VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-009-0125-4
Math. Slovaca **59** (2009), No. 3, 291–298

Σ_{n+1}-INVARIANT FORMS OF HIGHER DEGREE

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(*Communicated by Stanislav Jakubec*)

ABSTRACT. In [EGAWA, Y.—SUZUKI, H.: *Automorphism groups of* Σ_{n+1}*-invariant trilinear forms*, Hokkaido Math. J. **11**, (1985), 39–47] are explored Σ_{n+1}-invariant symmetric trilinear forms and their automorphisms. In the paper we generalize their results to d-linear symmetric forms for any $d \geq 3$.

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1. Introduction

Egawa and Suzuki [3] considered a trilinear form constructed in the following way. Let Σ_{n+1} be the symmetric group on the set $\{0, 1, \dots, n\}$ for $n \geq 2$ and let $V = \langle e_1, \dots, e_n \rangle$ be the natural n-dimensional irreducible Σ_{n+1}-module over the complex number field $\mathbb{C}$. That means that $(e_1, \dots, e_n)$ is a basis of V and Σ_{n+1} acts on $\{e_0, e_1, \dots, e_n\}$ in a standard way, where $e_0 = -(e_1 + \dots + e_n)$. A Σ_{n+1}-invariant symmetric trilinear form Θ_n on V was defined by

$$\begin{aligned} &\Theta_n(e_j, e_j, e_j) = n(n-1), && 1 \leq j \leq n, \\ &\Theta_n(e_j, e_j, e_k) = -(n-1), && 1 \leq j, k \leq n, \quad j \neq k, \\ &\Theta_n(e_j, e_k, e_l) = 2, && 1 \leq j, k, l \leq n, \ j \neq k \neq l \neq j. \end{aligned}$$

They proved two theorems.

Theorem A. ([3, Theorem 1]) *If Θ is an arbitrary nonzero Σ_{n+1}-invariant symmetric trilinear form over $\mathbb{C}$, then $\Theta = a\Theta_n$, $0 \neq a \in \mathbb{C}$.*

Theorem B. ([3, Theorem 2]) *If $n = 2$ or $n \geq 4$, then* $\mathrm{Aut}(\Theta_n)$*, the group of automorphisms of Θ_n, is isomorphic to $\mu_3 \times \Sigma_{n+1}$, where μ_3 is the group of complex 3rd roots of unity.*

2000 Mathematics Subject Classification: Primary 11E76.
Keywords: form of higher degree, automorphism group.

In the paper we generalize the above results to Σ_{n+1}-invariant forms of degree $d \geq 3$. In the next chapter we find all Σ_{n+1}-invariant forms of degree d, whereas in the last chapter we examine automorphisms of d-degree counterpart of Θ_n.

2. Σ_{n+1}-invariant forms

In the sequel we assume that K is a field of characteristic not dividing $d!$. Any d-linear symmetric form $\Theta\colon V^d \longrightarrow K$ on the K-vector space V determines the form (homogeneous polynomial) of degree d defined by

$$f_\Theta(X_1,\dots,X_n) := \Theta\left(\sum_{i=1}^n X_i e_i,\dots,\sum_{i=1}^n X_i e_i\right),$$

where $(e_1,\dots,e_n)$ is the basis of V. Then, by the polarization formula, the correspondence $\Theta \longmapsto f_\Theta$ is bijective.

Example 2.1. The trilinear form Θ_n determines

$$\begin{aligned}
&f_{\Theta_n}(X_1,\dots,X_n)\\
&=\Theta_n\left(\sum_{i=1}^n X_i e_i,\sum_{i=1}^n X_i e_i,\sum_{i=1}^n X_i e_i\right)\\
&=n(n-1)\sum_{i=1}^n X_i^3 - 3(n-1)\sum_{\substack{i,j=1\\ i\neq j}}^n X_i^2X_j + 12\sum_{\substack{i,j,k=1\\ i\neq j\neq k\neq i}}^n X_iX_jX_k\\
&= -\frac{1}{n+1}\big((X_1+X_2+\dots+X_n)^3 + (-nX_1+X_2+\dots+X_n)^3\\
&\qquad + (X_1-nX_2+\dots+X_n)^3+\dots+(X_1+\dots-nX_n)^3\big)\\
&= -\frac{1}{n+1}\big((-Y_1-\dots-Y_n)^3+Y_1^3+\dots+Y_n^3\big),
\end{aligned}$$

