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EVALUATION OF INFINITE SERIES INVOLVING SPECIAL PRODUCTS AND THEIR ALGEBRAIC CHARACTERIZATION

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ABSTRACT. The aim of the paper is the investigation of special infinite series of the form

$$\sum_{n=0}^{\infty}\left(\frac{\prod\limits_{s=0}^{m_1 n}(s+a)\prod\limits_{s=0}^{m_2 n}(s+b)}{\prod\limits_{s=0}^{(m_1+m_2)n+1}(s+a+b)}\right)^{c}\cdot\frac{P(n)}{\theta^n}\cdot f_n(n)$$

where $(a,b,m_1,m_2,\theta,c,P(n))\in\mathbb{R}^4\times\mathbb{C}\times\{\pm1\}\times\overline{\mathbb{Q}}[n]$ and $\{f_n(n)\}_{n\in\mathbb{N}_0}$ is a sequence of rational functions. A general summation method for the sum above in the case of the special choice of parameters a, b and $f_n(n)$ is included. We find the $2m$-tuple of rational numbers α_i, β_j $(1\le i\le m,\ 1\le j\le m)$ for which

$${}_{m+1}F_m\left(\begin{array}{cc}1, & \alpha_1,\,\dots,\,\alpha_m\\ & \beta_1+1,\,\dots,\,\beta_m+1\end{array}\middle|\,x\right)\notin\overline{\mathbb{Q}}$$

iff

$${}_{m+1}F_m\left(\begin{array}{cc}1, & \beta_1,\,\dots,\,\beta_m\\ & \alpha_1,\,\dots,\,\alpha_m\end{array}\middle|\,x\right)\in\overline{\mathbb{Q}}$$

and vice versa.

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1. Introduction

The generalized hypergeometric function ${}_pF_q$ with parameters $a_1, \ldots, a_p$, $b_1, \ldots, b_q \in \mathbb{C}$ is given by the infinite series

$$ {}_pF_q\left(\begin{array}{ccc} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{array} \middle| z \right) := \sum_{n=0}^{\infty} \frac{\prod\limits_{s=1}^{p} (a_s)_n}{\prod\limits_{s=1}^{q} (b_s)_n} \cdot \frac{z^n}{n!} \tag{1} $$

where the Pochhammer symbol $(\alpha)_n$ is defined by the relation

$$ (\alpha)_0 := 1, \qquad (\alpha)_n := (\alpha + n - 1) \cdot (\alpha)_{n-1}, \quad n \in \mathbb{N}. $$

If $p = q + 1$, then the series in (1) has a finite radius of convergence. Some basic results we can find in the reference [4] which gives the state of the art on the arithmetic natur of values of generalized hypergeometric functions, both from the qualitative point of view (irrationality, transcendence, linear and algebraic independence) and from the quantitative one (approximation measures, linear independence measures, measures of algebraic independence).

Some certain basic results concerning the function (1) can be found in [1], [11], [21], [22] or [23]. Factorial and Cantor series appear in [8], [9], [10] and [14] or in the book of Shidlovski [19].

2. Summation technique

In this section we present a summation technique which is the generalization of the papers [7], [12] or [18]. The results concerning the summation techniques for the series involving special products or the theory of Lehmer's or Gosper's sums and Apéry-like series we can find in [2, §1.7], [3], [5], [6], [16], [17], [20] and [24].

The following theorem gives the information about the structure of the values of the sums (2) and (3) and other infinite series involving special products in the form $\binom{n}{k}^{\pm 1}$, where $\binom{n}{k}$ denotes the generalized binomial coefficient.

Theorem 2.1. *Let* $(a, b, \theta, m_1, m_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{C} \times \mathbb{N}^2$ *and* $P(n) = \sum\limits_{j=0}^{J} a_j n^j \in \mathbb{C}[n]$ *be a polynomial. Set*

$$S_1 := \sum_{n=0}^{\infty} \frac{\prod\limits_{s=1}^{m_1 n}(s+a) \prod\limits_{s=1}^{m_2 n}(s+b)}{\prod\limits_{s=1}^{(m_1+m_2)n+1}(s+a+b)} \cdot \theta^n P(n) \tag{2}$$

and

$$S_2 := \sum_{n=0}^{\infty} \frac{\prod\limits_{s=0}^{(m_1+m_2)n}(s+a+b)}{\prod\limits_{s=0}^{m_1 n}(s+a) \prod\limits_{s=0}^{m_2 n}(s+b)} \cdot \theta^n P(n) \tag{3}$$

