VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-009-0141-4
Math. Slovaca **59** (2009), No. 4, 485–494

PARANORM I-CONVERGENT SEQUENCE SPACES

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(*Communicated by Pavel Kostyrko*)

ABSTRACT. In this article we introduced the sequence spaces $c^I(p)$, $c_0^I(p)$, $m^I(p)$ and $m_0^I(p)$ for $p = (p_k)$, a sequence of positive real numbers. We study some algebraic and topological properties of these spaces. We prove the decomposition theorem and obtain some inclusion relations.

1. Introduction

The notion of I-convergence was studied at initial stage by Kostyrko, Šalát, Wilczyński [3]. Later on it was studied by Šalát, Tripathy and Ziman [8], [9], Demirci [1] and many others.

Let X be a non-empty set. Then a family of sets $I \subseteq 2^X$ (power sets of X) is said to be an *ideal* if I is additive i.e. $A, B \in I \implies A \cup B \in I$ and hereditary i.e. $A \in I$, $B \subseteq A \implies B \in I$.

A non-empty family of sets $\mathscr{F} \subset 2^X$ is said to be a *filter* on X if and only if $\Phi \notin \mathscr{F}$, for each $A, B \in \mathscr{F}$ we have $A \cap B \in \mathscr{F}$ and for each $A \in \mathscr{F}$ and $B \supset A$, implies $B \in \mathscr{F}$.

An ideal $I \subseteq 2^X$ is called *non-trivial* if $I \neq 2^X$.

A non-trivial ideal $I \subset 2^X$ is called *admissible* if and only if $I \supset \big\{\{x\} : x \in X\big\}$.

A non-trivial ideal I is *maximal* if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset.

For each ideal I, there is a filter $\mathscr{F}(I)$ corresponding to I i.e. $\mathscr{F}(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} - K$.

Throughout w denote the class of all sequences.

2000 Mathematics Subject Classification: Primary 40A05, 46A45.
Keywords: ideal, filter, I-convergent, I-null, paranorm.

A sequence $(x_k) \in w$ is said to be I-*convergent* to the number L if for every $\varepsilon > 0$, $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in I$. We write $I\text{-}\lim x_k = L$.

A sequence $(x_k) \in w$ is said to be I-*null* if for every $\varepsilon > 0$, $\{k \in \mathbb{N} : |x_k| \geq \varepsilon\} \in I$. We write $I\text{-}\lim x_k = 0$.

A sequence $(x_k) \in w$ is said to be I-*Cauchy* if for every $\varepsilon > 0$, there exists a number $m = m(\varepsilon)$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \varepsilon\} \in I$.

A sequence $(x_k) \in w$ is said to be I-*bounded* if there exists $M > 0$ such that $\{k \in \mathbb{N} : |x_k| > M\} \in I$.

Let (x_k) and (y_k) be two sequences. We say that $x_k = y_k$ for *almost all k relative to I (a.a.k.r.I)*, if $\{k \in \mathbb{N} : x_k \neq y_k\} \in I$.

2. Definitions and notations

Throughout the article ℓ_∞, c^I, c_0^I, m^I, m_0^I represent the classes of *bounded*, I-*convergent*, I-*null*, *bounded I-convergent*, and *bounded I-null* sequence respectively.

A sequence space E is said to be *solid* (or *normal*) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequence α_k of scalars with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$.

A sequence space E is said to be *symmetric* if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where π is a permutation of $\mathbb{N}$.

A sequence space E is said to be *sequence algebra* if $(x_k) \star (y_k) = (x_k y_k) \in E$ whenever $(x_k), (y_k) \in E$.

A sequence space E is said to be $convergence free$ if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

Let $K = \{k_1 < k_2 < \dots\} \subseteq \mathbb{N}$ and E be a sequence space. A K-*step space* of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in w : (k_n) \in E\}$.

A canonical preimage of a sequence $\{(x_{k_n})\} \in \lambda_K^E$ is a sequence $\{y_n\} \in w$ defined as

$$y_n = \begin{cases} x_n, & \text{if } n \in K; \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all elements in λ_K^E, i.e. y is in canonical preimage of λ_K^E if and only if y is canonical preimage of some $x \in \lambda_K^E$.

A sequence space E is said to be *monotone* if it contains the canonical preimages of its step spaces.

The notion of paranormed sequence space was studied at the initial stage by Nakano [7] and Simons [10]. Later on it was further investigated by Maddox [6], Lascarides [4], [5], Tripathy and Sen [11] and many others.

