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IDEALS AND ATOMS OF BZ-ALGEBRAS

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ABSTRACT. Ideals and atoms are studied by various authors from different point of views. In different algebras there are studied different ideals, but obtained results are similar. Below we present a new method of study of ideals in BZ-algebras. Using this method we describe the connection between ideals of various types.

1. Introduction

In 1996 K. Iséki introduced in [19] the concept of BCI-algebras as algebras connected with some logics. Next, in 1984, Y. Komori used in [22] other type of algebras, introduced in [21] and called now BCC-algebras, to solve some problems on BCK-algebras. BCK-algebras are strongly connected with BCK logic ([2]). Connections between BCI-algebras and BCI logic is not such good. BCC-algebras (by some authors called also BIK-algebras) are an algebraic model of BCC-logic, i.e., implicational logic whose axioms scheme are the principal type-scheme of the combinators B, I, and K, and whose inference rules are modus ponens and modus ponens 2 [where $p \to q$ is inferred from $p \to (r \to q)$ and r]. Several years later some authors introduced independently more extensive algebraic system using different names. This new algebraic system has the same partial order as BCC-algebras and BCK-algebras but has no minimal element. Such obtained system is called a BZ-algebra ([29], [30]), GB-algebra ([39]) or a weak BCC-algebra ([9], [11]). From the mathematical point of view the last name is more corrected but more popular is the first name.

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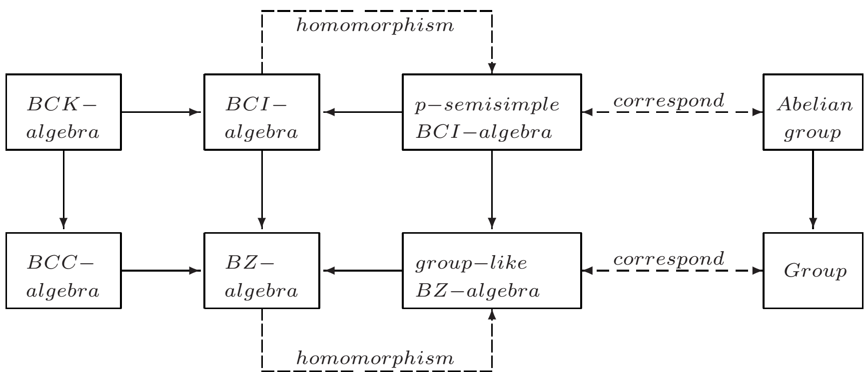
On the other hand many mathematicians, especially from China, Japan and Korea, independently studied such algebras as BCI-algebras ([4], [18], [20], [25]), B-algebras ([5]), difference algebras ([27]), implication algebras, G-algebras, Hilbert algebras, d-algebras and many others. All these algebras have one distinguished element and satisfy some common identities playing a crucial role in these algebras and, in fact, are generalization or a special case of BZ-algebras. So, results obtained for BZ-algebras are in some sense fundamental for these algebras, especially for $BCC/BCH/BCI/BCK$-algebras.

Group-like BZ-algebras (called also anti-grouped) are BZ-algebras with discrete natural order, i.e., BZ-algebras in which all elements are incomparable. Such BZ-algebras are strongly connected with groups ([36]). Each BZ-algebra has a least one subalgebra which is group-like. The maximal group-like subalgebra is uniquely determined and plays in BZ-algebras a similar role as a p-semisimple BCI-algebra in the theory of BCI-algebras (see for example [18], [26], [33]).

Relations between these algebras are illustrated by the following diagram, where $A \to B$ means that A is B but B may not be A:



A very important role in the theory of such algebras plays ideals. Ideals in BCK-algebras are induced by partial order or by homomorphisms. Ideals determine congruences. But in BCC-algebras there are congruences which are not determined by ideals ([16]). Moreover, relations determined by ideals in the same way as in BCK-algebras are not congruences of BCC-algebras, in general. So, in BCC-algebras we must introduce the new concept of ideals ([17]). Similarly in BZ-algebras. The main role in the theory of ideals play T-ideals (cf. [31], [38]) called also QA-ideals ([24]).

Below we present a new method of study of ideals. Our method is based on the map $\varphi(x) = 0x$ firstly used in [15] to the description of some decompositions of BCH-algebras. This method gives the possibility to describe the connections between different types of ideals and can be used to study similar connections in other types of algebras inspired by logic.

2. Preliminaries

Definition 2.1. A non-empty set X with a binary operation $\cdot$ denoted by juxtaposition and a distinguished element 0 is called a *BZ-algebra* (or a *weak BCC-algebra* [11]) if for any $x, y, z \in X$ the following axioms:

(1) $((xz)(yz))(xy) = 0,$

(2) $x0 = x,$

(3) $xy = yx = 0 \longrightarrow x = y$

are satisfied.

If a BZ-algebra $(X; \cdot, 0)$ satisfies also

(4) $0x = 0,$

then it is called a *BCC-algebra* or a *BIK^+-algebra* ([22]) and is a special case of *Komori algebras* studied in [23]. A BZ-algebra satisfying the identity $(xy)z = (xz)y$ is a *BCI-algebra.* A BZ-algebra is *proper* if it is not either a BCC-algebra or a BCI-algebra. A proper BZ-algebra has at least four elements ([11]). There are only two proper BZ-algebras of order four ([11]). Their multiplication tables are given in Example 3.3.

In the sequel, to save the simplicity of formulaes, the part of brackets will be replaced by dots. In this convention, the axiom (1) will be written in the form $(xz \cdot yz) \cdot xy = 0$; the condition $(xy)z = (xz)y$ in the form $xy \cdot z = xz \cdot y$.