as an absolutely convergent infinite series. Define

$$\phi_r(z) := \frac{(z+1)^{a+b} \cdot z^{rm_1-a-1}}{(z^{m_1} - \theta(z+1)^{(m_1+m_2)})^r}, \qquad \tau(a,b) := \frac{1}{(a+b+1)B(a+1,b+1)}$$

and let $\mathfrak{Z}$ *be the set defined as follows:*

$$\mathfrak{Z} := \left\{ \zeta_i \in \mathbb{C} : \ \zeta_i^{m_1} = \theta(\zeta_i+1)^{m_1+m_2}, \ \ |\zeta_i| < R_\theta \in \mathbb{R}^+, \ \ |\theta| < \tfrac{R_\theta^{m_1}}{(R_\theta+1)^{m_1+m_2}} \right\}.$$

Then

(1) *there exist the numbers* $c_{r,j}^{(1)} \in \mathbb{C}$, $0 \le j \le J$, $1 \le r \le J+1$, *such that*

$$S_1 = \tau(a,b) \sum_{j=0}^{J} \sum_{r=1}^{j+1} c_{r,j}^{(1)} \int\limits_0^1 \frac{x^a (1-x)^b}{(1-\theta x^{m_1}(1-x)^{m_2})^r} \ \mathrm{d}x; \tag{4}$$

(2) *there exist the numbers* $c_{r,j}^{(2)} \in \mathbb{C}$, $0 \le j \le J$, $1 \le r \le J+1$, *such that*

$$S_2 = B(a,b) \sum_{j=0}^{J} \sum_{r=1}^{j+1} c_{r,j}^{(2)} \sum_{\zeta \in \mathfrak{Z}} \operatorname{Res} \phi_r(\zeta). \tag{5}$$

Proof. First, recall that if $|Z| < 1$, $J \in \mathbb{N}_0$, $\{a_k\}_{k=0}^{J} \subset \mathbb{Z}$, then there exist the integer coefficients $C_{r,j}$ such that

$$\sum_{n=0}^{\infty} P(n) Z^n = \sum_{j=0}^{J} \sum_{r=1}^{j+1} \frac{C_{r,j}}{(1-Z)^r}.$$

Now, we consider the sums S_1 and S_2 separately.

(1) From the definition of the Beta function $B(x,y)$, assumptions on a, b, m_1, m_2, θ and in the virtue of the fact that

$$\begin{aligned}\int_0^1 x^{m_1 n+a}(1-x)^{m_2 n+b}\,\mathrm{d}x &= B(m_1 n+a+1, m_2 n+b+1)\\ &= \frac{\Gamma(a+1)\prod\limits_{s=1}^{m_1 n}(s+a)\cdot\Gamma(b+1)\prod\limits_{s=1}^{m_2 n}(s+b)}{\Gamma(a+b+2)\prod\limits_{s=1}^{(m_1+m_2)n+1}(s+a+b)}\\ &= \tau^{-1}(a,b)\frac{\prod\limits_{s=1}^{m_1 n}(s+a)\prod\limits_{s=1}^{m_2 n}(s+b)}{\prod\limits_{s=1}^{(m_1+m_2)n+1}(s+a+b)}\end{aligned}$$

together with the absolute convergence of the sum S_1 we obtain

$$\begin{aligned}S_1 &= \tau(a,b)\sum_{j=0}^{J} a_j \sum_{n=0}^{\infty}\theta^n n^j \int_0^1 x^a(1-x)^b\left(x^{m_1}(1-x)^{m_2}\right)^n\,\mathrm{d}x\\ &= \tau(a,b)\sum_{j=0}^{J}\sum_{r=1}^{j+1} c_{r,j}^{(1)}\int_0^1 \frac{x^a(1-x)^b}{\left(1-\theta x^{m_1}(1-x)^{m_2}\right)^r}\,\mathrm{d}x\end{aligned}$$

and (4) follows.

(2) To prove the identity (5) we use the integral representation of the generalized binomial coefficient:

$$\binom{n}{k} := \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} = \frac{1}{2\pi\mathrm{i}}\oint\limits_{|\zeta|=R}(\zeta+1)^n\cdot\zeta^{-(k+1)}\,\mathrm{d}\zeta, \qquad R\in\mathbb{R}^+$$

where $n,k\in\mathbb{R}$ and $k>-1$. This gives

$$\frac{\prod\limits_{s=0}^{(m_1+m_2)n}(s+a+b)}{\prod\limits_{s=0}^{m_1 n}(s+a)\cdot\prod\limits_{s=0}^{m_2 n}(s+b)} = \frac{B(a,b)}{2\pi\mathrm{i}}\oint\limits_{|\zeta|=R}(\zeta+1)^{(m_1+m_2)n+a+b}\cdot\zeta^{-(m_1 n+a+1)}\,\mathrm{d}\zeta$$