where

$$Y_i = (X_1+\dots+X_{i-1}-nX_i+X_{i+1}+\dots+X_n), \qquad i=1,\dots,n.$$

Suppose that V is n-dimensional Σ_{n+1}-module over the field K spanned by $(e_1,\dots,e_n)$ i.e. Σ_{n+1} acts on $\{e_0,e_1,\dots,e_n\}$ in a standard way, where $e_0 = -(e_1+\dots+e_n)$. Consider the ring

$$K[X_1,\dots,X_n]^{\Sigma_{n+1}} = \big\{f\in K[X_1,\dots,X_n]:\ \big(\forall\sigma\in\Sigma_{n+1}\big)\big(f(X)=f(\sigma(X))\big)\big\},$$

where

$$\sigma(X_1, \dots, X_n) = (Y_1, \dots, Y_n) \iff \sigma\left(\sum_{i=1}^{n} X_i e_i\right) = \sum_{i=1}^{n} Y_i e_i. \tag{1}$$

This is a graded ring

$$K[X_1, \dots, X_n]^{\Sigma_{n+1}} = \bigoplus_{d \geq 0} K[X_1, \dots, X_n]_d^{\Sigma_{n+1}}.$$

Theorem A says that $\mathbb{C}[X_1, \dots, X_n]_3^{\Sigma_{n+1}}$ is a 1-dimensional space spanned by f_{Θ_n}. Thus it is natural to ask about a base and $\dim_K K[X_1, \dots, X_n]_d^{\Sigma_{n+1}}$ for any $d \geq 3$.

Lemma 2.2. *A form f belongs to $K[X_1, \dots, X_n]_d^{\Sigma_{n+1}}$ if and only if f is a symmetric form of degree d and*

$$f(X_1, \dots, X_n) = f(X_1 - X_n, \dots, X_{n-1} - X_n, -X_n). \tag{2}$$

P r o o f. A form f is Σ_{n+1}-invariant if only if f is invariant with respect to the action of all permutations of the set $\{1, \dots, n\}$ on $\{e_0, e_1, \dots, e_n\}$, thus symmetric, and invariant with respect to the action of the transposition $(0, n)$ on $\{e_0, e_1, \dots, e_n\}$, i.e. satisfying the condition (2) □

Example 2.3. If

$$f(X) = (X_1 + \cdots + X_n)^d + (-nX_1 + \cdots + X_n)^d + \cdots + (X_1 + \cdots - nX_n)^d,$$

then $f(X) \in K[X_1, \dots, X_n]_d^{\Sigma_{n+1}}$.

Example 2.4. Let $s_1(X_0, \dots, X_n), \dots, s_{n+1}(X_0, \dots, X_n)$ be the fundamental symmetric forms in $X_0, \dots, X_n$ and let

$$l_0(X) = X_1 + \cdots + X_n, \qquad l_i(X) = X_1 + \cdots - nX_i + \cdots + X_n, \quad i = 1, \dots, n.$$

Then $s_1(l_0(X), \dots, l_n(X)) = 0$. Define

$$t_r(X) := s_r(l_0(X), \dots, l_n(X)), \qquad \text{for} \quad r = 2, \dots, n+1.$$

Since Σ_{n+1} permutes $l_0(X), \dots, l_n(X)$, we have

$$t_2(X), \dots, t_{n+1}(X) \in K[X_1, \dots, X_n]_d^{\Sigma_{n+1}}.$$

Notice that $f_{\Theta_n} = -\frac{3}{n+1}\, t_3$.

Theorem 2.5. $K[X_1, \dots, X_n]^{\Sigma_{n+1}} = K[t_2(X), \dots, t_{n+1}(X)]$.

P r o o f. Obviously $K[X_1, \dots, X_n]^{\Sigma_{n+1}} \supset K[t_2(X), \dots, t_{n+1}(X)]$.

Since the variety

$$V(t_2, \dots, t_{n+1}) = \big\{(x_1, \dots, x_n) \in K^n :\ t_r(x_1, \dots, x_n) = 0 \ \text{ for } \ r = 2, \dots, n+1\big\}$$

consists only of $(0, \dots, 0)$, by [8, Proposition 5.3.7], the forms $t_2(X), \dots, t_{n+1}(X)$ are a system of parameters. Now notice that

$$\deg(t_2) \cdots \deg(t_{n+1}) = (n+1)! = |\Sigma_{n+1}|,$$

so by [8, Proposition 5.5.5] we have $K[X_1, \dots, X_n]^{\Sigma_{n+1}} = K[t_2(X), \dots, t_{n+1}(X)]$. □

From the above theorem we get the following.

