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NON-OSCILLATORY CRITERIA FOR A CLASS OF SECOND ORDER NON-LINEAR FORCED NEUTRAL-DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, sufficient conditions are obtained, so that the second order neutral delay differential equation

$$\big(r(t)\,(y(t)-p(t)y(t-\tau))'\big)' + q(t)G\,(y(h(t))) = f(t)$$

has a positive and bounded solution, where $q, h, f \in C\,([0,\infty),\mathbb{R})$ such that $q(t) \geq 0$, but $\not\equiv 0$, $h(t) \leq t$, $h(t) \to \infty$ as $t \to \infty$, $r \in C^{(1)}\,([0,\infty),(0,\infty))$, $p \in C^{(2)}\,([0,\infty),\mathbb{R})$, $G \in C(\mathbb{R},\mathbb{R})$ and $\tau \in \mathbb{R}^+$. In our work $r(t) \equiv 1$ is admissible and neither we assume G is non-decreasing, $xG(x) > 0$ for $x \neq 0$, nor we take G is Lipschitzian. Hence the results of this paper improve many recent results.

1. Introduction

In this paper we find sufficient conditions for the neutral delay differential equation (NDDE in short) of second order

$$\big(r(t)\,(y(t)-p(t)y(t-\tau))'\big)' + q(t)G\,(y(h(t))) = f(t) \tag{E}$$

to have a bounded positive solution which does not tend to zero as $t \to \infty$, where $q, h, f \in C\,([0,\infty),\mathbb{R})$ such that $q(t) \geq 0$, but $\not\equiv 0$, $h(t) \leq t$, $h(t) \to \infty$ as $t \to \infty$, $r \in C^{(1)}\,([0,\infty),(0,\infty))$, $p \in C^{(2)}([0,\infty),\mathbb{R})$, $G \in C(\mathbb{R},\mathbb{R})$ and $\tau \in \mathbb{R}^+$.

We need some of the following assumptions in the sequel.

(H_1) There exists a bounded function $F(t)$ such that $F'(t) = f(t)$.

(H_2) $\int\limits_{t_0}^{\infty} q(t)\,\mathrm{d}t < \infty$.

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(H_3) $\int_{t_0}^{\infty} \frac{\mathrm{d}t}{r(t)} = \infty$.

(H_4) $\int_{t_0}^{\infty} \frac{\mathrm{d}t}{r(t)} < \infty$.

(H_5) $\int_{t_0}^{\infty} \left(\frac{1}{r(t)} \int_{t}^{\infty} q(s)\,\mathrm{d}s \right) \mathrm{d}t < \infty$.

Remark 1. Since $r(t) > 0$, therefore:

(i) either (H_3) or (H_4) holds exclusively.

(ii) If (H_3) holds then (H_5) implies (H_2) but not conversely.

(iii) If (H_4) holds then (H_2) implies (H_5) but not conversely.

In recent years there have been increasing interest among many authors all over the world to study oscillation and non-oscillation properties of neutral delay differential equations. We observe that the even order neutral differential equations are not so often studied as the odd order neutral differential equations have been. The authors have proved the existence of a bounded positive solution of neutral delay differential equations of various order in [1], [2], [5], [6], [7], [8], [9]. For that the authors assume the following hypothesis.

(H_6) There exists a function $F(t)$ such that $F(t) \to 0$ as $t \to \infty$ and $F'(t) = f(t)$.

(H_7) $\left| \int_{t_0}^{\infty} f(t)\,\mathrm{d}t \right| < \infty$.

(H_8) G is Lipschitzian in every interval of the form $[a, b]$, with $0 < a < b$.

(H_9) $xG(x) > 0$ for $x \neq 0$, and G is non-decreasing.