where $(a,b,m_1,m_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{N}^2$. Therefore, setting $R = R_\theta$, the properties of θ, R_θ and the definition of the set $\mathfrak{Z}$ allow us to write

$$\begin{aligned}
S_2 &= \sum_{n=0}^{\infty}\sum_{j=0}^{J} a_j n^j \oint\limits_{|\zeta|=R_\theta} \left(\frac{\theta(\zeta+1)^{m_1+m_2}}{\zeta^{m_1}}\right)^n \cdot \frac{(\zeta+1)^{a+b}}{\zeta^{a+1}}\,\mathrm{d}\zeta \\
&= \frac{B(a,b)}{2\pi\mathrm{i}} \oint\limits_{|\zeta|=R_\theta} \sum_{j=0}^{J} a_j \cdot \frac{(\zeta+1)^{a+b}}{\zeta^{a+1}} \sum_{n=0}^{\infty} n^j \left(\frac{\theta(\zeta+1)^{m_1+m_2}}{\zeta^{m_1}}\right)^n \,\mathrm{d}\zeta \\
&= \frac{B(a,b)}{2\pi\mathrm{i}} \oint\limits_{|\zeta|=R_\theta} \frac{(\zeta+1)^{a+b}}{\zeta^{a+1}} \sum_{j=0}^{J} a_j \sum_{r=1}^{j+1} \frac{C^{(2)}_{r,j}}{\left(1-\frac{\theta(\zeta+1)^{m_1+m_2}}{\zeta^{m_1}}\right)^r}\,\mathrm{d}\zeta \\
&= \frac{B(a,b)}{2\pi\mathrm{i}} \oint\limits_{|\zeta|=R_\theta} \frac{(\zeta+1)^{a+b}}{\zeta^{a+1}} \sum_{\substack{j=0 \\ c^{(2)}_{r,j}:=a_j C^{(2)}_{r,j}}}^{J}\sum_{r=1}^{j+1} \frac{c^{(2)}_{r,j}\zeta^{rm_1}}{(\zeta^{m_1}-\theta(\zeta+1)^{m_1+m_2})^r}\,\mathrm{d}\zeta \\
&= \frac{B(a,b)}{2\pi\mathrm{i}} \sum_{j=0}^{J}\sum_{r=1}^{j+1} c^{(2)}_{r,j} \oint\limits_{|\zeta|=R_\theta} \frac{(\zeta+1)^{a+b}\zeta^{rm_1-a-1}}{(\zeta^{m_1}-\theta(\zeta+1)^{m_1+m_2})^r}\,\mathrm{d}\zeta \\
&= B(a,b)\sum_{j=0}^{J}\sum_{r=1}^{j+1} c^{(2)}_{r,j} \sum_{\zeta\in\mathfrak{Z}} \operatorname{Res}\phi_r(\zeta).
\end{aligned}$$

The proof of Theorem 2.1 is complete. $\square$

Some summation results of Theorem 2.1 are presented in the next example.

Example 2.1.

$$\sum_{n=0}^{\infty} \frac{K^n}{\binom{2n+2}{n+1}(2n+3)} = \begin{cases} \frac{2}{\sqrt{5}}\ln\frac{3-\sqrt{5}}{2}+1 & \text{if} \quad K=-1, \\ \frac{2}{9}\pi\sqrt{3}-1 & \text{if} \quad K=1, \\ \frac{\pi}{4}-\frac{1}{2} & \text{if} \quad K=2, \\ \frac{4}{27}\pi\sqrt{3}-\frac{1}{3} & \text{if} \quad K=3. \end{cases}$$

Example 2.2. The following identities hold true

$$\sum_{n=0}^{\infty} \frac{T(n)\cdot n!}{\prod\limits_{s=0}^{n}(2s+3)} = \begin{cases} 2-\frac{\pi}{2} & \text{if} \quad T(n)\equiv 1, \\ \\ 1 & \text{if} \quad T(n)\equiv n(2n+3), \\ \\ \{\pi\} & \text{if} \quad T(n)\equiv n, \end{cases}$$

where $\{x\} := x-[x]$ denotes the fractional part of the real number x.