Similarly as in BCI-algebras in any BZ-algebra we can introduce a natural partial order $\leqslant$ putting

$$x \leqslant y \longleftrightarrow xy = 0.$$

Note that non-isomorphic BZ-algebras may have the same natural order. Moreover, the same order may be induced by BZ-algebra and BCI-algebra. In the case of BCC-algebras 0 is the smallest element.

The above definition of a BCC-algebra proposed in [10] is a dual form of an original definition of BCC-algebras used by Komori in [22] (cf. also [9]). In this original definition 0 is replaced by 1, xy by $y \to x$ and $x \leqslant y \longleftrightarrow x \to y = 1$. Such defined BCC-algebras are strongly connected with Hilbert algebras ([3]).

It is not difficult to see that in BZ-algebras

(5) $xx = 0,$

(6) $x \leqslant y \longrightarrow xz \leqslant yz$ and $zy \leqslant zx$

for all $x, y \in X$.

The map $\varphi(x) = 0x$, was formally introduced in [15] for BCH-algebras, but, in fact, different properties of this map were used in [6], [7] and [8] to characterizations of special classes of BCI-algebras connected with groups. Generally φ is not an endomorphism of these algebras but φ^2 always is an endomorphism. The same situation holds in the case of BZ-algebras. Namely, the following lemma is true.

Lemma 2.2. *In any BZ-algebra X for all $x, y, z \in X$ we have:*

(7) $\varphi(xy) \leqslant yx$,

(8) $\varphi^2(x) \leqslant x$,

(9) $\varphi(x) \cdot yx = \varphi(y)$,

(10) $\varphi(xy)\varphi(x) = \varphi^2(y)$,

(11) $\varphi^3(x) = \varphi(x)$,

(12) $\varphi^2(xy) = \varphi^2(x)\varphi^2(y)$,

(13) $x \leqslant y \longrightarrow \varphi(x) = \varphi(y)$,

(14) $\varphi^2(xy) = \varphi(yx)$,

(15) $(xy \cdot z)\varphi(z) \leqslant x(yz \cdot \varphi(z))$.

Proof. The first two conditions follow directly from the above axioms. From (1) we conclude also $\varphi(x) \cdot yx \leqslant \varphi(y)$ and $yx \cdot \varphi(x) \leqslant y$. Moreover, $xy \leqslant z$ implies $uy \cdot z \leqslant ux$ because, by (1) and (6), we have $uy \cdot z \leqslant uy \cdot xy \leqslant ux$. So, $yx \cdot \varphi(x) \leqslant y$ implies $\varphi(y) = \varphi(x)\varphi(y) \cdot y \leqslant \varphi(x) \cdot yx$, which together with $\varphi(x) \cdot yx \leqslant \varphi(y)$ proves (9). (10) is a consequence of (9). Another consequence of (9) is (13). From (13) we deduce (11) and (14). By (10) and (9) we obtain (12). Using (10) and (6) we can prove (15). □

Lemma 2.2 can be deduced also from results proved in [6], [9], [10], [15] and [36].

Definition 2.3. For any BZ-algebra X we consider three subsets

$\operatorname{Ker}\varphi = \{x \in X : \ \varphi(x) = 0\}$,

$G(X) = \{x \in X : \ \varphi^2(x) = x\}$,

$T(X) = \{x \in X : \ \varphi(x) \leqslant x\}$,

which are called the *BCC-part* ([10]), the *group-like* (or *p-semisimple* [18]) part and a *T-part* of X, respectively.

The role of these sets in BCI-algebras is described in [26] and [33]. $\operatorname{Ker}\varphi$ (denoted also by $B(X)$) and $G(X)$ are subalgebras. $T(X)$ is not a subalgebra, in general, but in some cases $T(X) = X$. A BZ-algebra with this property is called a *T-type* BZ-algebra. In this BZ-algebra $\varphi^2(x) = \varphi(x)$, by (13). On the other hand, if $\varphi^2(x) = \varphi(x)$, then $\varphi(x)x = \varphi^2(x)x = 0$, by (8), whence $\varphi(x) \leqslant x$. So, a BZ-algebra X is T-type if and only if $\varphi^2(x) = \varphi(x)$ holds for all $x \in X$. In a T-type BZ-algebra φ is an endomorphism. The converse is not true ([38]).

According to [36] we say that a BZ-algebra X is *anti-grouped* (*group-like* in other terminology [8]) if $\varphi^2(x) = x$ holds for all $x \in X$, i.e., if φ^2 is the identity endomorphism of this algebra. Such algebras are uniquely defined by some group ([36]). Namely, if $(X;\cdot,0)$ is an anti-grouped BZ-algebra then X with the operation $x \circ y = x\varphi(y)$ is a group. 0 is its neutral element. On the other hand, if $(X;\circ,0)$ is a group, then $(X;\cdot,0)$, where $xy = x \circ y^{-1}$, is a BZ-algebra ([35]). Since the identical result was firstly proved for BCI-algebras (cf. [6], [7] and [8]), and BCI-algebras with this property were called *group-like* we save this terminology for BZ-algebras.