It is obvious that (H_6) $\iff$ (H_7) and (H_1) is weaker than both (H_6) and (H_7). In this paper since we formulate our results with (H_1) and do not assume either (H_8) or (H_9) therefore our work improve some of the results of [1], [5], [6], [7], [8], [9]. Further one may observe an important point that the authors have found positive solutions of neutral delay differential equation

$$\left(y(t) - p(t)y(t-\tau)\right)^{(n)} + q(t)G\left(y(t-\sigma)\right) = f(t) \tag{1.1}$$

for $n \geq 2$, in different ranges of $p(t)$. But for $p(t) \equiv -1$ there is no result in these papers. However, in this work we consider $p(t)$ in different ranges including $p(t) = \pm 1$. Further the equation we consider i.e (E) is more general than (1.1) for $n = 2$.

Let $T_y > 0$ and $T_0 = \min\{h(T_y), T_y - \tau\}$. Suppose $\phi \in C\left([T_0 T_y], \mathbb{R}\right)$. By a solution of (E). We mean a real valued continuous function $y \in C^{(2)}\left([T_y, \infty), \mathbb{R}\right)$ such that $y(t) = \phi(t)$ for $T_0 \leq t \leq T_y$ and $\left(y(t) - p(t)y(t-\tau)\right)$ is differentiable, $r(t)\left(y(t) - p(t)y(t-\tau)\right)'$ is again differentiable and then (E) is satisfied. Such a solution is said to be oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory.

2. Main results

In this section we assume $p(t)$ to satisfy one of the following conditions.

(A_1) $\quad 0 \le p(t) \le p < 1.$

(A_2) $\quad -1 < -p \le p(t) \le 0.$

(A_3) $\quad -d < p(t) \le -c < -1.$

(A_4) $\quad 1 < c \le p(t) < d.$

For our work we need the following lemma from [3].

Lemma 2.1 (**Krasnoselskiis Fixed point Theorem**). ([3]) *Let X be a Banach space. Let Ω be a bounded closed convex subset of X and let S_1, S_2 be maps of Ω into X such that $S_1 x + S_2 y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contraction and S_2 is completely continuous, then the equation*

$$S_1 x + S_2 x = x$$

has a solution in Ω.

Theorem 2.2. *Let (A_1), (H_1), (H_4) and (H_5) hold. Then there exists a bounded solution of* (E) *which is bounded below by a positive constant i.e there exists a solution of* (E) *which neither oscillates nor tends to zero as $t \to \infty$.*

Proof. From (H_1) since $F(t)$ is bounded, we find $\alpha > 0$ and $t_1 > 0$ such that

$$F(t) < \alpha \qquad \text{for} \quad t \ge t_1. \tag{2.1}$$

Since $G \in C(\mathbb{R}, \mathbb{R})$, then let

$$\mu = \max\left\{G(x) : \ \tfrac{3}{4}(1-p) \le x \le 1\right\}. \tag{2.2}$$

Then using (2.1) and (H_4) we find t_2 such that $t \ge t_2$ implies

$$\left| \int_t^\infty \frac{F(s)}{r(s)} \, \mathrm{d}s \right| < \frac{1-p}{20}. \tag{2.3}$$

From (H_5) we find t_3 such that $t > t_3$ implies

$$\mu \int_t^\infty \frac{1}{r(s)} \left(\int_s^\infty q(u) \, \mathrm{d}u \right) \mathrm{d}s < \frac{1-p}{10}. \tag{2.4}$$

Let $T = \max\{t_1, t_2, t_3\}$ and $T_0 = \min\{T - \tau, h(T)\}$. Then for $t \ge T$, (2.1), (2.3) and (2.4) hold. Let $X = C\,([T_0, \infty), \mathbb{R})$ be the set of all continuous functions with norm $\|x\| = \sup\limits_{t \ge T_0} |x(t)| < \infty$. Clearly X is a Banach space. Let

$$S = \big\{u \in BC\,([T_0, \infty), \mathbb{R}) :\ \tfrac{3}{4}(1-p) \le u(t) \le 1\big\} \tag{2.5}$$

with supremum norm $\|u\| = \sup\{|u(t)| :\ t \ge T_0\}$. Clearly S is a closed, bounded and convex subset of $C\,([T_0, \infty), \mathbb{R})$.