Example 2.3. The following identities hold true

$$\sum_{n=0}^{\infty} \frac{n!}{\prod\limits_{s=1}^{n}(3s+1)} = \frac{\sqrt[3]{2}}{12}\left(3\ln\frac{\sqrt[3]{4}+2\sqrt[3]{2}+1}{\sqrt[3]{4}-\sqrt[3]{2}+1} + 6\sqrt{3}\arctan\frac{\sqrt[3]{4}-1}{\sqrt{3}} + \pi\sqrt{3}\right)$$

and

$$\sum_{n=0}^{\infty} \frac{n!}{\prod\limits_{s=1}^{n}(3s+2)} = \frac{\sqrt[3]{4}}{6}\left(3\ln\frac{\sqrt[3]{4}-\sqrt[3]{2}+1}{\sqrt[3]{4}+2\sqrt[3]{2}+1} + 6\sqrt{3}\arctan\frac{\sqrt[3]{4}-1}{\sqrt{3}} + \pi\sqrt{3}\right).$$

Example 2.4. The following identity holds true

$$\sum_{n=0}^{\infty} \frac{\prod\limits_{s=1}^{n}\left(s+\frac{1}{2}\right)}{(n+2)!}\cdot\left(\frac{3}{4}\right)^{n} = \frac{8}{9}.$$

Example 2.5. We have

$$\sum_{n=0}^{\infty} \frac{\binom{3n}{n}}{K^n} = \begin{cases} \frac{\sqrt{39}}{13}\left(\frac{1}{\sqrt[3]{\frac{\sqrt{39}}{6}+\frac{\sqrt{3}}{6}}} + \frac{1}{\sqrt[3]{\frac{\sqrt{39}}{6}-\frac{\sqrt{3}}{6}}}\right) & \text{if} \quad K=-81, \\ \\ \frac{2}{3}\sqrt{3}\cos\left(\frac{\pi}{18}\right) & \text{if} \quad K=27. \end{cases}$$

Example 2.6. Define

$$\begin{aligned}
p_1(r) &:= r^3 + r^2 - 2r - 3,\\
p_2(r) &:= 841r^6 - 1682r^5 + 2001r^4 - 1856r^3 + 1339r^2 - 705r + 225,\\
p_3(r) &:= 1067229r^5 - 10977573r^4 + 1858059r^3 - 1544859r^2\\
&\qquad\qquad\qquad\qquad\qquad + 2000936r + 2308535,\\
p_4(r) &:= 1067229r^5 - 10977573r^4 + 1858059r^3 + 1544859r^2\\
&\qquad\qquad\qquad\qquad\qquad + 2000936r - 352350
\end{aligned}$$

and

$$\varphi_n := \frac{\prod_{s=1}^{2n} (s+1) \prod_{s=1}^{n} \left(s+\frac{1}{3}\right)}{\prod_{s=1}^{3n+1} \left(s+\frac{4}{3}\right)}, \qquad \text{for all} \quad n \in \mathbb{N}_0. \tag{6}$$

Then

$$\sum_{n=0}^{\infty} \frac{\varphi_n}{27^n} = \frac{36}{5} \left(\sum_{p_1(r)=0} \frac{r(1+2r)}{p_1'(r)} \ln \frac{r-1}{r} - \sum_{p_2(r)=0} r \ln \frac{p_3(r)}{p_4(r)} \right) \tag{7}$$

where the finite sums on the right hand side of (7) are extended over all roots r of the equations $p_1(r) = 0$ and $p_2(r) = 0$ respectively.

Example 2.7. Let $\kappa = \sqrt[3]{148 + 4\mathrm{i}\sqrt{3}}$ and φ_n be defined as in (6). Then

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{27^n(9n+7)\varphi_n} &= \frac{3\sqrt[3]{2\left(343\mathrm{i}\kappa\sqrt{3} - 343\kappa + 4116 - 63\mathrm{i}\kappa^2\sqrt{3} - 70\kappa^2\right)}}{64\left(111\kappa^2 - 3108 - 683\mathrm{i}\sqrt{3}\kappa^2 + 19124\mathrm{i}\sqrt{3}\right)} \times\\
&\quad \times \Big(-222\mathrm{i}\kappa\sqrt{3} - 17092\mathrm{i}\sqrt{3} + 296\mathrm{i}\kappa\sqrt[6]{3^5} - 273\mathrm{i}\kappa^2\sqrt[6]{3^5}\\
&\qquad + 6016\mathrm{i}\sqrt[6]{3^5} + 658\mathrm{i}\kappa^2\sqrt{3} + 5464\kappa\sqrt[3]{3} - 497\kappa^2\sqrt[3]{3}\\
&\qquad - 16136\sqrt[3]{3} - 4098\kappa + 336\kappa^2 + 15180\Big).
\end{aligned} \tag{8}$$

Remark 2.1. For other trivial summation results see [12], [13] and [2, §1.7], for more complicated cases see [5] or [18]. The methods in all these papers are different or not so much general as the method presented in Theorem 2.1.