In the case when $\operatorname{Ker}\varphi = \{0\}$, by (14) and (12), we have

$$\varphi(x\varphi^2(x)) = \varphi^2(\varphi^2(x)x) = \varphi^2(x)\varphi^2(x) = \varphi^2(xx) = 0,$$

which implies $x\varphi^2(x) \in \operatorname{Ker}\varphi$. Thus $\varphi^2(x) = x$, i.e., X is a group-like BZ-algebra. In the case when φ is the identity endomorphism of X, from (14) we conclude $xy = yx$, whence, by (8), we get $x\varphi^2(x) = \varphi^2(x)x = 0$. This means that X is group-like and the corresponding group is a Boolean group. Moreover, $xy = x \circ y$.

The BZ-algebra defined by the symmetric group S_3 is an example of a BZ-algebra with $\operatorname{Ker}\varphi = \{0\}$ for which φ is not the identity endomorphism.

Lemma 2.4. *Let $\varphi^2(A) = \{\varphi^2(a) : a \in A\}$, where A is a non-empty subset of a BZ-algebra X. Then $\varphi^2(A) \subseteq G(X)$ and $\varphi^2(X) = G(X)$.*

Proof. Indeed, for $y \in \varphi^2(A)$ there exists $a \in A$ such that $y = \varphi^2(a)$. Then $\varphi^2(y) = \varphi^4(a) = \varphi^2(a) = y$, by (11). So, $y \in G(X)$. Thus $\varphi^2(A) \subseteq G(X)$. Consequently $\varphi^2(X) \subseteq G(X)$. The inclusion $G(X) \subseteq \varphi(X)$ is obvious. $\square$

3. Ideals

As it is well known, a non-empty subset I of a BCK/BCI-algebra (BZ-algebra, too) is called an *ideal* if it contains 0 and $xy, y \in I$ imply $x \in I$.

Definition 3.1. A non-empty subset I of a BZ-algebra X is called a BZ-ideal of X if

(i) $0 \in I$,

(ii) $xy \cdot z \in I$ and $y \in I$ imply $xz \in I$ for all $x, y, z \in X$.

BZ-ideals were introduced in [29]. Such ideals are mainly investigated in BCC-algebras ([12]) and play an important role in fuzzyfication of BCC-algebras ([14], [37]). In BCC-algebras such ideals are called *BCC-ideals* and are used to the description of congruences ([17]).

Putting $z = 0$ in (ii) we see that a BZ-ideal is an ideal and the following lemma is true.

Lemma 3.2. *If I is an ideal of a BZ-algebra X, then $xy \in I$ and $y \in I$ imply $x \in I$. In particular, $x \leqslant y$, $y \in I$ imply $x \in I$.*

Example 3.3. On the set $X = \{0, 1, 2, 3\}$ we can define two proper BZ-algebras ([11]). Their multiplication tables are as follows:

$\cdot$	0	1	2	3
0	0	0	2	2
1	1	0	2	2
2	2	2	0	0
3	3	3	1	0

$\cdot$	0	1	2	3
0	0	0	2	2
1	1	0	3	3
2	2	2	0	0
3	3	3	1	0

$(X; \cdot, 0)$ is not either a BCI-algebra or a BCC-algebra because $32 \cdot 1 \neq 31 \cdot 2$. The subset $\{0, 2\}$ is an ideal but not a BZ-ideal. Indeed, $32 \cdot 1 = 0 \in \{0, 2\}$, $2 \in \{0, 2\}$ but $31 = 3 \notin \{0, 2\}$.

Different authors investigated in BZ-algebras ideals of different types ([30], [35], [28]). The same ideals are investigated in BCI-algebras ([4], [18], [20], [25], [26], [31], [1], [13], [32]), BCC-algebras ([24], [12], [34]) and in other algebras. Obtained results are similar but not identical.

Below we remind more popular types of ideals and describe connections between these ideals.

Definition 3.4. An ideal (BZ-ideal) I of a BZ-algebra X is called

- *closed* if $\varphi(I) \subseteq I$,
- an $(^*)$*-ideal* ($(^*)$*-BZ-ideal*) if $xa \in I$ for all $x \in I$ and $a \in X - I$,
- *anti-grouped* if $\varphi^2(x) \in I$ implies $x \in I$,
- *strong* if $x \in I$, $a \in X - I$ imply $xa \in X - I$,
- *regular* if $xy \in I$, $x \in I$ imply $y \in I$,

- *associative* if $x\varphi(z) \in I$ implies $zx \in I$,
- *T-ideal* if $x \cdot yz \in I$ and $y \in I$ implies $xz \in I$,
- *H-ideal* (*horizontal ideal*) if $I \cap \operatorname{Ker}\varphi = \{0\}$.

For any ideal I of a BZ-algebra X we can define a binary relation θ_I on X putting:

$$(x, y) \in \theta_I \longleftrightarrow xy, yx \in I.$$

Such defined relation is an equivalence relation. It is a congruence if I is a BZ-ideal ([17]). In BCC-algebras all congruences are defined in this way. Namely, as it is proved in [17], each congruence of a BCC-algebra is uniquely determined by its equivalence class containing 0.

Let I be a BZ-ideal of a BZ-algebra X. The set $X/I = X/\theta_I$ of all equivalence classes

$$C_x = \{y \in X : \ (x, y) \in \theta_I\}$$

is a BZ-algebra with respect to the operation $C_x \cdot C_y = C_{xy}$. The role of 0 plays the class C_0 which is a BZ-ideal of X. Indeed, if $xy \cdot z \in C_0$ and $y \in C_0$ then $(xy \cdot z, 0) \in \theta_I$ and $(y, 0) \in \theta_I$, whence we obtain $(xy, x) = (xy, x0) \in \theta_I$. Similarly $(xy \cdot z, xz) \in \theta_I$. Therefore $(0, xz) \in \theta_I$, i.e., $xz \in C_0$. So, C_0 is a BZ-ideal of X. Clearly $C_0 = \{x \in I : \ \varphi(x) \in I\} \subseteq I$. For closed BZ-ideals we have $C_0 = I$.