Define two maps A and $B\colon S \to X$ as follows. For $x \in S$ define

$$Ax(t) = \begin{cases} Ax(T), & t \in [T_0, T] \\ p(t)x(t-\tau) + \frac{9(1-p)}{10}, & t \ge T. \end{cases} \tag{2.6}$$

$$Bx(t) = \begin{cases} Bx(T), & \text{for } \ t \in [T_0, T] \\ -\int\limits_t^\infty \frac{1}{r(s)} \left(\int\limits_s^\infty q(u) G(x(h(u)))\, \mathrm{d}u \right) \mathrm{d}s & \\ -\int\limits_t^\infty \frac{F(S)}{r(s)}\, \mathrm{d}s, & \text{for } \ t \ge T. \end{cases} \tag{2.7}$$

First we show that if $x, y \in S$ then $Ax + By \in S$.

In fact, for every $x, y \in S$ and $t \ge T$, we get

$$\begin{aligned} (Ax)(t) + (By)(t) &\le p(t)x(t-\tau) + \frac{9(1-p)}{10} - \int\limits_t^\infty \frac{F(s)}{r(s)}\, \mathrm{d}s \\ &\le p + \frac{9(1-p)}{10} + \frac{1-p}{20} \\ &\le 1. \end{aligned}$$

On the other hand for $t \ge T$

$$\begin{aligned} (Ax)(t) + (By)(t) &\ge \frac{9(1-p)}{10} - \int\limits_t^\infty \frac{1}{r(s)} \left(\int\limits_s^\infty q(u) G(y(h(u)))\, \mathrm{d}u \right) \mathrm{d}s - \int\limits_t^\infty \frac{F(s)}{r(s)}\, \mathrm{d}s \\ &\ge \frac{9(1-p)}{10} - \mu \int\limits_t^\infty \frac{1}{r(s)} \left(\int\limits_s^\infty q(u)\, \mathrm{d}u \right) \mathrm{d}s - \int\limits_t^\infty \frac{F(s)}{r(s)}\, \mathrm{d}s \\ &\ge \frac{9(1-p)}{10} - \frac{1-p}{10} - \frac{1-p}{20} = \frac{3}{4}(1-p). \end{aligned}$$

Hence

$$\frac{3}{4}(1-p) \le (Ax)(t) + (By)(t) \le 1 \qquad \text{for} \quad t \ge T.$$

So that $Ax + By \in S$ for all $x, y \in S$.

Next we show that A is a contraction in S. In fact, for $x, y \in S$ and $t \geq T$, we have

$$\begin{aligned}|(Ax)(t) - (Ay)(t)| &\leq |p(t)\{x(t-\tau) - y(t-\tau)\}| \\ &\leq |p(t)||x(t-\tau) - y(t-\tau)| \\ &\leq p\|x - y\|.\end{aligned}$$

Since $0 < p < 1$ we conclude that A is a contraction mapping on S.

We now show that B is completely continuous. First, we shall show that B is continuous. Let $x_k = x_k(t) \in S$ be such that $\sup_{t \geq T} |x_k(t) - x(t)| \to 0$ as $k \to \infty$.

Because S is closed, $x = x(t) \in S$. For $t \geq T$, we have

$$\begin{aligned}&|(Bx_k)(t) - (Bx)(t)| \\ &\qquad \leq \int_t^\infty \frac{1}{r(s)} \left(\int_s^\infty q(u) |G(x(h(u))) - G(x_k(h(u)))| \,\mathrm{d}u \right) \mathrm{d}s.\end{aligned}$$

Since for all $t \geq T$, $x_k(t)$, $k = 1, 2, \ldots$, tend uniformly to $x(t)$ as $k \to \infty$, it follows that for $t \geq T$, $G(x_k(h(t)))$ tend uniformly to $G(x(h(t)))$ as $k \to \infty$. Hence $\lim_{k \to \infty} |(Bx_k)(t) - (Bx)(t)| = 0$ for $t \geq T$. This means that B is continuous.

