounded in $(Y, \beta(\mu, \sigma_1))$ for each $x \in X$. Consequently, using Theorem 2, we deduce that $\{T_n\}$ is (σ_1, μ)-bounded. Hence,

$$\forall A \in F_1 \ \ \exists C_1 > 0 \ \ \forall x \in \mathscr{B}_1(X) \ \ \forall f \in A \ \ \forall n \in \mathbb{N} \ \ |f(T_n x)| \le C_1.$$

Let $A \in \mathscr{F}_1$. Then the set $\{f \circ T_n : \ f \in A, \ n \in \mathbb{N}\}$ is $\zeta_\sigma(\lambda)$-bounded.

Now let $\{x_k\}$ be a sequence in X such that $x_k \to x$ in $(X, \beta(\lambda, \sigma))$, and $\varepsilon > 0$. Since $\{f \circ T_n : \ f \in A, \ n \in \mathbb{N}\}$ is $\zeta_\sigma(\lambda)$-bounded, it follows that

$\lim_{k\to\infty} f(T_n x_k) = f(T_n x)$ uniformly in $f \in A$, and $n \in \mathbb{N}$. Hence, there is $k_0 \in \mathbb{N}$ such that

$$\forall f \in A \ \ \forall n \in \mathbb{N} \ \ \forall k \geq k_0 \ \ |f(T_n x_k) - f(T_n x)| < \frac{\varepsilon}{3}.$$

Fix $k \geq k_0$. Since $\lim_{n\to\infty} f(T_n x_k) = f(T x_k)$ uniformly in $f \in A$ and $\lim_{n\to\infty} f(T_n x) = f(Tx)$ uniformly in $f \in A$, there is $n_0 \in \mathbb{N}$ such that

$$|f(T_{n_0} x_k) - f(T x_k)| < \frac{\varepsilon}{3}, \quad |f(T_{n_0} x) - f(Tx)| < \frac{\varepsilon}{3}, \qquad \text{for all} \quad f \in A.$$

Therefore,

$$|f(Tx_k) - f(Tx)| \leq |f(Tx_k) - f(T_{n_0} x_k)| + |f(T_{n_0} x_k) - f(T_{n_0} x)| + |f(T_{n_0} x) - f(Tx)|.$$

This shows that $|f(Tx_k) - f(Tx)| < \varepsilon$, for all $f \in A$. Since $A \in \mathscr{F}_1$ is arbitrary, then $\lim_{k\to\infty} Tx_k = Tx$ in $(Y, \beta(\mu, \sigma_1))$. Thus, T is $(\beta(\lambda,\sigma), \beta(\mu,\sigma_1))$-sequentially continuous. In particular, if X, Y are two Banach spaces, setting $\lambda = \|\cdot\|_X$, $\sigma = \mathscr{B}(X)$, $\mu = \|\cdot\|_Y$, $\sigma_1 = \mathscr{B}(Y)$, we recapture the Banach-Steinhaus theorem in Banach spaces. In this case $\beta(\lambda,\sigma) = \|\cdot\|_X$, $\beta(\mu,\sigma_1) = \|\cdot\|_Y$. $\square$

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