In this article, we introduce the following sequence spaces. Let $p = (p_k)$ be a sequence of positive real numbers. Then for a given $\varepsilon > 0$,

$$\begin{aligned} c^I(p) &= \big\{(x_k) \in w : \ \{k \in \mathbb{N} : \ |x_k - L|^{p_k} \geq \varepsilon\} \in I, \ \text{ for some } \ L \in C\big\}; \\ c_0^I(p) &= \big\{(x_k) \in w : \ \{k \in \mathbb{N} : \ |x_k|^{p_k} \geq \varepsilon\} \in I\big\}; \\ \ell_\infty &= \big\{(x_k) \in w : \ \sup_k |x_k|^{p_k} < \infty\big\}. \end{aligned}$$

We write

$$\begin{aligned} m^I(p) &= c^I(p) \cap \ell_\infty(p); \\ m_0^I(p) &= c_0^I(p) \cap \ell_\infty(p). \end{aligned}$$

Lascarides [4] defined the following sequence spaces:

$$\begin{aligned} \ell_\infty\{p\} &= \Big\{(x_k) \in w : \ \text{there exists } \ r > 0 \ \text{ such that } \ \sup_k |x_k r|^{p_k} t_k < \infty\Big\}; \\ c_0\{p\} &= \Big\{(x_k) \in w : \ \text{there exists } \ r > 0 \ \text{ such that } \ \lim_{k\to\infty} |x_k r|^{p_k} t_k = 0\Big\}; \\ \ell\{p\} &= \Big\{(x_k) \in w : \ \text{there exists } \ r > 0 \ \text{ such that } \ \sum_{k=1}^{\infty} |x_k r|^{p_k} t_k > \infty\Big\}, \end{aligned}$$

where $t_k = p_k^{-1}$, for all $k \in \mathbb{N}$.

The following results will be used for establishing some results of this article.

Lemma 1. *A sequence space E is said solid implies E is monotone.*

(One may refer to Kamthan and Gupta [2, p. 53].)

Lemma 2. (Šalát, Tripathy and Ziman [8, Lemma 2.5]) *Let $K \in \mathscr{F}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.*

Lemma 3. (Kostyrko, Šalát, Wilczyński [3, Lemma 5.1]) *If $I \subset 2^{\mathbb{N}}$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.*

Lemma 4. (Lascarides [4, Proposition 1]) *Let $h = \inf_k p_k$, $H = \sup_k p_k$. Then the following conditions are equivalent:*

(i) $H < \infty$ *and* $h > 0$;

(ii) $c_0(p) = c_0$ *or* $\ell_\infty(p) = \ell_\infty$;

(iii) $\ell_\infty\{p\} = \ell_\infty(p)$;

(iv) $c_0\{p\} = c_0(p)$;

(v) $\ell\{p\} = \ell(p)$.

3. Main results

Theorem 1. *Let $(p_k) \in \ell_\infty$. Then $c^I(p)$, $c_0^I(p)$, $m^I(p)$ and $m_0^I(p)$ are linear spaces.*

Proof. Let $(x_k), (y_k) \in c^I(p)$ and α, β be two scalars. Then for a given $\varepsilon > 0$, we have

$$\Big\{k \in \mathbb{N}: \ |x_k - L_1|^{p_k} \geq \tfrac{\varepsilon}{2M_1}, \ \text{ for some } \ L_1 \in C\Big\} \in I;$$
$$\Big\{k \in \mathbb{N}: \ |y_k - L_2|^{p_k} \geq \tfrac{\varepsilon}{2M_2}, \ \text{ for some } \ L_2 \in C\Big\} \in I,$$

where

$$M_1 = D \cdot \max\Big\{1, \sup_k |\alpha|^{p_k}\Big\};$$
$$M_2 = D \cdot \max\Big\{1, \sup_k |\beta|^{p_k}\Big\}$$

and

$$D = \max\{1, 2^{G-1}\} \qquad \text{and} \qquad G = \sup_k p_k \geq 0.$$

Let

$$A_1 = \Big\{k \in \mathbb{N}: \ |x_k - L_1|^{p_k} < \tfrac{\varepsilon}{2M_1}, \ \text{ for some } \ L_1 \in C\Big\} \in I;$$
$$A_2 = \Big\{k \in \mathbb{N}: \ |y_k - L_2|^{p_k} < \tfrac{\varepsilon}{2M_2}, \ \text{ for some } \ L_2 \in C\Big\} \in I,$$

be such that $A_1^c, A_2^c \in I$.