Theorem 3.5. *An ideal I of a BZ-algebra X is closed if and only if it is a subalgebra.*

Proof. Assume that an ideal I is a subalgebra. Then $0 \in I$ and $0x \in I$ for every $x \in I$. So, $\varphi(x) \in I$, i.e., an ideal I is closed.

Conversely, if an ideal I is closed, then, by (1) and (2), for any $x, y \in I$, we have

$$(xy \cdot 0y) \cdot x = (xy \cdot 0y) \cdot x0 = 0 \in I,$$

which by Lemma 3.2 implies $xy \cdot 0y \in I$ and $xy \in I$, because $0y \in I$. This proves that I is a subalgebra. □

Corollary 3.6. $\operatorname{Ker}\varphi$ *is a closed anti-grouped BZ-ideal.*

Proof. From (11) and (14). □

Lemma 3.7. $\varphi^2(I) = I \cap G(X)$ *for any ideal I of X.*

Proof. If I is an ideal and $y \in \varphi^2(I)$, then $y = \varphi^2(x)$ for some $x \in I$. This, by (8) and Lemma 3.2, implies $y \in I$. Therefore $\varphi^2(I) \subseteq I \cap G(X)$. The converse inclusion is obvious. □

Theorem 3.8. *A BZ-ideal (ideal) I of a BZ-algebra X is closed if and only if $\varphi^2(I)$ is a closed BZ-ideal (ideal) of $\varphi^2(X)$.*

Proof. Let I be a closed BZ-ideal of X. By Lemma 3.2, $\varphi^2(I) \subseteq I$. Let $xy \cdot z \in \varphi^2(I)$, where $y \in \varphi^2(I)$, $x, z \in \varphi^2(X)$. Then $xy \cdot z \in I$ and $y \in I$, consequently $xz \in I$. Since $x = \varphi^2(a)$, $z = \varphi^2(b)$ for some $a, b \in X$, by (12) and Lemma 2.4, we obtain $xz = \varphi^2(a)\varphi^2(b) = \varphi^2(ab) \in \varphi^2(X) = G(X)$. Hence $xz \in \varphi^2(I)$ (Lemma 3.7). So, $\varphi^2(I)$ is a BZ-ideal of $\varphi^2(X)$. It is closed because $\varphi(\varphi^2(I)) = \varphi^2(\varphi(I)) \subseteq \varphi^2(I)$.

Conversely, let I be a BZ-ideal of X. If a BZ-ideal $\varphi^2(I)$ is closed then by (11) and Lemma 3.7 we have $\varphi(I) = \varphi^3(I) = \varphi(\varphi^2(I)) \subseteq \varphi^2(I) = I \cap G(X) \subseteq I$, which means that I is closed. □

Theorem 3.9. *A non-empty subset I of a BZ-algebra X such that $\varphi(X) \subseteq I$ is a closed BZ-ideal of X if and only if*

(i) *$ay \in I$ for all $a \in I$ and $y \in X$,*

(ii) *$x(xa \cdot b) \in I$ for $a, b \in I$ and $x \in X$.*

Proof. If I is a closed BZ-ideal of X and $a, b \in I$, $x, y \in X$, then from $aa \cdot y = \varphi(y) \in I$ we conclude $ay \in I$. Similarly, from $xa \cdot xa = 0 \in I$ we obtain $x \cdot xa \in I$. But $xb \cdot (xa \cdot b) \leqslant x \cdot xa$, whence, by Lemma 3.7, we get $xb \cdot (xa \cdot b) \in I$. Consequently $x(xa \cdot b) \in I$. This proves that I satisfies (i) and (ii).

Conversely, for any closed ideal I satisfying (i) and (ii) an element $b = x \cdot xc$, where $x \in X$, $c \in I$, lies in I because $x \cdot xc = x(xc \cdot 0) \in I$ by (ii). Now, if $xc \cdot y = a \in I$, then, applying (1) and (2), we get

$$xy = xy \cdot 0 = xy \cdot ((xy \cdot (xc \cdot y)) \cdot (x \cdot xc)) = xy \cdot ((xy \cdot a)b) \in I,$$

which completes the proof. □

As a simple consequence we obtain the following characterization of BCC-ideals proved in [12].

Corollary 3.10. *A subset I containing 0 is a BCC-ideal of a BCC-algebra X if and only if it satisfies the conditions (i) and (ii) of the above theorem.*

The identical result (by dualism) can be proved for Hilbert algebras ([3]).

Theorem 3.11. *An ideal I of a BZ-algebra X is a closed $(^*)$-ideal if and only if $\varphi(X) \subseteq I$.*

Proof. Let I be an ideal of X. If $\varphi(X) \subseteq I$, then obviously $\varphi(I) \subseteq I$, i.e., an ideal I is closed. Since $\varphi(a) \in I$ and $xa \cdot 0a \leqslant x$, for any $a \in X - I$ and $x \in I$, by Lemma 3.2, we get $xa \in I$. Hence I is an $(^*)$-ideal of X.

The converse is obvious. □

Theorem 3.12. *A closed BZ-ideal I of a BZ-algebra X is an $(^*)$-BZ-ideal if and only if X/I is a BCC-algebra.*

Proof. Let I be a closed $(^*)$-ideal and let $x \in X$. Then $\varphi(x) \in I$, by Theorem 3.11. Thus $\varphi(x)0 \in I$, $0\varphi(x) \in I$, which gives $C_0 \cdot C_x = C_{0x} = C_0$. Hence X/I is a BCC-algebra.