Next, we show that BS is relatively compact. It suffices to show that the family of functions $\{Bx : \ x \in S\}$ is uniformly bounded and equicontinuous on $[T_0, \infty)$. The uniform boundedness is obvious. For the equicontinuity, according to Levitan's result we only need to show that, for any given $\varepsilon > 0$, $[T_0, \infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have change of amplitude less than ε. From (H_5) and (H_4) it follows that for any $\varepsilon > 0$, we can find T large enough so that for any $T^* \geq T$ implies

$$\mu \int_{T^*}^\infty \frac{1}{r(s)} \left(\int_s^\infty q(u) \,\mathrm{d}u \right) \mathrm{d}s < \frac{\varepsilon}{4}$$

and

$$\alpha \int_{T^*}^\infty \frac{\mathrm{d}s}{r(s)} < \frac{\varepsilon}{4}.$$

Then for $x \in S$ and $t_2 > t_1 \geq T^*$,

$$\begin{aligned}|(Bx)(t_2) - (Bx)(t_1)| &= \Bigg| -\int_{t_2}^{\infty} \frac{1}{r(s)} \Bigg(\int_{s}^{\infty} q(u) G(x(h(u)))\,\mathrm{d}u \Bigg)\,\mathrm{d}s - \int_{t_2}^{\infty} \frac{F(s)}{r(s)}\,\mathrm{d}s \\ &\quad + \int_{t_1}^{\infty} \Bigg(\int_{s}^{\infty} q(u) G(x(h(u)))\,\mathrm{d}u \Bigg)\,\mathrm{d}s + \int_{t_1}^{\infty} \frac{F(s)}{r(s)}\,\mathrm{d}s \Bigg| \\ &\leq \mu \int_{t_2}^{\infty} \frac{1}{r(s)} \Bigg(\int_{s}^{\infty} q(u)\,\mathrm{d}u \Bigg)\,\mathrm{d}s + \alpha \int_{t_2}^{\infty} \frac{\mathrm{d}s}{r(s)} \\ &\quad + \mu \int_{t_1}^{\infty} \frac{1}{r(s)} \Bigg(\int_{s}^{\infty} q(u)\,\mathrm{d}u \Bigg)\,\mathrm{d}s + \alpha \int_{t_1}^{\infty} \frac{\mathrm{d}s}{r(s)} < \varepsilon.\end{aligned}$$

For $x \in S$ and $T \leq t_1 < t_2 \leq T^*$,

$$\begin{aligned}|(Bx)(t_2) - (Bx)(t_1)| &\leq \mu \int_{t_1}^{t_2} \frac{1}{r(s)} \Bigg(\int_{s}^{\infty} q(u)\,\mathrm{d}u \Bigg)\,\mathrm{d}s + \alpha \int_{t_1}^{t_2} \frac{1}{r(s)}\,\mathrm{d}s \\ &\leq \max_{T \leq s \leq T^*} \Bigg[\Bigg(\frac{\mu}{r(s)} \int_{s}^{\infty} q(u)\,\mathrm{d}u \Bigg) + \frac{\alpha}{r(s)} \Bigg] (t_2 - t_1).\end{aligned}$$

Thus there exists a $\delta > 0$ such that

$$|(Bx)(t_2) - (Bx)(t_1)| < \varepsilon \qquad \text{if} \quad 0 < |t_2 - t_1| < \delta.$$

For any $x \in S$, $T_0 \leq t_1 < t_2 \leq T$, it is easy to see that

$$|(Bx)(t_2) - (Bx)(t_1)| = 0 < \varepsilon.$$

Therefore $\{Bx : x \in S\}$ is uniformly bounded and equicontinuous on $[T_0, \infty)$ and hence BS is relatively compact. By Lemma 2.1, there is an $x_0 \in S$ such that $Ax_0 + Bx_0 = x_0$. It is easy to see that $x_0(t)$ is the required non oscillatory solution of the equation (E), which is bounded below by the positive constant $\frac{3(1-p)}{4}$. □

Corollary 2.3. *Let (A_1), (H_1), (H_2) and (H_4) hold. Then there exists a bounded solution of* (E) *which is bounded below by a positive constant.*

P r o o f. By Remark 1(iii), (H_2) and (H_4) imply (H_5). Hence the proof follows from the proof of the above theorem. □