Then $A_3 = \big\{k \in \mathbb{N}: \ |(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|^{p_k} < \varepsilon\big\} \supseteq \big\{k \in \mathbb{N}: |\alpha|^{p_k}|x_k - L_1|^{p_k} < \frac{\varepsilon}{2M_1}|\alpha|^{p_k} D\big\} \cap \big\{k \in \mathbb{N}: \ |\beta|^{p_k}|y_k - L_2|^{p_k} < \frac{\varepsilon}{2M_2}|\beta|^{p_k} D\big\}$.

Thus $A_3^c = A_1^c \cap A_2^c \in I$. Hence $(\alpha(x_k) + \beta(y_k)) \in c^I(p)$. Therefore $c^I(p)$ is a linear space. The rest of the results follows similarly. □

Note 1. Let $I = I_\delta$ and $(p_k) \notin \ell_\infty$. Consider $J = \{n_i \in \mathbb{N}: \ i \in \mathbb{N}\}$ be an infinite set such that $\lim_{i\to\infty} p_{k_i} = \infty$. Let $D = \{n_i : \ i \in \mathbb{N}\} \subset J$ such that $\delta(D) = 0$. Consider the sequence (x_k) defined by

$$x_k = \begin{cases} 0, & \text{if } \ k \in D; \\ 1, & \text{otherwise.} \end{cases}$$

Then $\text{stat-}\lim x_k = 1$, so $(x_k) \in c^I(p)$. For any scalar $\lambda \geq 1$, we have $\text{stat-}\lim(\lambda x_k) = \infty$. Therefore $\lambda(x_k) \notin c^I(p)$. Thus $c^I(p)$ is not a linear space. Therefore the condition $(p_k) \in \ell_\infty$ is necessary in Theorem 1.

Theorem 2. *Let $(p_k) \in \ell_\infty$, then $m^I(p)$ and $m_0^I(p)$ are paranormed spaces, paranormed by $g((x_k)) = \sup_k |x_k|^{\frac{p_k}{M}}$, where $M = \max\big\{1, \sup_k p_k\big\}$.*

The proof of the result is easy, so omitted.

Theorem 3. *$m^I(p)$ is a closed subspace of $\ell_\infty(p)$.*

Proof. Let $(x_k^{(n)})$ be a Cauchy sequence in $m^I(p)$ such that $x^{(n)} \to x$. We show that $x \in m^I(p)$.

Since $(x_k^{(n)}) \in m^I(p)$, then there exists a_n such that

$\{k \in \mathbb{N}: \ |x_k^{(n)} - a_n|^{p_k} \geq \varepsilon\} \in I$.

We need to show that

(i) (a_n) converges to a,

(ii) if $U = \{k \in \mathbb{N}: \ |x_k - a|^{p_k} < \varepsilon\}$, then $U^c \in I$.

(i) Since $(x_k^{(n)})$ is a Cauchy sequence in $m^I(p)$ then for a given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\sup_k \big|x_k^{(n)} - x_k^{(m)}\big|^{\frac{p_k}{M}} < \frac{\varepsilon}{3}, \qquad \text{for all} \quad n, m \geq k_0.$$

Given $\varepsilon > 0$, we have

$$\begin{aligned} B_{nm} &= \big\{k \in \mathbb{N}: \ |x_k^{(n)} - x_k^{(m)}|^{p_k} < \big(\tfrac{\varepsilon}{3}\big)^M\big\}; \\ B_m &= \big\{k \in \mathbb{N}: \ |x_k^{(m)} - a_m|^{p_k} < \big(\tfrac{\varepsilon}{3}\big)^M\big\} \\ B_n &= \big\{k \in \mathbb{N}: \ |x_k^{(n)} - a_n|^{p_k} < \big(\tfrac{\varepsilon}{3}\big)^M\big\}. \end{aligned}$$

Then $B_{nm}^c, B_m^c, B_n^c \in I$.

Let $B^c = B_{nm}^c \cap B_m^c \cap B_n^c$, where $B = \{k \in \mathbb{N}: \ |a_m - a_n|^{p_k} < \varepsilon\}$. Then $B^c \in I$.