Conversely, if X/I is a BCC-algebra, then $C_0 \cdot C_x = C_{0x} = C_0$ for any $x \in X$. Hence $0x \in I$. Theorem 3.11 completes the proof. □

Theorem 3.13. *$G(X)$ is a BZ-ideal of X if and only if*

(16) *$y \in G(X)$ implies $x \cdot xy \in G(X)$,*

(17) *$xb = ab$ implies $x = a$ for $a, b \in G(X)$.*

Proof. Assume that $G(X)$ is a BZ-ideal of X and consider an arbitrary element $y \in G(X)$. Then $xy \cdot xy = 0 \in G(X)$, which implies $x \cdot xy \in G(X)$. Thus (16) holds. Now let $xb = ab$ for some $a, b \in G(X)$. Since $G(X)$ is a subalgebra and a BZ-ideal, $xb = ab \in G(X)$, whence $x \in G(X)$ by Lemma 3.2. Applying (10) and (1) we obtain

$$x = \varphi^2(x) = \varphi(ax)\varphi(a) = ((ab \cdot xb) \cdot ax)\varphi(a) = 0\varphi(a) = \varphi^2(a) = a.$$

This proves (17).

To prove the converse statement assume that $G(X)$ satisfies (16) and (17). Clearly $0 \in G(X)$. If $xy \in G(X)$, $y \in G(X)$, then $xy = \varphi^2(xy) = \varphi^2(x)\varphi^2(y) = \varphi^2(x)y$, which, by (17), implies $x = \varphi^2(x)$, because $\varphi^2(\varphi^2(x)) = \varphi^2(x) \in G(X)$. So, $x \in G(X)$. Thus $G(X)$ is an ideal. In fact it is a BZ-ideal. Indeed, for $xy \cdot z \in G(X)$, $y \in G(X)$, from $(xz \cdot (xy \cdot z)) \cdot (x \cdot xy) = 0 \in G(X)$, (16) and Lemma 3.2 we deduce $xz \in G(X)$. This completes the proof. □

Theorem 3.14. *For a BZ-ideal (ideal) I the following conditions are equivalent:*

(a) *I is anti-grouped,*

(b) *$x \leqslant y$, $x \in I$ imply $y \in I$,*

(c) *$xz \cdot yz \in I$, $y \in I$ imply $x \in I$,*

(d) *$xz \cdot \varphi(z) \in I$ implies $x \in I$.*

Proof.

(a) $\longrightarrow$ (b). Let $x \leqslant y$ and $x \in I$. Then $\varphi^2(yx) = \varphi(xy) = \varphi(0) = 0 \in I$, by (14). This, according to the definition of an anti-grouped ideal, implies $yx \in I$. From this, by Lemma 3.2, we obtain $y \in I$. This means that (b) holds.

(b) $\longrightarrow$ (c). Since $xz \cdot yz \leqslant xy$ and $xz \cdot yz \in I$ we have $xy \in I$, whence by Lemma 3.2 we obtain $x \in I$. This means that (c) holds.

(c) $\longrightarrow$ (d). Putting $y = 0$ we get (d).

(d) $\longrightarrow$ (a). Putting $z = x$ we get (a). $\square$

Theorem 3.15. *A BZ-ideal (ideal) I of a BZ-algebra X is closed and anti-grouped if and only if for every $x \in X$ both x and $\varphi(x)$ belong or not belong to I.*

Proof. Let I be a closed anti-grouped ideal. Then $x \in I$ implies $\varphi(x) \in I$. Similarly $\varphi(x) \in I$ implies $\varphi^2(x) \in I$. But I is anti-grouped, so, $x \in I$. Hence both x and $\varphi(x)$ belong or not belong to I.

Conversely, any ideal I with the property that both x and $\varphi(x)$ belong or not belong to I, is obviously closed. Moreover, if $\varphi^2(x) \in I$, then also $\varphi(x) \in I$, whence $x \in I$. So, an ideal I is anti-grouped. $\square$

Theorem 3.16. *A BZ-ideal (ideal) I is strong if and only if it is closed and anti-grouped.*

Proof. A strong ideal I is anti-grouped because in the case $\varphi^2(x) \in I$, $x \in X - I$ must be $\varphi^2(x)x \in X - I$, which by (8) gives $0 \in I$. This contradiction means that $x \in I$. So, I is anti-grouped. To prove that I is closed we select arbitrary $x \in I$. Then $\varphi^2(x) \in I$, by (11) and Lemma 3.2. For $\varphi(x) \in X - I$, by $0 \in I$ and the definition of a strong ideal, we get $\varphi^2(x) = 0\varphi(x) \in X - I$. This is a contradiction. Hence $\varphi(x) \in I$, i.e., an ideal I is closed.

On the other hand, if an ideal I is closed and anti-grouped then for any $a \in X - I$, $x \in I$, from $xa \in I$ we obtain $\varphi(xa) \in I$. Also $\varphi(x) \in I$. Thus $\varphi^2(a) = \varphi(xa)\varphi(x) \in I$, by (10) and Theorem 3.5. But I is anti-grouped, so $\varphi^2(a) \in I$ implies $a \in I$, which is a contradiction. Hence $xa \in X - I$, i.e., an ideal I is strong. $\square$

Theorem 3.17. *A BZ-ideal (ideal) I is regular if and only if it is closed and anti-grouped.*

Proof. Let I be a regular ideal. If $\varphi^2(x) \in I$, then, by (8), we have $\varphi^2(x)x = 0 \in I$, whence, by the regularity of I, we get $x \in I$. This proves that I is anti-grouped. Moreover, for $x \in I$, by (8) and Lemma 3.2, we have $\varphi^2(x) \in I$. But $0\varphi(x) = \varphi^2(x) \in I$, according to the regularity, gives $\varphi(x) \in I$. Thus I is closed.