Theorem 2.2 (**Mertens' multiplication theorem**). *If at least one of two convergent series* $\alpha := \sum_{n=0}^{+\infty} \alpha_n$ *and* $\beta := \sum_{n=0}^{+\infty} \beta_n$ *converges absolutely, then their Cauchy product* $\sum_{n=0}^{+\infty} \sum_{k=0}^{n} \alpha_k \beta_{n-k}$ *converges to* $\alpha\beta$.

Using Theorem 2.1 and 2.2 we can sum new infinite series. Certain summations are presented in the following example.

Example 2.8.

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{8^n} {}_3F_2\left(\left.\begin{matrix} 1, & 1, & -n \\ \frac{1}{2}, & \frac{1}{2}-n & \end{matrix}\right| \frac{1}{16}\right) = \frac{32\sqrt{2}}{31}\left(1 + \frac{1}{\sqrt{31}} \arcsin \frac{\sqrt{2}}{8}\right),$$

$$\sum_{n=0}^{+\infty} \frac{\binom{2n}{n}}{8^n} {}_3F_2\left(\left.\begin{matrix} 1, & 1, & -n \\ \frac{1}{2}, & \frac{1}{2}-n & \end{matrix}\right| 1\right) = \frac{\pi}{\sqrt{2}} + 2\sqrt{2},$$

$$\sum_{n=0}^{\infty} \frac{\binom{3n}{n}}{9^n} {}_3F_2\left(\left.\begin{matrix} \frac{1}{2}, & -n, & \frac{1}{2}-n \\ \frac{1}{3}-n, & \frac{2}{3}-n & \end{matrix}\right| \frac{2}{3}\right) = 2\sqrt{2}\cos\frac{\pi}{9},$$

$$\sum_{n=0}^{\infty} \frac{\binom{3n}{n}}{9^n} {}_4F_3\left(\left.\begin{matrix} 1, & 1, & \frac{1}{2}-n, & -n \\ \frac{3}{2}, & \frac{1}{3}-n, & \frac{2}{3}-n & \end{matrix}\right| \frac{2}{3}\right) = \pi\cos\frac{\pi}{9}.$$

Example 2.9. Define

$$F(n) := {}_7F_6\left(\left.\begin{matrix} 1, & 1, & \frac{4}{3}, & \frac{3}{2}, & -n, & -n-\frac{1}{2}, & -n-\frac{1}{3} \\ \frac{10}{9}, & \frac{13}{9}, & \frac{16}{9}, & -n-\frac{4}{9}, & -n-\frac{1}{9}, & -n+\frac{2}{9} & \end{matrix}\right| \frac{16}{729}\right).$$

Then

$$\sum_{n=0}^{\infty} \frac{\prod_{s=n+1}^{3n+1}(3s+1)}{243^n \cdot (2n+1)!} F(n) = \frac{7}{3} S_1 S_2$$

where S_1 and S_2 are the sums in (7) and (8) respectively.

3. Corollaries for the transcendence

In this section we shall discuss the special values of the generalized hypergeometric function ${}_{M+1}F_M(x)$ for $x \in \overline{\mathbb{Q}}$ using Theorem 2.1.

Remark 3.1 (Transformation). The convergent infinite series of the form

$$\sum_{n=0}^{\infty}\left(\frac{\prod_{s=0}^{m_1 n}(s+a)\prod_{s=0}^{m_2 n}(s+b)}{\prod_{s=0}^{(m_1+m_2)n+1}(s+a+b)}\right)^{\pm 1}\cdot P(n)\theta^n$$

can be transformed and written as the linear combination of generalized hypergeometric functions ${}_{M+1}F_M(a(\theta))$, where $a(\theta)$ is a suitable algebraic function and $M = m_1 + m_2$. This and Theorem 2.1 enables us to study the algebraic character of such a special hypergeometric functions at algebraic points. In view of [15] we obtain

$$\sum_{\substack{n=0\\ J\in\mathbb{N}_0}}^{\infty}\frac{\prod_{s=1}^{m_1 n}(s+a)\cdot\prod_{s=1}^{m_2 n}(s+b)}{\theta^n\prod_{s=1}^{(m_1+m_2)n+1}(s+a+b)}n^J$$

