VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-009-0124-5
Math. Slovaca **59** (2009), No. 3, 283–290

ON ESTIMATIONS OF DISPERSION OF RATIO BLOCK SEQUENCES

József Bukor — Peter Csiba

(*Communicated by Stanislav Jakubec*)

ABSTRACT. Using new characteristics of an infinite subset of positive integers we give some estimations of the dispersion of the related block sequence.

©2009
Mathematical Institute
Slovak Academy of Sciences

1. Introduction

Denote by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and positive real numbers, respectively. For $X \subset \mathbb{N}$ let $X(n) = \#\{x \in X : \ x \leq n\}$. In the whole paper we will assume that X is infinite. Denote by $R(X) = \left\{\frac{x}{y} : \ x \in X, \ y \in X\right\}$ the *ratio set of* X and say that a set X is (R)-dense if $R(X)$ is (topologically) dense in the set $\mathbb{R}^+$. Let us notice that the concept of (R)-density was defined and first studied in papers [9] and [10]. Define

$$\underline{d}(X) = \liminf_{n\to\infty} \frac{X(n)}{n}, \qquad \overline{d}(X) = \limsup_{n\to\infty} \frac{X(n)}{n}$$

$$d(X) = \lim_{n\to\infty} \frac{X(n)}{n}$$

the *lower asymptotic density, upper asymptotic density*, and *asymptotic density* (if defined), respectively. Relations between (R)-density and asymptotic density were studied, among others, in papers [3], [4], [6] and [7].

Now let $X = \{x_1, x_2, \dots\}$ where $x_n < x_{n+1}$ are positive integers. The following sequence

$$\frac{x_1}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_2}, \frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{x_3}{x_3}, \dots, \frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}, \dots \tag{1}$$

2000 Mathematics Subject Classification: Primary 11B05.
Keywords: dispersion, block sequence, asymptotic density.
Supported by VEGA Grant no. 1/4006/07.

is called the *block sequence* of the set X. Thus the block sequence is formed by blocks $X_1, X_2, \dots, X_n, \dots$ where $X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right)$.

Distribution functions of block sequences were studied in [8].

For every $n \in \mathbb{N}$ let

$$D(X_n) = \max\left\{\frac{x_1}{x_n}, \frac{x_2 - x_1}{x_n}, \dots, \frac{x_{i+1} - x_i}{x_n}, \dots, \frac{x_n - x_{n-1}}{x_n}\right\}$$

the maximum distance between two consecutive terms in the nth block. In this paper we will consider the following characteristics, called the *dispersion* of the sequence X

$$\underline{D}(X) = \liminf_{n\to\infty} D(X_n).$$

Relations between the density of the block sequence and the dispersion of the block sequence were studied in [11], [1] and [2]. Much more information about the mentioned concepts and their relations can be found in the monograph [5]. Notice that the (R)-density of the set X is equivalent to the density of the block sequence (1) in the set $(0,1)$.

Let $X \subset \mathbb{N}$. Then X can be written in the form $X = \bigcup\limits_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N}$, where $c_n < d_n < c_{n+1}$ are positive integers for every $n \in \mathbb{N}$.

Let us introduce the following characteristics of the set X:

$$\underline{b}(X) = \liminf_{n\to\infty} \frac{d_n - c_n}{c_{n+1} - d_n}, \qquad \overline{b}(X) = \limsup_{n\to\infty} \frac{d_n - c_n}{c_{n+1} - d_n},$$

and

$$\underline{l}(X) = \liminf_{n\to\infty} \frac{c_{n+1}}{d_n}, \qquad \overline{l}(X) = \limsup_{n\to\infty} \frac{c_{n+1}}{d_n}.$$

The aim of this note is to estimate the dispersion of X by $\underline{b}(X)$, $\overline{b}(X)$, $\underline{l}(X)$, $\overline{l}(X)$.

2. Results

We will start with the following simple property of $\underline{D}(X)$.