On the other hand, for any closed and anti-grouped ideal I for $xy \in I$, $x \in I$ must be $\varphi(xy), \varphi(x) \in I$. Whence, by (10) and Theorem 3.5, we obtain $\varphi^2(y) =$

$\varphi(xy)\varphi(x) \in I$. So, $y \in I$, because I is anti-grouped. This proves that I is regular. □

Corollary 3.18. *For BZ-ideals (ideals) of BZ-algebras the following statements are equivalent:*

(a) *strong ideal,*

(b) *regular ideal,*

(c) *closed anti-grouped ideal.*

Proposition 3.19. *An associative ideal is closed and anti-grouped.*

Proof. Indeed, for any $x \in I$ we have $x\varphi(0) = x \in I$, which gives $\varphi(x) \in I$. So, I is closed. It is also anti-grouped because $0\varphi(x) = \varphi^2(x) \in I$ implies $x = x0 \in I$. □

Theorem 3.20. *An ideal I of a BZ-algebra X is a T-ideal if and only if one of the following conditions is satisfied:*

(a) $x\varphi(z) \in I$ *implies* $xz \in I$ *for all* $x, z \in X$,

(b) $x\varphi(z) \in I$ *implies* $x\varphi^2(z) \in I$ *for all* $x, z \in X$,

(c) $\varphi(x)x \in I$ *for all* $x \in X$.

Proof. Putting $y = 0$ in the definition of a T-ideal we obtain (a). To prove the converse assume that (a) holds and $x \cdot yz \in I$ for some $y \in I$. Since $yz \cdot 0z \leqslant y$ and $x\varphi(z) \cdot (yz \cdot \varphi(z)) \leqslant x \cdot yz$ by (1), Lemma 3.2 implies $yz \cdot \varphi(z) \in I$ and $(x\varphi(z)) \cdot (yz \cdot \varphi(z)) \in I$, whence $x\varphi(z) \in I$. So, $xz \in I$.

(b) is a consequence of (a) and (11). Applying (6), (8) and Lemma 3.2 to (b) we obtain (a).

To prove (c) observe that $0x \cdot 0x = 0 \in I$ for all $x \in X$. Whence, according to the definition of a T-ideal, we conclude $0x \cdot x \in I$. This proves (c). Conversely, if (c) holds and $x\varphi(z) \in I$, then, by (1), we have $xz \cdot \varphi(z)z \leqslant x\varphi(z)$, which, by Lemma 3.2, gives $xz \cdot \varphi(z)z \in I$. So, $xz \in I$, i.e., (c) implies (a). Therefore an ideal I satisfying (c) is a T-ideal. □

As a consequence of Theorem 3.20 (c) we obtain the following results proved in [38].

Corollary 3.21. *In T-type BZ-algebras all ideals are T-ideals.*

Corollary 3.22. *A BZ-algebra is T-type if and only if $\{0\}$ is its T-ideal.*

Corollary 3.23. *Any ideal containing T-ideal is a T-ideal.*

Theorem 3.24. *A T-ideal is anti-grouped if and only if it is associative.*

Proof. If I is an anti-grouped T-ideal and $x\varphi(z) \in I$, then $xz \in I$, by Theorem 3.20(a). Whence, by the last condition of the same theorem, we deduce $\varphi(xz) \cdot xz \in I$, which, by Lemma 3.2, gives $\varphi(xz) \in I$. From this, applying (14), we obtain $\varphi^2(zx) \in I$. But I is anti-grouped, so, $zx \in I$. This proves that I is associative.

The converse statement is a consequence of Proposition 3.19. □

Theorem 3.25. *Any closed H-ideal of X is contained in $G(X)$.*

Proof. By (7) and (8), for any $x \in X$, we have $\varphi(x\varphi^2(x)) \leqslant \varphi^2(x)x = 0$. So, $x\varphi^2(x) \in \operatorname{Ker}\varphi$. If x is an arbitrary element of a closed H-ideal I, then $\varphi^2(x) \in I$. Thus $x\varphi^2(x) \in I$, by Theorem 3.5. So, $x\varphi^2(x) \in I \cap \operatorname{Ker}\varphi = \{0\}$, whence $x \leqslant \varphi^2(x)$. Applying (8) we obtain $x = \varphi^2(x)$. Therefore $I \subseteq G(X)$. □

Theorem 3.26. *Let X be a BZ-algebra. A subalgebra $S \subseteq G(X)$ is a BZ-ideal which is an H-ideal of X if and only if*

(18) *$y \in S$ implies $x \cdot xy \in S$,*

(19) *$xb = ab$ implies $x = a$ for $a, b \in S$.*

Proof. Assume that a subalgebra $S \subseteq G(X)$ is an H-ideal of X. Then obviously $xy \cdot xy = 0 \in S$. Thus $y \in S$ implies $x \cdot xy \in S$, which proves (18). Now let $xb = ab$ for some $a, b \in S$. Since S is a subalgebra $ab \in S$, whence $xb \in S$. Consequently $x \in S$ (Lemma 3.2). Applying (10) and (1) we obtain

$$x = \varphi^2(x) = \varphi(ax)\varphi(a) = (0 \cdot ax)\varphi(a) = ((ab \cdot xb) \cdot ax)\varphi(a) = 0\varphi(a) = \varphi^2(a) = a.$$

This completes the proof of (19).