Corollary 2.6.

$$f(X) \in K[X_1, \dots, X_n]_d^{\Sigma_{n+1}} \iff f(X) = \sum_{\substack{i_1+\dots+i_k=d \\ 2\le i_1,\dots,i_k \le n+1}} a_{i_1\dots i_k} t_{i_1}(X) \cdots t_{i_k}(X).$$

By Theorem 2.5 and [8, p. 75] the Poincaré series

$$\sum_{d=0}^{\infty} \dim_K K[X_1, \dots, X_n]_d^{\Sigma_{n+1}} t^d$$

of the graded algebra $K[X_1, \dots, X_n]^{\Sigma_{n+1}}$ equals

$$\prod_{i=2}^{n+1} \frac{1}{1-t^i}.$$

Since $t_2(X), \dots, t_{n+1}(X)$ are algebraically independent, the representation of $f(X)$ in the above corollary is unique. Thus $b(d, n) := \dim_K K[X_1, \dots, X_n]_d^{\Sigma_{n+1}}$ equals the number of unordered partitions of d with summands in $\{2, \dots, n+1\}$. One cannot expect an explicit formula for $b(d, n)$, however it is clear that

$$b(d, n) = b(d, d-1) \qquad \text{for} \quad n \ge d-1.$$

It is not difficult to calculate $b(d, n)$ for small d, but with increasing d the calculations become more tedious.

The table below shows the value of $b(d, d-1)$ for $d \le 100$.

d	$b(d,d-1)$	d	$b(d,d-1)$	d	$b(d,d-1)$	d	$b(d,d-1)$
3	1	28	708	53	48342	78	1512301
4	2	29	847	54	56224	79	1716486
5	2	30	1039	55	65121	80	1947826
6	4	31	1238	56	75547	81	2207851
7	4	32	1507	57	87331	82	2501928
8	7	33	1794	58	101066	83	2832214
9	8	34	2167	59	116600	84	3205191
10	12	35	2573	60	134647	85	3623697
11	14	36	3094	61	155038	86	4095605
12	21	37	3660	62	178651	87	4624711
13	24	38	4378	63	205343	88	5220436
14	34	39	5170	64	236131	89	5887816
15	41	40	6153	65	270928	90	6638248
16	55	41	7245	66	310962	91	7478186
17	66	42	8591	67	356169	92	8421448
18	88	43	10087	68	408046	93	9476370
19	105	44	11914	69	466610	94	10659543
20	137	45	13959	70	533623	95	11981699
21	165	46	16424	71	609237	96	13462885
22	210	47	19196	72	695578	97	15116626
23	253	48	22519	73	792906	98	16967206
24	320	49	26252	74	903811	99	19031739
25	383	50	30701	75	1028764	100	21339417
26	478	51	35717	76	1170827		
27	574	52	41646	77	1330772		

In the next table the reader finds the values of $b(d,n)$ for $n \leq 11$ and $d \leq 12$

$d \setminus n$	3	4	5	6	7	8	9	10	11
3	1	1	1	1	1	1	1	1	1
4	2	2	2	2	2	2	2	2	2
5	1	2	2	2	2	2	2	2	2
6	3	3	4	4	4	4	4	4	4
7	2	3	3	4	4	4	4	4	4
8	4	5	6	6	7	7	7	7	7
9	3	5	6	7	7	8	8	8	8
10	5	7	9	10	11	11	12	12	12
11	4	7	9	11	12	13	13	14	14
12	7	10	14	16	18	19	20	20	21

3. Automorphism group of a certain form of degree d.

Now suppose that K is an algebraically closed field of characteristic 0. Wołowiec-Musiał and the author of this paper [7] showed a different proof of Theorem B using the fact that $f_{\Theta_n}(X)$ is isomorphic to the form

$$g_{3,n}(X) = (-X_1 - \cdots - X_n)^3 + X_1^3 + \cdots + X_n^3$$

which has a unique representation as a sum of powers of linear forms. A higher degree counterpart of $g_{3,n}(X)$ is

$$g_{d,n}(X) = (-X_1 - \cdots - X_n)^d + X_1^d + \cdots + X_n^d = l_0(X)^d + l_1(X)^d + \cdots + l_n(X)^d.$$

Using the method from [7] and certain result of Schinzel we obtain the following.