$$=\begin{cases}\mu_0\cdot{}_{M+1}F_M\left(\left.\begin{matrix}\frac{k_1+a}{m_1}\Big|_{1\le k_1\le m_1}, & \frac{k_2+b}{m_2}\Big|_{1\le k_2\le m_2}, 1\\ & \frac{k_3+a+b}{M}\Big|_{2\le k_3\le M+1}\end{matrix}\right| a_J(\theta)\right) & \text{for} \quad J=0,\\[2ex] \sum_{\lambda=1}^{J}\mu_\lambda\cdot{}_{M+1}F_M\left(\left.\begin{matrix}\lambda+\frac{k_1+a}{m_1}\Big|_{1\le k_1\le m_1}, & \lambda+\frac{k_2+b}{m_2}\Big|_{1\le k_2\le m_2}, \lambda+1\\ & \lambda+\frac{k_3+a+b}{M}\Big|_{2\le k_3\le M+1}\end{matrix}\right| a_J(\theta)\right) & \text{for} \quad J>0\end{cases}$$

and

$$\sum_{\substack{n=0\\ J\in\mathbb{N}_0}}^{\infty}\frac{\prod_{s=0}^{(m_1+m_2)n}(s+a+b)}{\theta^n\prod_{s=0}^{m_1 n}(s+a)\cdot\prod_{s=0}^{m_2 n}(s+b)}n^J$$

$$=\begin{cases}\mu_0\cdot{}_{M+1}F_M\left(\left.\begin{matrix}\frac{a+b+k_1}{M}\Big|_{1\le k_1\le M}, 1\\ \frac{a+k_2}{m_1}\Big|_{1\le k_2\le m_1}, \frac{b+k_3}{m_2}\Big|_{1\le k_3\le m_2}\end{matrix}\right| a_J(\theta)\right) & \text{for} \quad J=0,\\[2ex] \sum_{\lambda=1}^{J}\mu_\lambda\cdot{}_{M+1}F_M\left(\left.\begin{matrix}\lambda+\frac{a+b+k_1}{M}\Big|_{1\le k_1\le M}, \lambda+1\\ \lambda+\frac{a+k_2}{m_1}\Big|_{1\le k_2\le m_1}, \lambda+\frac{b+k_3}{m_2}\Big|_{1\le k_3\le m_2}\end{matrix}\right| a_J(\theta)\right) & \text{for} \quad J>0\end{cases}$$

where $M := m_1 + m_2$, $\mu_j \in \mathbb{C}$ $(0 \le j \le J)$ and $a_J(\theta)$, $J \in \mathbb{N}_0$, is a suitable algebraic function of θ.

Theorem 3.1. *Let $a, b \in \mathbb{Q}^+$, $P(X) \in \overline{\mathbb{Q}}[x]$ be a polynomial, $\theta \in \overline{\mathbb{Q}} \setminus \{0\}$ and $\lambda \in \mathbb{N}_0$. Set*

$$F^{(1)}_{m_1,m_2,\lambda}(a,b \mid \theta) := {}_{M+1}F_M \left(\begin{array}{c} \lambda + \frac{k_1+a}{m_1}\Big|_{1\le k_1\le m_1}, \ \lambda + \frac{k_2+b}{m_2}\Big|_{1\le k_2\le m_2}, \ \lambda+1 \\ \lambda + \frac{k_3+a+b}{M}\Big|_{2\le k_3\le M+1} \end{array} \Bigg| \, \theta \right)$$

and

$$F^{(2)}_{m_1,m_2,\lambda}(a,b \mid \theta) := {}_{M+1}F_M \left(\begin{array}{c} \lambda + \frac{a+b+k_1}{M}\Big|_{1\le k_1\le M}, \ \lambda+1 \\ \lambda + \frac{a+k_2}{m_1}\Big|_{1\le k_2\le m_1}, \ \lambda + \frac{b+k_3}{m_2}\Big|_{1\le k_3\le m_2} \end{array} \Bigg| \, \theta \right).$$

Assume that $F^{(i)}_{m_1,m_2,\lambda}(a,b \mid \theta)_{i=1,2}$ are two absolutely convergent series which sum can be computed using Theorem 2.1. *Then*