Assume now that a subalgebra $S \subseteq G(X)$ satisfies (18) and (19). Clearly $0 \in S$. If $xy \in S$, $y \in S$, then

$$\varphi^2(x) = \varphi(yx)\varphi(y) = \varphi^2(xy)\varphi(y) = xy \cdot \varphi(y),$$

by (10) and (14). So, $\varphi^2(x) = xy \cdot \varphi(y) \in S$ because S is a subalgebra. But $S \subseteq G(X)$, hence $x = \varphi^2(x) \in S$. Thus S is an ideal of X. Moreover, for $xy \cdot z \in S$ and $y \in S$, by (18), we get $x \cdot xy \in S$, which, together with (1), implies

$$(xz \cdot (xy \cdot z)) \cdot (x \cdot xy) = 0 \in S.$$

From this, according to Lemma 3.2, we obtain $xz \cdot (xy \cdot z) \in S$. Consequently $xz \in S$. So, S is a BZ-ideal of X.

It is also an H-ideal because for $x \in S \cap \operatorname{Ker}\varphi$ we have $\varphi(x) = 0$. Thus $xx = \varphi(x) = 0x$. From this, by (19), we obtain $x = 0$. Hence $S \cap \operatorname{Ker}\varphi = \{0\}$. □

The following lemma is obvious.

Lemma 3.27. *A non-empty subset of a group-like BZ-algebra is its subalgebra if and only if it is a subgroup of the corresponding group.*

Proposition 3.28. *A non-empty subset A of a group-like BZ-algebra is its subalgebra if and only if it is a closed ideal.*

Proof. Indeed, if A is a subalgebra then $0 \in A$ and A is a subgroup of the corresponding group. Therefore, $0a = a^{-1} \in A$ for every $a \in A$ and $z = xa \in A$ implies $x = z \cdot a^{-1} \in A$. So, A is a closed ideal.

Conversely, if A is a closed ideal then $ya^{-1} = 0a \in A$ for every $a \in A$. Moreover, for all $x, y \in A$, from $z = xy = x \cdot y^{-1}$ we obtain $A \ni x = z \cdot y = z(0y)$, which implies $z = xy \in A$. Therefore A is a subalgebra. □

Corollary 3.29. *A non-empty subset of a group-like BZ-algebra is its closed ideal if and only if it is a subgroup of the corresponding group.*

Corollary 3.30. *A non-empty subset of a group-like BZ-algebra is its closed BZ-ideal if and only if it is a normal subgroup of the corresponding group.*

Proof. Let A be a closed BZ-ideal. Then, according to (1), for all $a \in A$ and $x \in X$ we have

$$((x(0a))a)x = ((x(0a))(0(0a)))(x0) = 0 \in A,$$

and consequently $x \cdot a \cdot x^{-1} = (x(0a))x \in A$. So, A is a normal subgroup.

On the other hand, if A is a normal subgroup, then for every $x \in X$ and $a \in A$ there is $c \in A$ such that $x \cdot a = c \cdot x$. Thus $a \in A$ and $(xa)y \in A$ imply $x \cdot a^{-1} \cdot y^{-1} = b \in A$, whence $b \cdot y = x \cdot a^{-1} = c \cdot x$ for some $c \in A$. Therefore $xy = x \cdot y^{-1} = c^{-1} \cdot b \in A$, which proves that A is a BZ-ideal. □

In general, an ideal (BZ-ideal too) is not a subalgebra. For example, the set of all non-negative integers is a BZ-ideal of a group-like BZ-algebra induced by the additive group of all integers. This BZ-ideal is not closed. It is not a subalgebra, too.

4. Atoms

Definition 4.1. An element a of a BZ-algebra X is called an *atom* if $x \leqslant a$ implies $x = a$ for all $x \in X$, that is, a is a minimal element of $(X; \leqslant)$. Obviously, 0 is an atom. The set of all atoms of X is denoted by $L(X)$.

Theorem 4.2. *$L(X) = G(X) = \varphi(X) = \{x \in X : x = \varphi(y)$ for some $y \in X\}$.*

P r o o f. Let $a \in L(X)$. Then $\varphi^2(a) = a$ according to (8). This shows that $a \in G(X)$. Obviously $G(X) \subseteq \varphi(X)$. Thus $L(X) \subseteq G(X) \subseteq \varphi(X)$.

Conversely, for any $a \in \varphi(X)$ there exists $y \in X$ such that $a = \varphi(y)$. Whence, applying (11), we obtain $\varphi^2(a) = \varphi^3(y) = \varphi(y) = a$, which gives $\varphi(X) \subseteq G(X)$. Thus $G(X) = \varphi(X)$.

Now let $a \in G(X)$ and $x \leqslant a$ for some $x \in X$. Then $xa = 0$ and $ax = 0$ because

$$ax = \varphi^2(a)x = (xa \cdot \varphi(a))x = (xa \cdot 0a) \cdot x0 = 0.$$

This implies $x = a$, i.e., a is an atom. So, $G(X) \subseteq L(X)$, consequently $G(X) = L(X)$. □

Corollary 4.3. *$L(X)$ is a subalgebra of X.*

Corollary 4.4. *$L(X) = X$ if and only if X is a group-like BZ-algebra.*

Example 3.3 proves that $L(X)$ is not an ideal.