Theorem 2.4. *Let* (A_1), (H_3) *and* (H_5) *hold. Suppose there exists* $\alpha > 0$ *such that for large* t

$$r(t) > \frac{1}{\alpha} \tag{2.8}$$

and

$$\left|\int_0^\infty F(t)\,\mathrm{d}t\right| < \infty \qquad \text{with} \quad F'(t) = f(t). \tag{2.9}$$

Then there exists a bounded solution of (E) *which is bounded below by a positive constant.*

Proof. Using (2.8) and (2.9) we can get (2.3). Rest of the proof is similar to that of the Theorem2.2. □

Remark 2. (2.9) implies (H_1).

Corollary 2.5. *Let* (A_1), (H_5), (2.8), (2.9) *hold. Then there exists a bounded solution of* (E) *which is bounded below by a positive constant.*

Proof. By Remark 1(i) we have either (H_3) holds or (H_4) holds. If (H_3) holds we proceed as in the proof of Theorem 2.4 and if (H_4) holds then since (2.9) gives (H_1) we use Theorem 2.2, and get the result. □

Remark 3. If in (H_5) we take $r(t) \equiv 1$ then it reduces to

$$\int_{t_0}^\infty \int_t^\infty q(s)\,\mathrm{d}s\,\mathrm{d}t < \infty. \tag{2.10}$$

The above condition is required for our next result which follows from Corollary 2.5 when $r(t) \equiv 1$.

Corollary 2.6. *Inequality* (2.10) *is a sufficient condition for the second order NDDE*

$$\big(y(t) - p(t)y(t-\tau)\big)'' + q(t)G\left(y(t-\sigma)\right) = f(t) \tag{2.11}$$

to have a solution bounded below by a positive constant under the assumptions (A_1), (2.8) *and* (2.9).

Remark 4. Corollary 2.6 improves [1, Theorem 1] and [5, Theorem 4.3] for $n = 2$ because the authors assumed G to be Lipschizian and satisfies (H_9).

Theorem 2.7. *Let* (A_2), (H_1), (H_4) *and* (H_5) *hold. Then there exists a bounded solution of* (E) *which is bounded below by a positive constant.*

P r o o f. We proceed as in the proof of the Theorem 2.2 with the following changes:

$$\mu = \max\left\{|G(x)| : \tfrac{1-p}{20} \le x \le 1\right\}.$$

By (H_1), (H_4) and (H_5), we find T such that for $t \ge T$

$$\mu \int_t^\infty \left(\frac{1}{r(s)} \int_s^\infty q(u)\,\mathrm{d}u \right) \mathrm{d}s < \frac{1-p}{10}$$

and

$$\int_t^\infty \frac{F(s)}{r(s)}\,\mathrm{d}s < \alpha \int_t^\infty \frac{\mathrm{d}s}{r(s)} < \frac{1-p}{20}.$$

Let $S = \left\{y \in X : \frac{1-p}{10} \le y(t) \le 1,\ \ t \ge T_0\right\}$. Define the mappings

$$(Ay)(t) = \begin{cases} \frac{4p+1}{5} + p(t)y(t-\tau), & \text{for } t \ge T \\ Ay(T), & \text{for } T_0 \le t \le T. \end{cases}$$

$$(By)(t) = \begin{cases} By(T), & \text{for } T_0 \le t \le T \\ -\int\limits_t^\infty \left(\frac{1}{r(s)} \int\limits_s^\infty q(u)G(y(h(u)))\,\mathrm{d}u \right) \mathrm{d}s & \\ -\int\limits_t^\infty \frac{F(s)}{r(s)}\,\mathrm{d}s, & \text{for } t \ge T. \end{cases}$$

Then as in Theorem 2.2 we prove

(i) $Ax + By \in S$,

(ii) A is a contraction,

and finally

(iii) B is completely continuous.

Then by Lemma 2.1 there is a fixed point x_0 in S such that $Ax_0 + Bx_0 = x_0$ which is required solution bounded below by a positive constant. □

Remark 5. The above theorem substantially improves [9, Theorem 3.1] where the authors obtained a positive bounded solution of (E) with assumptions (A_2), (H_2), (H_4), (H_6), (H_8) and (H_9).