1. *if $(a,b) \in \mathbb{N} \times \mathbb{Q}^+$ or $(a,b) \in \mathbb{Q}^+ \times \mathbb{N}$ and $a+b \notin \mathbb{N}$ then $F^{(1)}_{m_1,m_2,\lambda}(a,b \mid \theta)$ is a transcendental number or a computable algebraic number;*
2. *if $(a,b) = \left(\frac{\alpha_1}{\alpha_2}, \frac{\beta_1}{\beta_2}\right) \in \mathbb{Q}^+ \times \mathbb{Q}^+$, $a+b \in \mathbb{N}$, $\beta_1 = 2u_1+1$, $\beta_2 = 2u_2$ with $(u_1,u_2) \in \mathbb{N}_0 \times \mathbb{N}$ then $F^{(1)}_{m_1,m_2,\lambda}(a,b \mid \theta)$ is an algebraic number;*
3. *if $(a,b) \in \mathbb{N} \times \mathbb{Q}^+$ or $(a,b) \in \mathbb{Q}^+ \times \mathbb{N}$ and $m_1 > a+1$, then $F^{(2)}_{m_1,m_2,\lambda}(a,b \mid \theta)$ is an algebraic number;*
4. *if $(a,b) \in \mathbb{Q}^+ \times \mathbb{Q}^+$, $a \notin \mathbb{N}$, $b \notin \mathbb{N}$ and $m_1 > a+1$, then $F^{(2)}_{m_1,m_2,\lambda}(a,b \mid \theta)$ is a transcendental number.*

Proof.

(1) In virtue of Theorem 2.1 we discuss the value of the integral

$$\int_0^1 \frac{x^a(1-x)^b}{(1-\theta x^{m_1}(1-x)^{m_2})^r} \, \mathrm{d}x \tag{9}$$

and the value $B(a+1,b+1)$. Without lost of the generality, we may assume that $(a,b) \in \mathbb{N} \times \mathbb{Q}^+$. Then $B(a+1,b+1) = \frac{a!}{\prod_{\ell=1}^{a}(b+\ell)} \in \mathbb{Q}$. Next, suppose that $b = \frac{\beta_1}{\beta_2}$ with $(\beta_1,\beta_2) \in \mathbb{N}^2$, $(\beta_1,\beta_2) = 1$. This allows us to use the transformation $1-x = t^{\beta_2}$ in (9). The coefficients $c^{(1)}_{r,j}$ in Theorem 2.1 are algebraic. Then Baker's theorem concerning the linear forms in logarithms, see [15], and Remark 3.1 completes the proof of statement (1) of this theorem.

(2) Suppose that $a = \frac{\alpha_1}{\alpha_2}$, $b = \frac{\beta_1}{\beta_2}$ with $(\alpha_i, \beta_j)_{i,j=1,2} \in \mathbb{N}^4$ and $(\alpha_1, \alpha_2) = 1 = (\beta_1, \beta_2)$. Using the transformation $1 - x = t^{\beta_2} x$ the integral (9) gives the form

$$\mathfrak{I}_r := \beta_2 \int_0^\infty \frac{t^{\beta_1+\beta_2-1} \cdot (t^{\beta_2}+1)^{r(m_1+m_2)-\frac{\alpha_1}{\alpha_2}-\frac{\beta_1}{\beta_2}-2}}{\left((t^{\beta_2}+1)^{m_1+m_2} - \theta t^{\beta_2 m_2}\right)^r} \,\mathrm{d}t.$$

Now, assume that $\beta_1 = 2u_1 + 1$ and $\beta_2 = 2u_2$ with $(u_1, u_2) \in \mathbb{N}_0 \times \mathbb{N}$. Then the integral (9) can be easy computed with the residues theorem. Thus, we obtain

$$\mathfrak{I}_r = \pi\beta_2 \mathrm{i} \sum_{p=1}^{P} \operatorname*{Res}_{t=\zeta_p} \frac{t^{\beta_1+\beta_2-1} \cdot (t^{\beta_2}+1)^{r(m_1+m_2)-\frac{\alpha_1}{\alpha_2}-\frac{\beta_1}{\beta_2}-2}}{\left((t^{\beta_2}+1)^{m_1+m_2} - \theta t^{\beta_2 m_2}\right)^r}$$

where $(\zeta_p)_{1\le p\le P}$ are all solutions of the equation

$$\left((t^{\beta_2}+1)^{m_1+m_2} - \theta t^{\beta_2 m_2}\right)(t^{\beta_2}+1)^{-r(m_1+m_2)+\frac{\alpha_1}{\alpha_2}+\frac{\beta_1}{\beta_2}+2} = 0.$$