We choose $k_0 \in B^c$. Then for each $n, m \geq k_0$, we have

$$\begin{aligned} \big\{k \in \mathbb{N}: \ |a_m - a_n|^{p_k} < \varepsilon\big\} \supseteq \Big[&\big\{k \in \mathbb{N}: \ |a_m - x_k^{(m)}|^{p_k} < \big(\tfrac{\varepsilon}{3}\big)^M\big\} \\ &\cap \big\{k \in \mathbb{N}: \ |x_k^{(m)} - x_k^{(n)}|^{p_k} < \big(\tfrac{\varepsilon}{3}\big)^M\big\} \\ &\cap \big\{k \in \mathbb{N}: \ |x_k^{(n)} - a_n|^{p_k} < \big(\tfrac{\varepsilon}{3}\big)^M\big\}\Big]. \end{aligned}$$

Then (a_n) is a Cauchy sequence of scalars in C, so there exists a scalar a in C such that $a_n \to a$, as $n \to \infty$.

(ii) Let $0 < \delta < 1$ be given. Then we show that if $U = \{k \in \mathbb{N}: \ |x_k - a|^{p_k} < \delta\}$, then $U^c \in I$. Since $x^{(n)} \to x$, then there exists $q_0 \in \mathbb{N}$ such that

$$P = \Big\{k \in \mathbb{N}: \ \big|(x_k^{(q_0)}) - (x_k)\big| < \big(\tfrac{\delta}{3D}\big)^M\Big\}, \tag{1}$$

implies $P^c \in I$.

The number q_0 can be so chosen that together with (1), we have $Q = \{k \in \mathbb{N}: |a_{q_0} - a|^{p_k} < \big(\tfrac{\delta}{3D}\big)^M\}$ such that $Q^c \in I$.

Since $\{k \in \mathbb{N}: \ |x_k^{(q_0)} - a_{q_0}|^{p_k} \geq \delta\} \in I$. Then we have a subset S of $\mathbb{N}$ such that $S^c \in I$, where $S = \{k \in \mathbb{N}: \ |x_k^{(q_0)} - a_{q_0}|^{p_k} < \left(\frac{\delta}{3D}\right)^M\}$.

Let $U^c = P^c \cup Q^c \cup S^c$, when $U = \{k \in \mathbb{N}: \ |x_k - a|^{p_k} < \delta\}$.

Therefore for each $k \in U^c$, we have

$$\begin{aligned}\{k \in \mathbb{N}: \ |x_k - a|^{p_k} < \delta\} \supseteq \Big[& \Big\{k \in \mathbb{N}: \ \big|(x_k) - \big(x_k^{(q_0)}\big)\big|^{p_k} < \left(\tfrac{\delta}{3D}\right)^M\Big\} \\ & \cap \Big\{k \in \mathbb{N}: \ \big|\big(x_k^{(q_0)}\big) - a_{q_0}\big|^{p_k} < \left(\tfrac{\delta}{3D}\right)^M\Big\} \\ & \cap \Big\{k \in \mathbb{N}: \ |a_{q_0} - a|^{p_k} < \left(\tfrac{\delta}{3D}\right)^M\Big\}\Big].\end{aligned}$$

Then the result follows. □

Since the inclusions $m^I(p) \subset \ell_\infty(p)$ and $m_0^I(p) \subset \ell_\infty(p)$ are strict, so in view of Theorem 3, we have the following result.

THEOREM 4. *The spaces $m^I(p)$ and $m_0^I(p)$ are nowhere dense subsets of $\ell_\infty(p)$.*

THEOREM 5. *The spaces $c_0^I(p)$ and $m_0^I(p)$ are both solid and monotone.*

Proof. Let $(x_k) \in c_0^I(p)$ and (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$, for all $k \in \mathbb{N}$. Since $|\alpha_k|^{p_k} \leq \max\{1, |\alpha_k|^G\} \leq 1$, for all $k \in \mathbb{N}$, we have,

$$|\alpha_k x_k|^{p_k} \leq |x_k|^{p_k}, \qquad \text{for all} \quad k \in \mathbb{N}.$$

The space $c_0^I(p)$ is solid follows from the following inclusion relation.

$$\{k \in \mathbb{N}: \ |x_k|^{p_k} \geq \varepsilon\} \supseteq \{k \in \mathbb{N}: \ |\alpha_k x_k|^{p_k} \geq \varepsilon\}.$$

The space $c_0^I(p)$ is monotone follows by Lemma 1.

The other part follows similarly. □

RESULT 1. *The spaces $c^I(p)$ and $m^I(p)$ are neither monotone nor solid, if I is neither maximal nor $I = I_f$.*

Proof. We prove this result with the help of the following example. □

Example 1. Let $I = I_\delta$. Let $p_k = 1$ if k is even and $p_k = 2$, if k is odd.