Lemma 1. *Let $X = \{x_1 < x_2 < \cdots\} \subset \mathbb{N}$. Then*

$$\liminf_{n\to\infty} \frac{x_n - x_{n-1}}{x_n} \le \underline{D}(X) \le \limsup_{n\to\infty} \frac{x_n - x_{n-1}}{x_n} \tag{2}$$

P r o o f. By the definition of $D(X_n)$, $D(X_n) \ge \frac{x_n - x_{n-1}}{x_n}$. Therefore

$$\underline{D}(X) \ge \liminf_{n\to\infty} \frac{x_n - x_{n-1}}{x_n}.$$

On the other hand, if $D(X_n) = \frac{x_i - x_{i-1}}{x_n}$ for $i \le n$, we have

$$\frac{x_i - x_{i-1}}{x_n} \le \frac{x_i - x_{i-1}}{x_i}.$$

Hence $\underline{D}(X)$ is not greater then the limes inferior of some subsequence of $\left(\frac{x_n - x_{n-1}}{x_n}\right)$ which is clearly not greater then the limes superior of the sequence $\left(\frac{x_n - x_{n-1}}{x_n}\right)$. □

When estimating the value $\underline{D}(X)$ the following lemma is often useful.

Lemma 2. *Let $X = \{x_1, x_2, \dots\} = \bigcup_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N}$ and let $c_n < d_n < c_{n+1}$ be positive integers for $n \in \mathbb{N}$. Then*

$$\liminf_{n\to\infty} \frac{c_{n+1} - d_n}{d_{n+1}} \le \underline{D}(X) \le \limsup_{n\to\infty} \frac{c_{n+1} - d_n}{d_{n+1}}. \tag{3}$$

P r o o f. By Theorem 1 (see [11])

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{\max\{c_{i+1} - d_i : \ i = 1, 2, \dots, n\}}{d_{n+1}}. \tag{4}$$

Using this result we immediately have the lower bound in (3).

Let

$$M(X) = \big\{n \in \mathbb{N} : \ c_{n+1} - d_n = \max\{c_{i+1} - d_i : \ i = 1, 2, \dots, n\}\big\}.$$

From (4) we have

$$\underline{D}(X) \le \liminf_{m\to\infty,\, m\in M(X)} \frac{c_{m+1} - d_m}{d_{m+1}}$$

(see [11, Remark 1]). Clearly

$$\liminf_{m\to\infty,\, m\in M(X)} \frac{c_{m+1} - d_m}{d_{m+1}} \le \limsup_{n\to\infty} \frac{c_{n+1} - d_n}{d_{n+1}}$$

and the estimation of the upper bound follows. □

The following corollary is a straightforward consequence of the Lemma 1. It gives a partial characterization of sets whose dispersion is equal to zero.

Corollary 1. *The following implications hold for every set $X = \{x_1 < x_2 < \dots\} \subset \mathbb{N}$.*

(i) *If $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$ then $\underline{D}(X) = 0$.*

(ii) *If $\underline{D}(X) = 0$ then $\liminf\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$.*

P r o o f.

(i) Suppose that the set $X = \{x_1 < x_2 < \dots\} \subset \mathbb{N}$ is such that $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$. Consequently

$$\limsup_{n\to\infty} \frac{x_n - x_{n-1}}{x_n} = \limsup_{n\to\infty} \left(1 - \frac{x_{n-1}}{x_n}\right) = 0,$$

then by Lemma 1 we have $\underline{D}(X) = 0$.

(ii) Let $\underline{D}(x) = 0$. Hence

$$\liminf_{n\to\infty} \frac{x_n - x_{n-1}}{x_n} = 1 - \frac{1}{\liminf\limits_{n\to\infty} \frac{x_n}{x_{n-1}}},$$

then from Lemma 1 it follows that $\liminf\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$. □

Remark 1. The reverse implications to (i) and (ii) in the previous theorem do not hold. This is shown by the following examples.

(i) There exists a set $X = \{x_1 < x_2 < \dots\} \subset \mathbb{N}$ such that $\underline{D}(X) = 0$ and $\limsup\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \infty$.

Put $X = \bigcup\limits_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N}$ where $c_1 = 1$, $d_1 = 3$ and

$$c_n = n \cdot d_{n-1}, \qquad \text{and} \qquad d_n = (n-1) \cdot d_{n-1}^2$$

for every $n \geq 2$. Then

$$\underline{D}(X) \leq \limsup_{n\to\infty} \frac{c_{n+1} - d_n}{d_{n+1}} = \lim_{n\to\infty} \frac{(n+1) \cdot d_n - d_n}{n \cdot d_n^2} = \lim_{n\to\infty} \frac{1}{d_n} = 0.$$

On the other hand

$$\limsup_{n\to\infty} \frac{x_{n+1}}{x_n} = \limsup_{n\to\infty} \frac{c_{n+1}}{d_n} = \lim_{n\to\infty} \frac{(n+1) \cdot d_n}{d_n} = \infty.$$

(ii) There exists a set $X = \{x_1 < x_2 < \dots\} \subset \mathbb{N}$ such that $\liminf\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$ and $\underline{D}(X) \neq 0$, moreover for every $\alpha \in (0, 1\rangle$ there exists a set $X = \{x_1 < x_2 < \dots\} \subset \mathbb{N}$ such that $\liminf\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$ and $\underline{D}(X) = \alpha$.