Theorem 2.8. *Let (A_2), (H_3), (H_5), (2.8) and (2.9) hold. Then there exists a bounded solution of* (E) *which is bounded below by a positive constant.*

P r o o f. The proof of the above Theorem is similar to that of Theorem 2.7. □

Theorem 2.9. *Suppose that (A_3) holds, such that $d < c^2$. Let (H_1), (H_4) and (H_5) hold. Then there exists a bounded solution of* (E) *which is bounded below by a positive constant.*

P r o o f. Suppose h be any small positive constant. Choose two more scalars H and λ such that.

$$H = h(2d + 2c + cd)/(c^2 - d), \quad \text{and} \quad \lambda = cH - 2h.$$

Then $H > h > 0$. Set $\mu = \max\{|G(x)| : \ h \le x \le H\}$. From (H_4) and (H_5) one can find $T > 0$ such that $t \ge T$ implies

$$\int_t^\infty \frac{F(s)}{r(s)} \, \mathrm{d}s < h$$

and

$$\mu \int_t^\infty \frac{1}{r(s)} \left(\int_s^\infty q(u) \, \mathrm{d}u \right) \mathrm{d}s < h.$$

Let

$$S = \{y(t) \in X : \ h \le y(t) \le H, \ \ t \ge T_0\}.$$

Define

$$Ax(t) = \begin{cases} Ax(T), & \text{if } \ t \in [T_0, T] \\ \frac{x(t+\tau)}{p(t+\tau)} - \frac{\lambda}{p(t+\tau)}, & \text{if } \ t \ge T. \end{cases}$$

$$Bx(t) = \begin{cases} Bx(T), & \text{if } \ t \in [T_0, T] \\ \frac{1}{p(t+\tau)} \int\limits_{t+\tau}^\infty \frac{1}{r(s)} \left(\int\limits_s^\infty G(y(h(u))) \, \mathrm{d}u \right) \mathrm{d}s & \\ + \frac{1}{p(t+\tau)} \int\limits_{t+\tau}^\infty \frac{F(s)}{r(s)} \, \mathrm{d}s, & \text{if } \ t \ge T. \end{cases}$$

Then proceeding as in the proof of Theorem 2.2, one may complete the proof. □

Theorem 2.10. *Let* (A_3), (H_3), (H_5), (2.8), (2.9) *hold. Then there exists a bounded solution of* (E) *which is bounded below by a positive constant.*

P r o o f. The proof is similar to that of the above theorem. □

Corollary 2.11. *Let* (A_3), (H_5), (2.8), (2.9) *hold. Then there exists a bounded solution of* (E) *which is bounded below by a positive constant.*

P r o o f. In view of Remark 1(i) the proof follows lines similar to those in Theorems 2.9 and 2.10.

The results for the range (A_4) are similar to those under condition (A_3). □

Hence we skip all proofs except the following one.

Theorem 2.12. *Let* (A_4), (H_1), (H_4), (H_5) *hold. Then there exists a bounded solution of* (E) *which is bounded below by a positive constant.*

Proof. We proceed as in the proof of the Theorem 2.7 with the following changes. Choose

$$\mu = \max\left\{|G(x)| : \frac{3}{d} \le x \le \frac{3c}{c-1}\right\}.$$

$$S = \left\{y \in X :\ \tfrac{3}{d} \le y \le \tfrac{3c}{c-1}\right\}.$$

From (H_4) and (H_5) we can find $T > 0$ such that $t \ge T$ implies

$$\int_t^\infty \frac{F(s)}{r(s)}\,\mathrm{d}s < c-1 \quad \text{and} \quad \mu \int_t^\infty \frac{1}{r(s)} \int_s^\infty q(u)\,\mathrm{d}u\,\mathrm{d}s < c-1.$$