Thus $(\zeta_p)_{1\le p\le P}$ are, obviously, algebraic numbers. Further, we express the term $B(a+1, b+1)$ for $a \in (0,1)$. From the properties of the Beta function $B(x,y)$, we obtain immediately that

$$\begin{aligned} B(a+1,b+1) &:= \frac{\Gamma(a+1)\cdot\Gamma(b+1)}{\Gamma(a+b+2)} = \frac{\Gamma(a+1)\cdot\Gamma(a+b+1-a)}{(a+b+1)!} \\ &= \frac{a\cdot\Gamma(a)\cdot\Gamma(1-a)\prod\limits_{\ell=1}^{a+b}(\ell-a)}{(a+b+1)!} \\ &= \frac{\pi a}{\sin(\pi a)} \cdot \frac{\prod\limits_{\ell=1}^{a+b}(\ell-a)}{(a+b+1)!} \in \pi\cdot\overline{\mathbb{Q}} \end{aligned}$$

where $\pi \cdot \overline{\mathbb{Q}} = \left\{x \in \mathbb{C} : \ x = \pi \cdot A, \ A \in \overline{\mathbb{Q}}\right\}$. Thus the sum of the series $F^{(1)}_{m_1,m_2,\lambda}(a, b \mid \theta)$ is an algebraic number in view of Theorem 2.1 and Remark 3.1.

For $a > 1$ the proof is analogous as in the case $a \in (0,1)$.

(3) Theorem 2.1, Remark 3.1 and the algebraic properties of the Beta function $B(x,y)$ imply the algebraicity of the sum $F^{(2)}_{m_1,m_2,\lambda}(a, b \mid \theta)$.

(4) Theorem 2.1, Remark 3.1 and the algebraic properties of the Beta function $B(x,y)$ imply the transcendence of the sum $F^{(2)}_{m_1,m_2,\lambda}(a, b \mid \theta)$. □

Remark 3.2. The proof of the fact that the integral $\mathfrak{I}_r$ in the proof of Theorem 3.1 for general $(\alpha_i, \beta_i, m_i, r)_{i=1,2} \in \mathbb{N}^7$ is an algebraic multiple of the number π may be very hard. On the other hand, the residues theorem implies that $\mathfrak{I}_r = \sum_j A_j \cdot \ln B_j$ where A_j and B_j are suitable complex numbers.

Corollary 3.1. *Let $x \in \overline{\mathbb{Q}}$ and $M := m_1 + m_2$, $(m_1, m_2) \in \mathbb{N}^2$. Define*

$$\alpha_k := \begin{cases} \frac{a+k}{m_1} & \iff \quad 1 \le k \le m_1, \\ \\ \frac{b+k-m_1}{m_2} & \iff \quad m_1 + 1 \le k \le m_1 + m_2, \end{cases}$$
$$\beta_k := \tfrac{a+b+k}{M} \iff 1 \le k \le M.$$

(1) *Assume that $(a, b) \in \mathbb{N} \times \mathbb{Q}^+$ or $(a, b) \in \mathbb{Q}^+ \times \mathbb{N}$ and $a + b \notin \mathbb{N}$, $m_1 > a + 1$. Then*

$${}_{M+1}F_M\left(\begin{matrix} 1, & \alpha_1, \dots, \alpha_M \\ & \beta_1 + 1, \dots, \beta_M + 1 \end{matrix} \,\middle|\, x \right) \in \overline{\mathbb{Q}} \quad \textit{iff} \quad {}_{M+1}F_M\left(\begin{matrix} 1, \ \beta_1, \dots, \beta_M \\ \alpha_1, \dots, \alpha_M \end{matrix} \,\middle|\, x \right) \notin \overline{\mathbb{Q}}.$$

(2) *Assume that $(a, b) \in \mathbb{Q}^+ \times \mathbb{Q}^+$, $b = \frac{\beta_1}{\beta_2}$, $a + b \in \mathbb{N}$, $\beta_1 = 2u_1 + 1$, $\beta_2 = 2u_2$ with $(u_1, u_2) \in \mathbb{N}_0 \times \mathbb{N}$ and $m_1 > a + 1$. Then*

$${}_{M+1}F_M\left(\begin{matrix} 1, & \alpha_1, \dots, \alpha_M \\ & \beta_1 + 1, \dots, \beta_M + 1 \end{matrix} \,\middle|\, x \right) \notin \overline{\mathbb{Q}} \quad \textit{iff} \quad {}_{M+1}F_M\left(\begin{matrix} 1, \ \beta_1, \dots, \beta_M \\ \alpha_1, \dots, \alpha_M \end{matrix} \,\middle|\, x \right) \in \overline{\mathbb{Q}}.$$

Proof. Apply Theorem 2.1, Theorem 3.1 and Remark 3.1 to obtain Corollary 3.1. □

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