Theorem 4.5. *In a BZ-algebra X the following conditions are equivalent:*

(a) *a is an atom,*

(b) *$a = \varphi(x)$,*

(c) *$\varphi^2(a) = a$,*

(d) *$\varphi(xa) = ax$,*

(e) *$\varphi^2(ax) = ax$,*

(f) *ax is an atom,*

(g) *$y \leqslant z$ implies $ay = az$,*

where x, y, z are arbitrary elements of X.

P r o o f. (a), (b), (c) are equivalent by Theorem 4.2.

(c) $\longrightarrow$ (d). If $\varphi^2(a) = a$, then for any $z \in X$, by (14), (12), (8) and (6), we have

$$\varphi(xa) = \varphi^2(ax) = \varphi^2(a)\varphi^2(x) = a\varphi^2(x) \geqslant ax.$$

On the other hand, $\varphi(xa) \leqslant ax$, by (7). So, $\varphi(xa) = ax$.

(d) $\longrightarrow$ (e). Applying (14) to (d) we obtain (e).

(e) $\longrightarrow$ (f). $\varphi^2(ax) = ax$ means that $ax \in G(X)$. This implies (Theorem 4.2) that ax is an atom.

(f) $\longrightarrow$ (g). Since for $y \leqslant z$ we have $ay \leqslant az$, the assumption that az is an atom implies $ay = az$.

(g) $\longrightarrow$ (a). $\varphi^2(a) \leqslant a$ according to (8). Whence, applying (g), we obtain $a\varphi^2(a) = aa = 0$. So, $a \leqslant \varphi^2(a)$. Thus $\varphi^2(a) = a$. Theorem 4.2 completes the proof. □

The set $V(a) = \{x \in X : \ a \leqslant x\}$, where a is an atom of X, is called a *branch* of X. Clearly $V(0) = \operatorname{Ker}\varphi$ is a BCC-algebra. By Corollary 3.6 it is a closed anti-grouped BZ-ideal.

Theorem 4.6. *Let a, b be two atoms of a BZ-algebra X. Then*

(a) $xy \in V(ab)$ *for* $x \in V(a)$, $y \in V(b)$,

(b) $xy \in V(0) \longleftrightarrow yx \in V(0)$,

(c) $xy \in V(0) \longleftrightarrow x, y \in V(a)$ *for some atom* a,

(d) $ax = ab$ *for* $x \in V(b)$,

(e) $V(a) \cap V(b) = \emptyset$ *for* $a \neq b$,

(f) X *is a set-theoretic union of branch initiated by atoms.*

Proof.

(a) For $x \in V(a)$, $y \in V(b)$, we have $a \leqslant x$ and $b \leqslant y$, whence, by (6) we obtain $ay \leqslant xy$ and $ay \leqslant ab$. But ab is an atom (Theorem 4.5), so, $ay = ab$. Thus $ab \leqslant xy$, that is, $xy \in V(ab)$.

(b) For $xy \in V(0)$ we have $0 \leqslant xy$. Whence, applying (13) and (7), we obtain $0 = \varphi(0) = \varphi(xy) \leqslant yx$, which proves $yx \in V(0)$.

(c) Let $xy \in V(0)$ and $x \in V(a)$ for some atom a. Then $yx \in V(0)$ and $0 = 0 \cdot yx = ax \cdot yx \leqslant ay$, by (1). Since ay is an atom (Theorem 4.5), $0 = ay$. So, $y \in V(a)$. The converse statement is a consequence of (a).

Theorem 4.5 (g) implies (d). (e) is a consequence of (d). (f) is obvious. □

Corollary 4.7. *Comparable elements are in the same branch.*

Corollary 4.8. *Any bounded BZ-algebra is a BCC-algebra.*

Proof. Indeed, if X is bounded then for any $x \in X$ there exists an element $c \in X$ such that $0 \leqslant c$ and $x \leqslant c$. Then $0 = xc \cdot 0c \leqslant x0 = x$, i.e., $x \in V(0)$. Thus $X = V(0)$. □

Since $V(0) = \operatorname{Ker}\varphi$ is a closed BZ-ideal, the relation $\theta_{\operatorname{Ker}\varphi}$ is a congruence. Equivalence classes of this relation coincide with branches induced by atoms. This together with Theorem 4.6 proves the following proposition.

Proposition 4.9. $X/\operatorname{Ker}\varphi \cong G(X) = \varphi(X)$.

Theorem 4.10. *For a BZ-ideal I the following statements are equivalent:*

(a) *I is an anti-grouped ideal,*

(b) *$V(\varphi^2(x)) \subseteq I$ for $x \in I$,*

(c) *$I = \bigcup\{V(a) :\ a \in \varphi^2(I)\}$.*

Proof.

(a) $\longrightarrow$ (b). Let $x \in I$. Then, according to (8) and Lemma 3.2, $\varphi^2(x) \in I$. For $y \in V(\varphi^2(x))$ we have $\varphi^2(x) \leqslant y$, whence, $y \in I$, by Theorem 3.14 (b). Thus $V(\varphi^2(x)) \subseteq I$ for $x \in I$.

(b) $\longrightarrow$ (c). Obvious.

(c) $\longrightarrow$ (a). If $\varphi^2(x) \in I$, then $a \leqslant \varphi^2(x) \leqslant x$ for some $a \in \varphi^2(I)$. Hence $a \leqslant x$, i.e., $x \in V(a) \subseteq I$. So, a BZ-ideal I is anti-grouped. □

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