First, let us consider the case $0 < \alpha < 1$. Put $X = \bigcup\limits_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N}$ where $c_n = nd_{n-1}$ and $d_n = \left[\frac{1}{\alpha} c_n\right]$ for every $n > 1$ (d_1 is given). Then $\lim\limits_{n\to\infty} \frac{c_{n+1} - d_n}{d_{n+1}} = \alpha$, since from Lemma 2 we immediately have $\underline{D}(X) = \alpha$.

Now let $\alpha = 1$. Then put $X = \bigcup\limits_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N}$ where $c_n = nd_{n-1}$ and $d_n = c_n + 2$ for every $n > 1$.

Theorem 1. *Let* $X = \{x_1, x_2, \dots\} = \bigcup_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N}$ *and let* $c_n < d_n < c_{n+1}$ *be positive integers for* $n \in \mathbb{N}$*. Then*

$$\frac{\underline{b}(X)}{\underline{b}(X)+1} \le \underline{d}(X) \le \frac{\overline{b}(X)}{\overline{b}(X)+1} \tag{5}$$

and

$$\frac{\underline{b}(X)}{\underline{b}(X)+1}\overline{l}(X) \le \underline{d}(X) \le \frac{\overline{b}(X)}{\overline{b}(X)+1}\overline{l}(X). \tag{6}$$

P r o o f. First, we are going to prove (5). Let $\varepsilon > 0$ be given. Then there exists such a $k_0 \in \mathbb{N}$ that for all $k \ge k_0$ the inequalities

$$\underline{b}(X) - \varepsilon < \frac{d_k - c_k}{c_{k+1} - d_k} < \overline{b}(X) + \varepsilon$$

hold. These inequalities can be rewritten to the form

$$\begin{aligned}
&\underline{b}(X)(c_{k+1} - c_k) - \varepsilon(c_{k+1} - d_k) \\
&< (d_k - c_k)\big(\underline{b}(X) + 1\big) \\
&\le (d_k - c_k)\big(\overline{b}(X) + 1\big) \\
&< \overline{b}(X)(c_{k+1} - c_k) + \varepsilon(c_{k+1} - d_k)
\end{aligned}$$

which, taking into account that $c_{k+1} - d_k < c_{k+1} - c_k$, yield

$$\frac{\underline{b}(X) - \varepsilon}{\underline{b}(X)+1}(c_{k+1} - c_k) < d_k - c_k < \frac{\overline{b}(X) + \varepsilon}{\overline{b}(X)+1}(c_{k+1} - c_k). \tag{7}$$

Let us denote $K = \sum_{i=1}^{k_0-1} (d_i - c_i)$. Then for all sufficiently large $n \in \mathbb{N}$ we have

$$X(c_n) = K + \sum_{i=k_0}^{n-1} (d_i - c_i).$$

By summing up the inequalities (7) for $k = k_0, k_0 + 1, \dots, n - 1$ we get

$$K + \frac{\underline{b}(X) - \varepsilon}{\underline{b}(X)+1}(c_n - c_{k_0}) < X(c_n) < K + \frac{\overline{b}(X) + \varepsilon}{\overline{b}(X)+1}(c_n - c_{k_0}). \tag{8}$$

Now, as $\underline{d}(X) = \liminf_{n\to\infty} \frac{X(c_n)}{c_n}$, the limit process for $n \to \infty$ applied to (8) proves (5).

To prove (6), first notice that $\overline{d}(X) = \limsup_{n\to\infty} \frac{X(d_n)}{d_n}$ and for $n > k_0$ it is

$$X(d_n) = K + \sum_{i=k_0}^{n} (d_i - c_i).$$

Again, summing up the inequalities (7), now for $k = k_0, k_0+1, \dots, n$ we have

$$K + \frac{\underline{b}(X) - \varepsilon}{\underline{b}(X) + 1}(c_{n+1} - c_{k_0}) < X(d_n) < K + \frac{\overline{b}(X) + \varepsilon}{\overline{b}(X) + 1}(c_{n+1} - c_{k_0}) \tag{9}$$

and the limit process for $n \to \infty$ applied to (9) proves (6). □

Corollary 2. *If $\underline{b}(X) \geq 1$ then X is (R)-dense, i.e. the block sequence is dense in $(0,1)$.*