Consider the Kth-step spaces E_K of E defined as follows:

Let $(x_k) \in E$ and $(y_k) \in E_K$ be such that

$$y_k = \begin{cases} x_k, & \text{if } k \text{ is odd;} \\ 1, & \text{otherwise.} \end{cases}$$

Consider the sequence (x_k) as $x_k = k^{-1}$, for all $k \in \mathbb{N}$. Then $(x_k) \in Z(p)$, but its Kth-step space preimage does not belong to $Z(p)$, where $Z = c^I$ and m^I.

Thus $c^I(p)$ and $m^I(p)$ are not monotone. The spaces $c^I(p)$ and $m^I(p)$ are not solid follows by Lemma 1.

Result 2. *If I is neither maximal nor $I = I_f$, then the spaces $Z(p)$ are not symmetric, where $Z = c_0^I, c^I, m_0^I$ and m^I.*

P r o o f. We prove the result with the help of the following example. □

Example 2. Let $I = I_c$. Let $A_0 = \{k : \ k = s^2 \text{ or } t^3, \text{ for } s, t \in \mathbb{N}\} = \{k \in \mathbb{N} : k = s^2, \text{ for } s \in \mathbb{N}\} \cup \{k \in \mathbb{N} : \ k = t^3, \text{ for } t \in \mathbb{N}\}$, then $\sum\limits_{k \in A_0} k^{-1} < \infty$.

Let $p_k = 1$, if k is even and $p_k = 2$, if k is odd.

Consider the sequence (x_k) defined as follows:

$$y_k = \begin{cases} k^{-1} & \text{if } k = t^3, \ t \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the rearrangement (y_k) of (x_k) defined by

$$(y_k) = (x_1, x_2, x_3, x_8, x_4, x_5, x_{27}, x_6, x_7, x_{64}, x_8, x_9, \dots).$$

Then $(y_k) \notin Z(p)$, but $(x_k) \in Z(p)$, where $Z = c_0^I, c^I, m_0^I$ and m^I.

Theorem 6. *Let (p_k) and (q_k) be two sequences of positive real numbers. Then $m_0^I(p) \supseteq m_0^I(q)$ if and only if $\liminf\limits_{k \in K} \frac{p_k}{q_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.*

P r o o f. Let $\liminf\limits_{k \in K} \frac{p_k}{q_k} > 0$ and $(x_k) \in m_0^I(q)$. Then there exists $\beta > 0$ such that $p_k > \beta q_k$, for all sufficiently large $k \in K$.

Since $(x_k) \in m_0^I(q)$, for a given $\varepsilon > 0$, we have

$$B_0 = \{k \in \mathbb{N} : \ |x_k|^{q_k} \geq \varepsilon\} \in I.$$

Let $G_0 = K^c \cup B_0$. Then $G_0 \in I$.

Then for all sufficiently large $k \in G_0$,

$$\{k \in \mathbb{N} : \ |x_k|^{p_k} \geq \varepsilon\} \subseteq \{k \in \mathbb{N} : \ |x_k|^{\beta q_k} \geq \varepsilon\} \in I.$$

Therefore $(x_k) \in m_0^I(p)$.

The converse part of the result follows obviously. □

The following result follows from Theorem 6.

Corollary 1. *Let (p_k) and (q_k) be two sequences of positive real numbers. Then $m_0^I(p) = m_0^I(q)$ if and only if $\liminf\limits_{k \in K} \frac{p_k}{q_k} > 0$, and $\liminf\limits_{k \in K} \frac{q_k}{p_k} > 0$ where $K \subseteq \mathbb{N}$ such that $K^c \in I$.*

Theorem 7. *Let $h = \inf\limits_k p_k$ and $H = \sup\limits_k p_k$, then the following results are equivalent:*

(a) *$G < \infty$ and $h > 0$;*

(b) *$c_0^I(p) = c_0^I$.*

P r o o f. Suppose first that $h > 0$ and $G < \infty$, then the inequalities $\min\{1, s^h\} \leq s^{p_k} \leq \max\{1, s^G\}$ hold for any $s > 0$ and for all $k \in \mathbb{N}$.

Therefore the equivalent of (a) and (b) is obvious. $\square$

Result 3. *The spaces $m_0^I(p)$ and $m^I(p)$ are not separable.*

P r o o f. Let $M = \{m_1 < m_2 < \dots\}$ be an infinite subset of of $\mathbb{N}$ such that $M \in I$.