Define

$$Ax(t) = \begin{cases} Ax(T), & \text{for } T_0 \le t \le T \\ \frac{x(t+\tau)}{p(t+\tau)} + \frac{c+2}{p(t+\tau)}, & \text{for } t \ge T. \end{cases}$$

$$By(t) = \begin{cases} By(T), & \text{for } T_0 \le t \le T \\ \frac{1}{p(t+\tau)} \int\limits_{t+\tau}^\infty \frac{1}{r(s)} \int\limits_s^\infty q(u)G(y(h(u)))\,\mathrm{d}u\,\mathrm{d}s & \\ +\frac{1}{p(t+\tau)} \int\limits_{t+\tau}^\infty \frac{F(s)}{r(s)}\,\mathrm{d}s, & \text{for } t \ge T. \end{cases}$$

Rest of the proof is similar to that of the Theorem 2.2. □

3. Positive solution for $p(t) = \pm 1$

In this section we find sufficient condition for the NDDE

$$(r(t)(y(t) + y(t-\tau))')' + q(t)G(y(h(t))) = f(t) \tag{3.1}$$

or

$$(r(t)(y(t) - y(t-\tau))')' + q(t)G(y(h(t))) = f(t) \tag{3.2}$$

to have a bounded positive solution.

The results with NDDE (3.1) are rare in the literature. We don't find such a result in [1], [2] or [5]–[9]. To achieve our result we need the following Lemma.

Lemma 3.1 (**Schauder's Fixed Point Theorem**). ([3]) *Let Ω be a closed convex and nonempty subset of a Banach space X. Let $S\colon \Omega \to \Omega$ be a continuous mapping such that $S(\Omega)$ is a relatively compact subset of X. Then S has at least one fixed point in Ω. That is there exists an $x \in \Omega$ such that $Sx = x$.*

Theorem 3.2. *Suppose* (H_1), (H_4) *and* (H_5) *hold. Then there exists a solution of* (3.1) *which is bounded below by a positive constant, that is, it neither oscillates nor tends to zero as* t *tends to* ∞.

P r o o f. We proceed as in the proof of Theorem 2.2 with the following changes. Let

$$\mu = \max\{|G(x)| : \ 1 \le x \le 4\}.$$

From (H_1), (H_4) and (H_5) there exists $T > 0$ such that $t \ge T$ implies

$$\left| \int_t^\infty \frac{1}{r(s)} \left(\int_s^\infty q(u) G(h(u))\,\mathrm{d}u \right) \mathrm{d}s \right| < 1$$

and

$$\left| \int_t^\infty \frac{F(s)}{r(s)}\,\mathrm{d}s \right| < 1.$$

Define

$$S = \{y \in X : \ 1 \le y(t) \le 4, \ \ t \ge T_0\}$$

and define the mapping $B\colon S \to X$ as

$$By(t) = \begin{cases} By(T), & \text{for } T_0 \le t \le T \\ 3 - \sum\limits_{l=1}^{\infty} \int\limits_{t+(2l-1)\tau}^{t+2l\tau} \left(\frac{1}{r(s)} \int\limits_s^\infty q(u) G(y(h(u)))\,\mathrm{d}u \right) \mathrm{d}s & \\ - \sum\limits_{l=1}^{\infty} \int\limits_{t+(2l-1)\tau}^{t+2l\tau} \left(\frac{F(s)}{r(s)} \right) \mathrm{d}s, & \text{for } t \ge T. \end{cases}$$

Then as in Theorem 2.2 we prove

(i) $By \in S$ for $y \in S$,

(ii) BS is relatively compact.

Then by Lemma 3.1 there is a fixed point y_0 in S such that $By_0 = y_0$. Hence the theorem is proved. □

Corollary 3.3. *If* (H_1), (H_2), (H_4) *hold, then there exists a positive solution of* (3.1) *which is bounded below by a positive constant.*

P r o o f. The proof follows from Remark 1 and above Theorem. □

Theorem 3.4. *Let* (H_3), (H_5), (2.8) *and* (2.9) *hold. Then there exists a positive solution of* (3.1) *which is bounded below by a positive constant that is, it neither oscillates nor tends to zero as* t *tends to* ∞.