Proof. Let $\underline{b}(X) \geq 1$. Then (5) implies $\underline{d}(X) \geq \frac{1}{2}$. Then, by [6, Theorem 1], X is (R)-dense. □

Corollary 3. *If $\underline{b}(X) \geq \frac{1}{\overline{l}(X)}$ then X is (R)-dense, i.e. the block sequence is dense in $(0,1)$.*

Proof. It is proved in [6, Theorem 1] that if $\underline{d}(X) + \overline{d}(X) \geq 1$ then X is (R)-dense. Summing up the left sides of inequalities (5) and (6) we have

$$\underline{d}(X) + \overline{d}(X) \geq \frac{\underline{b}(X)}{\underline{b}(X) + 1}(1 + \overline{l}(X)).$$

Notice that for $\underline{b}(X) \geq \frac{1}{\overline{l}(X)}$ the right-hand side of the above inequality is greater than or equal to one which yields the (R)-density of X. □

The following theorem shows that the upper bound in Corollary 2 cannot be improved.

Theorem 2. *For every $\varepsilon > 0$ there exists a set $X \subset \mathbb{N}$ such that $\underline{b}(X) > 1 - \varepsilon$ and X is not (R)-dense.*

Proof. Let $\varepsilon > 0$. Then there are real numbers $1 < a < b$ such that $\frac{a-1}{a(b-1)} > 1 - \varepsilon$. Let us consider the set

$$X = \bigcup_{n=1}^{\infty} \left([a^n b^n], [a^{n+1} b^n] \right) \cap \mathbb{N}.$$

In [4, Theorem 1] it is shown that $R(X) \cap (a,b) = \emptyset$, i.e. X is not (R)-dense set. An easy calculation shows that

$$\underline{b}(X) = \lim_{n\to\infty} \frac{a^{n+1}b^n - a^n b^n}{a^{n+1}b^{n+1} - a^{n+1}b^n} = \frac{a-1}{a(b-1)} > 1 - \varepsilon.$$

□

The next theorems states the upper bounds for dispersion $\underline{D}(X)$.

Theorem 3. *Let $X = \{x_1, x_2, \dots\} = \bigcup_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N}$ and let $c_n < d_n < c_{n+1}$ be positive integers for $n \in \mathbb{N}$. Then*

(i) $\underline{D}(X) \le 1 - \frac{1}{\overline{l}(X)}$

(ii) *If* $\underline{b}(X) > 0$ *then* $\underline{D}(X) \le \frac{1}{\underline{l}(X).\underline{b}(X)}$.

P r o o f.

(i) Let $\varepsilon > 0$ be given. Then there exists an n_0 such that for all $n > n_0$ we have $\overline{l}(X) + \varepsilon > \frac{c_{n+1}}{d_n}$. According to Lemma 1

$$\underline{D}(X) \le \limsup_{n\to\infty} \frac{c_{n+1} - d_n}{c_{n+1}} = \limsup_{n\to\infty} \left(1 - \frac{1}{\frac{c_{n+1}}{d_n}}\right) \le 1 - \frac{1}{\overline{l}(X) + \varepsilon}$$

and so $\underline{D}(X) \le 1 - \frac{1}{\overline{l}(X)}$.

(ii) Similarly, for given $\varepsilon > 0$ (where $\varepsilon < \underline{b}(X)$) and sufficiently large n we have

$$\underline{b}(X) - \varepsilon < \frac{d_n - c_n}{c_{n+1} - d_n} \qquad \text{and} \qquad \underline{l}(X) - \varepsilon < \frac{c_{n+1}}{d_n}.$$

Since

$$\underline{D}(X) \le \limsup_{n\to\infty} \frac{c_{n+1} - d_n}{c_{n+1}} = \limsup_{n\to\infty} \frac{1}{\frac{d_n - c_n}{c_{n+1} - d_n}} \cdot \left(\frac{1}{\frac{c_{n+1}}{d_n}} - \frac{c_n}{c_{n+1}}\right)$$

it follows that

$$\underline{D}(X) < \frac{1}{\underline{b}(X) - \varepsilon} \cdot \frac{1}{\underline{l}(X) - \varepsilon}$$

which proves the upper bound for $\underline{D}(X)$, as ε was arbitrary. □

In the next theorem we improve the estimations and give lower bound for $\underline{D}(X)$.