Let

$$p_k = \begin{cases} 1, & \text{if } k \in M; \\ 2, & \text{otherwise.} \end{cases}$$

Let $P_0 = \big\{(x_k) : \ x_k = 0, \text{ or } 1, \text{ for } k = m_j, \ j \in \mathbb{N} \text{ and } x_k = 0, \text{ otherwise}\big\}$. Since M is infinite, so P_0 is uncountable.

Consider the class of open balls $B_1 = \big\{B\big(z, \frac{1}{2}\big) : \ z \in P_0\big\}$. Let C_1 be an open cover of $m_0^I(p)$ or $m^I(p)$ containing B_1. Since B_1 is uncountable, so C_1 cannot be reduced to a countable subcover for $m_0^I(p)$ as well as $m^I(p)$. Thus $m_0^I(p)$ and $m^I(p)$ are not separable. $\square$

Theorem 8. *Let $G = \sup_k p_k < \infty$ and I an admissible ideal. Then the followings are equivalent:*

(a) $(x_k) \in c^I(p)$;

(b) *there exists $(y_k) \in c(p)$ such that $x_k = y_k$, for a.a.k.r.I;*

(c) *there exists $(y_k) \in c(p)$ and $(z_k) \in c_0^I(p)$ such that $x_k = y_k + z_k$, for all $k \in \mathbb{N}$ and $\big\{k \in \mathbb{N} : \ |y_k - L|^{p_k} \geq \varepsilon\big\} \in I$;*

(d) *there exists a subset $K = \{k_1 < k_2 < \dots\}$ of $\mathbb{N}$ such that $K \in \mathscr{F}(I)$ and $\lim\limits_{n\to\infty} |x_{k_n} - L|^{p_{k_n}} = 0$.*

P r o o f.

(a) $\implies$ (b). Let $(x_k) \in c^I(p)$. Then there exists $L \in C$ such that

$$\big\{k \in \mathbb{N} : \ |x_k - L|^{p_k} \geq \varepsilon\big\} \in I.$$

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\big\{k \leq m_t : \ |x_k - L|^{p_k} \geq \tfrac{1}{t}\big\} \in I.$$

Define a sequence (y_k) as follows;

$$y_k = x_k, \qquad \text{for all} \quad k \leq m_1.$$

For $m_t < k \leq m_{t+1}$, $t \in \mathbb{N}$,

$$y_k = \begin{cases} x_k, & \text{if } |x_k - L|^{p_k} < t^{-1}; \\ L, & \text{otherwise.} \end{cases}$$

Then $(y_k) \in c(p)$ and form the following inclusion

$$\left\{k \le m_t : x_k \ne y_k\right\} \subseteq \left\{k \in \mathbb{N} : \ |x_k - L|^{p_k} \ge \varepsilon\right\} \in I.$$

We get $x_k = y_k$, for *a.a.k.r.I.*

(b) $\Longrightarrow$ (c). For $(x_k) \in c^I(p)$, then there exists $(y_k) \in c(p)$ such that $x_k = y_k$, for *a.a.k.r.I.* Let $K = \left\{k \in \mathbb{N} : \ x_k \ne y_k\right\}$, then $K \in I$.

Define (z_k) as follows:

$$z_k = \begin{cases} x_k - y_k, & \text{if } k \in K; \\ 0, & \text{if } k \notin K. \end{cases}$$

Then $(z_k) \in c_0^I(p)$ and $(y_k) \in c(p)$.

(c) $\Longrightarrow$ (d). Suppose (c) holds. Let $\varepsilon > 0$ be given.

Let $P_1 = \left\{k \in \mathbb{N} : \ |z_k|^{p_k} \ge \varepsilon\right\}$ and $K = P_1^c = \{k_1 < k_2 < \dots\} \in \mathscr{F}(I)$.

Then we have

$$\lim_{n\to\infty} |x_{k_n} - L|^{p_{k_n}} = 0.$$

(d) $\Longrightarrow$ (a). Let $K = \{k_1 < k_2 < \dots\} \in \mathscr{F}(I)$ and $\lim\limits_{n\to\infty} |x_{k_n} - L|^{p_{k_n}} = 0$. Then for any $\varepsilon > 0$, and Lemma 2, we have

$$\left\{k \in \mathbb{N} : \ |x_k - L|^{p_k} \ge \varepsilon\right\} \subseteq K^c \cup \left\{k \in K : \ |x_k - L|^{p_k} \ge \varepsilon\right\}.$$

Thus $(x_k) \in c^I(p)$. □

Acknowledgement. The authors thank the referee for his comments that improved the presentation of the paper.

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Received 4. 9. 2006

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