P r o o f. The proof is similar to that of Theorem 3.2. □

Theorem 3.5. *Suppose* (H_1) *hold. Assume*

$$\sum_{n=1}^{\infty} \int_{t+n\tau}^{\infty} \frac{1}{r(s)} \left(\int_{s}^{\infty} q(u)\,\mathrm{d}u \right) \mathrm{d}s < \infty \tag{3.3}$$

$$\sum_{n=1}^{\infty} \int_{t+n\tau}^{\infty} \frac{1}{r(s)}\,\mathrm{d}s < \infty. \tag{3.4}$$

Then (3.2) *has a solution bounded below by a positive constant.*

P r o o f. We proceed as in the proof of Theorem 3.2 with the following changes. Let

$$\mu = \max\{|G(x)| :\ 2 \le x \le 5\}.$$

Then from (H_1), (3.3) and (3.4), there exists $T > 0$ such that for $t \ge T$

$$\sum_{n=1}^{\infty} \int_{t+n\tau}^{\infty} \frac{1}{r(s)} \left(\int_{s}^{\infty} q(u)\,\mathrm{d}u \right) \mathrm{d}s < 1$$

and

$$\sum_{n=1}^{\infty} \int_{t+n\tau}^{\infty} \frac{F(s)}{r(s)}\,\mathrm{d}s < 1.$$

Let

$$S = \{y \in X :\ 2 \le y \le 5,\ \ t \ge T_0\}.$$

Then define

$$By(t) = \begin{cases} By(T), & \text{for } t \in [T_0, T] \\ 3 + \sum\limits_{n=1}^{\infty} \int\limits_{t+n\tau}^{\infty} \left(\frac{1}{r(s)} \int\limits_{s}^{\infty} q(u) G(y(h(u)))\,\mathrm{d}u \right) \mathrm{d}s & \\ + \sum\limits_{n=1}^{\infty} \int\limits_{t+n\tau}^{\infty} \frac{F(s)}{r(s)}\,\mathrm{d}s, & \text{for } t \ge T. \end{cases}$$

Then as in Theorem 2.2 we prove

(i) $By \in S$ for $y \in S$, and

(ii) BS is relatively compact.

Then by Lemma 3.1 there exists a fixed point $y_0 \in S$ such that $By_0 = y_0$, Putting $y_0 = y(t)$, we get

$$y(t) = 3 + \sum_{n=1}^{\infty} \int_{t+n\tau}^{\infty} \left(\frac{1}{r(s)} \int_{s}^{\infty} q(u)G(y(h(u)))\,\mathrm{d}u \right) \mathrm{d}s + \sum_{n=1}^{\infty} \int_{t+n\tau}^{\infty} \frac{F(s)}{r(s)}\,\mathrm{d}s.$$

Then for $t \geq T$,

$$y(t) - y(t-\tau) = -\int_{t}^{\infty} \frac{1}{r(s)} \int_{s}^{\infty} q(u)G(y(h(u)))\,\mathrm{d}u\,\mathrm{d}s - \int_{t}^{\infty} \frac{F(s)}{r(s)}\,\mathrm{d}s.$$

We may differentiate the above and then multiply by $r(t)$ and then again differentiate to arrive at (3.2). This solution is bounded below by a positive constant. □

Remark 6. All results of this paper hold for the homogeneous equation corresponding to (E).

Before we close this article we present an interesting example which illustrates all the results of this paper.

Example 1. Consider NDDE

$$(r(t)(y(t) - py(t-\tau))')' + q(t)G(y(h(t))) = 0 \qquad \text{for} \quad t \geq t_0. \tag{3.5}$$

Here let us assume p to be any constant, $r(t)$ is a continuous function satisfying either (H_3) or (H_4) or (3.4). $q(t)$ satisfies (H_2), (H_5) or (3.3). $G(u) = 1 - u^m$, with m any positive odd integer, is decreasing. The equation (3.5) has a positive solution $y(t) \equiv 1$.

Hence this example illustrates all the results of this paper. However since G is decreasing the existing results of [1], [2], [5], [6], [7], [8], [9] are not applicable to this NDDE (3.5).

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