Theorem 4. *Let* $X = \{x_1, x_2, \dots\} = \bigcup_{n=1}^{\infty} (c_n, d_n\rangle \cap \mathbb{N}$ *and let* $c_n < d_n < c_{n+1}$ *be positive integers for* $n \in \mathbb{N}$. *Then*

$$\underline{D}(X) \le \left(1 - \frac{1}{\overline{l}(X)}\right) \cdot \max\left\{0, \left(1 - \left(\underline{l}(X) - 1\right) \cdot \underline{b}(X)\right)\right\} \tag{10}$$

and

$$\underline{D}(X) \ge \left(1 - \frac{1}{\underline{l}(X)}\right) \cdot \max\left\{0, \left(1 - \left(\overline{l}(X) - 1\right) \cdot \overline{b}(X)\right)\right\}. \tag{11}$$

P r o o f. First observe that

$$\frac{c_{n+1} - d_n}{d_{n+1}} = \left(1 - \frac{1}{\frac{c_{n+1}}{d_n}}\right) \cdot \left(1 - \left(\frac{c_{n+2}}{d_{n+1}} - 1\right) \frac{d_{n+1} - c_{n+1}}{c_{n+2} - d_{n+1}}\right).$$

Taking into account (3), in virtue of the definition of $\underline{l}(X)$, $\overline{l}(X)$, $\underline{b}(X)$, $\overline{b}(X)$ we can complete the proof by the same way as in the proof of the previous theorem. □

Remark, in the case $l(X) = \underline{l}(X) = \overline{l}(X)$ and $b(X) = \underline{b}(X) = \overline{b}(X)$ we have

$$\underline{D}(X) = \left(1 - \frac{1}{l(X)}\right) \cdot \big(1 - (l(X) - 1) \cdot b(X)\big).$$

For example, let us consider the set $X = \bigcup_{n=1}^{\infty} \big([s^n t^n], [s^{n+1} t^n]\big) \cap \mathbb{N}$ for arbitrary real numbers $1 < s < t$. An easy calculation shows that

$$b(X) = \frac{s-1}{s(t-1)}, \qquad l(X) = t \qquad \text{and} \qquad \underline{D}(X) = \frac{t-1}{s.t}.$$

REFERENCES

[1] FILIP, F.—TÓTH, J. T.: *On estimations of dispersions of certain dense block sequences*, Tatra Mt. Math. Publ. **31** (2005), 65–74.

[2] FILIP, F.—MIŠÍK, L.—TÓTH, J. T.: *Dispersion of ratio block sequences and asymptotic density*, Acta Arith. (To appear).

[3] MIŠÍK, L.: *Sets of positive integers with prescribed values of densities*, Math. Slovaca, **52** (2002), 289–296.

[4] MIŠÍK, L.—TÓTH, J. T.: *Logarithmic density of sequence of integers and density of its ratio set*, J. Théor. Nombres Bordeaux **15** (2003), 309–318.

[5] STRAUCH, O.—PORUBSKÝ, Š: *Distribution of Sequences: A Sampler*, Peter Lang, Frankfurt am Main, 2005.

[6] STRAUCH, O.—TÓTH, J. T.: *Asymptotic density of $A \subset \mathbb{N}$ and density of the ratio set $R(A)$*, Acta Arith. **87** (1998), 67–78.

[7] STRAUCH, O.—TÓTH, J. T.: *Corrigendum to Theorem 5 of the paper "Asymptotic density of $A \subset \mathbb{N}$ and density of the ratio set $R(A)$"* (Acta Arith. **87** (1998), 67–78), Acta Arith. **103** (2002), 191–200.

[8] STRAUCH, O.—TÓTH, J. T.: *Distribution functions of ratio sequences*, Publ. Math. Debrecen **58** (2001), 751–778.

[9] ŠALÁT, T.: *On ratio sets of sets of natural numbers*, Acta Arith. **15** (1969), 273–278.

[10] ŠALÁT, T.: *Quotientbasen und (R)-dichte Mengen*, Acta Arith. **19** (1971), 63–78.

[11] TÓTH, J. T.—MIŠÍK, L.—FILIP, F.: *On some properties of dispersion of block sequences of positive integers*, Math. Slovaca **54** (2004), 453–464.

Received 7. 1. 2008

Department of Mathematics
J. Selye University
P.O.Box 54
SK–945 01 Komárno
SLOVAKIA

E-mail: bukor.jozsef@selyeuni.sk
csiba.peter@selyeuni.sk