Theorem 3.1. *If $n \geq 4$ and $d \geq 3$, then $\mathrm{Aut}(g_{d,n})$, the group of automorphisms of $g_{d,n}$, is isomorphic to $\mu_d \times \Sigma_{n+1}$, where μ_d is the group of complex dth roots of unity.*

Proof. Suppose $n \geq 4$, $d \geq 3$. It can be readily verified that $g_{d,n}$ is nondegenerate which means that $(0, \ldots, 0)$ is the only common zero of the partial derivatives $\partial^2 g_{d,n}/\partial X_i \partial X_j$, for $i, j = 1, \ldots, n$.

Now we shall show that $g_{d,n}$ is indecomposable, that is, $g_{d,n}$ cannot be transformed by a nonsingular linear substitution to a form

$$g(X_1, \ldots, X_k) + h(X_{k+1}, \ldots, X_n),$$

for some forms g, h and $k \in \{1, \ldots, n-1\}$.

Suppose that $g_{d,n}$ is decomposable and for some nonsingular linear substitution φ we have

$$f(X_1, \ldots, X_n) := g_{d,n}\big(\varphi(X_1, \ldots, X_n)\big) = g(X_1, \ldots, X_k) + h(X_{k+1}, \ldots, X_n).$$

Applying the hessian H to the both sides of the above equality we have

$$H_f(X_1, \ldots, X_n) = H_g(X_1, \ldots, X_k) H_h(X_{k+1}, \ldots, X_n). \tag{3}$$

Now taking into account that H is a covariant of weight 2 we get

$$H_f(X_1, \ldots, X_n) = \det(\varphi)^2 H_{g_{d,n}}(\varphi(X_1, \ldots, X_n))$$

which together with (3) means that $H_{g_{d,n}}$ is a reducible polynomial. However, this is not true, because $\frac{H_f(X)}{d(d-1))^n}$ equals

$$(X_1 \cdots X_n)^{d-2} + (-X_1 - \cdots - X_n)^{d-2} \sum_{i=1}^{n} (X_1 \cdots X_{i-1} X_{i+1} \cdots X_n)^{d-2}, \tag{4}$$

whereas S c h i n z e l [6] proved that polynomial (4) is irreducible for $n \geq 4$ and $d \geq 3$. By [2, Theorem 3.4] that says:

> *If a nondegenerate and indecomposable form of degree $d \geq 3$ in $n \geq 4$ variables is a sum of $n+1$ dth powers of linear forms, then the linear forms in this representation are unique up to reordering and multiplying by dth roots of unity,*

the above representation of $g_{d,n}$ as a sum of $n+1$ dth powers of linear forms $l_0, l_1, \dots, l_n$ is unique and thus every automorphism permutes the summands in this representation. Consider the group homomorphism

$$\begin{aligned} &\Phi\colon \operatorname{Aut}(g_{d,n}) \longrightarrow \Sigma_{n+1},\\ &\Phi(\varphi) = \sigma \iff l_i^d(\varphi(X_1, \dots, X_n)) = l_{\sigma(i)}^d(X_1, \dots, X_n). \end{aligned}$$

Notice that cycles $(0, i)$ and $(1, \dots, n)$ belong to $\operatorname{im}\Phi$, so Φ is an epimorphism. Moreover, $\ker\Phi = \mu_d \operatorname{id}_V$. In this way we get the exact sequence

$$0 \longrightarrow \mu_d \operatorname{id}_V \longrightarrow \operatorname{Aut}(g_{d,n}) \longrightarrow \Sigma_{n+1} \longrightarrow 0$$

which splits. Since $\mu_d \operatorname{id}_V$ is contained in the center of $\operatorname{Aut}(g_{d,n})$ we have

$$\operatorname{Aut}(g_{d,n}) \cong \mu_d \times \Sigma_{n+1}.$$

$\square$

Since (1) implies $\operatorname{Aut}(\Theta) \cong \operatorname{Aut}(f_\Theta)$ (thus $\operatorname{Aut}(\Theta_n) \cong \operatorname{Aut}(g_{3,n})$), the above theorem is a generalization of Theorem B of E g a w a and S u z u k i.

Acknowledgement. We thank Prof. M. Kula for programming a computer to count $b(d, d-1)$ for big d. The values up to $d = 100$ in our table were obtained by his computer program.

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Received 14. 9. 2007

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