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DIVISIBITY, ITERATED DIGIT SUMS, PRIMALITY TESTS

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ABSTRACT. The divisibility of numbers is obtained by iteration of the weighted sum of their integer digits. Then evaluation of the related congruences yields information about the primality of numbers in certain recursive sequences. From the row elements in generalized Delannoy triangles, we can verify the primality of any constellation of numbers. When a number set is not a prime constellation, we can identify factors of their composite numbers. The constellation primality test is proven in all generality, and examples are given for twin primes, prime triplets, and Sophie Germain primes.

We consider the divisibility of integers at various bases in connection with the iteration of their weighted digit sums, which are shown in Section 1 to correspond to residues of congruences whose moduli are related to the selected base. Lucas and Lehmer used congruences and second order recurrences to develop deterministic primality tests for Mersenne numbers. In Section 2, we discuss an extension of these primality tests to numbers of a more general form. Recently Dilcher and Stolarsky [1] described a Pascal-type triangle to identify twin primes. In Section 3, we develop generalized triangles which identify prime constellations for any arrangement of primes.

1. Divisibility

1.1. Introduction to divisibility

We write a natural number N in decimal base and the sum of its weighted digits $S(N)$, with decimal coefficients c_i and weights w_i:

$$N = \sum_{i\geq 0} c_i 10^i \qquad \text{and} \qquad S(N) = \sum_{i\geq 0} w_i c_i. \tag{1}$$

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Divisibility tests in [11] determine whether an integer p is a factor of a decimal number N. For specific weights w_i associated with each factor p, divisibility rules state: if p divides $S(N)$, then p also divides N. The Rule of $p = 9$ has weights $w_i = 1$ and the Rule of 11 has weights $w_i = (-1)^i$. And apparently Pascal knew of the Rule for $p = 7$. We summarize these three examples of divisibility tests in the nomenclature of congruences:

$$\begin{aligned} N &\equiv 0 \pmod 9 \iff S(N) \equiv 0 \pmod 9 \quad \text{if} \quad w_i = 1, \\ N &\equiv 0 \pmod{11} \iff S(N) \equiv 0 \pmod{11} \quad \text{if} \quad w_i = (-1)^i, \\ N &\equiv 0 \pmod 7 \iff S(N) \equiv 0 \pmod 7 \quad \text{if} \quad w_i \in \{1, 3, 2, -1, -3, -2\}. \end{aligned}$$

As per [11], the sequence of weights comes from $w_i \equiv 10^i \pmod p$.

1.2. Generalized divisibility

Take an integer N represented in base B with digit coefficients c_i, and take an integer modulus p written as $p = B + \Delta$. Define the sum $S_{B,\Delta}(N)$ of the weighted digits with coefficient weights $\psi_{\Delta,i}$ in (3). So if the sum $S_{B,\Delta}(N)$ is divisible by p, then the number N is also divisible by p:

$$N \equiv 0 \pmod p \iff S_{B,\Delta}(N) \equiv 0 \pmod p \tag{2}$$

$$\text{for} \quad S_{B,\Delta}(N) = \sum_{i \geq 0} c_i \psi_{\Delta,i}, \tag{3}$$

where we define weights $\psi_{\Delta,i}$ by $\psi_{\Delta,i} \equiv (-\Delta)^i \pmod p$. As an aid to the summation convergence, we further define the weights $\psi_{\Delta,i}$ in (3) by the equivalent signed format as

$$\psi_{\Delta,i} \quad = \quad \begin{cases} \lambda_{\Delta,i} & \text{if } \lambda_{\Delta,i} \leq p/2 \\ \lambda_{\Delta,i} - p & \text{if } \lambda_{\Delta,i} > p/2 \end{cases}$$

for $\lambda_{\Delta,i} \equiv (-\Delta)^i \pmod p$.

The proof follows from a binomial expansion of base B in (4) and takes the congruence to get the desired result for the coefficient weight $\psi_{\Delta,i}$.

$$S_{B,\Delta}(N) \pmod p = \sum_{i \geq 0} c_i B^i = \sum_{i \geq 0} c_i (p - \Delta)^i = \sum_{i \geq 0} c_i (-\Delta)^i. \tag{4}$$

1.3. Iterated Sum of Digits

The Iterated Sum of Digits can be given as an algorithm where we again represent a natural number N in base B as $N = \sum_{i \geq 0} c_i B^i$. We add the B-ary coefficients to get their weighted digit sum $S_{B,\Delta}(N) = \sum_{i \geq 0} c_i \psi_{\Delta,i}$ of (3). We repeat the process by using this digit summation $S_{B,\Delta}(N)$ as our new number N, and only stop iterating when the digit sum is $\leq B$. We designate the Iterated Digit Sum $\$_{B,\Delta}(N)$ as the symbol I over S, where the final iteration of $S_{B,\Delta}(N)$

is notated as $\$_{B,\Delta}(N)$. This definition of the Iterated Sum of Digits $\$_{B,\Delta}(N)$, with its weighted coefficients and arbitrary Δ values, generalizes the Iterated Sum of Digits function in [2].

By induction we can see that the congruences at each iteration step must yield the same residue value. Thus we can extend the congruence of (2) to the Iterated Sum of Digits $\$_{B,\Delta}(N)$; and for small values of Δ with $|\Delta| < B$ and $B = p - \Delta$, we can know the exact value of $\$_{B,\Delta}(N)$, given by

$$\begin{aligned} N \equiv 0 \pmod p \quad &\iff \quad \$_{B,\Delta}(N) \equiv 0 \pmod p \\ &\iff \quad \$_{B,\Delta}(N) = 0 \text{ or } p. \end{aligned} \tag{5}$$

Standard primality tests consider what sequence of recursive numbers N_n will satisfy this congruence $N_n \equiv 0 \pmod p$.

2. Primality tests

2.1. Lucas-Lehmer test

Second order recursions can be used to test primality of special number systems ([8]), like the Mersenne numbers and related number sequences. In the Lucas-Lehmer test ([7], [12]), Mersenne numbers $M_n = 2^n - 1$ are checked for primality by the recursion $s_n \equiv s_{n-1}^2 - 2$ with initial condition $s_0 = 4$. The Mersenne number M_p is prime, if and only if $s_{p-2} \equiv 0 \pmod{M_p}$. The $\{s_n\}_{n\geq 0}$ sequence in [9] begins $\{4, 14, 194, 37634, 1416317954, \ldots\}$. This primality result corresponds to the Iterated Sum case in [2] with base $B = M_p + 1 = 2^p$, $\Delta = -1$ with Iterated Sum of Digits $\$_{2^p,-1}(\{s_n\})$ to give

$$s_{p-2} \equiv 0 \pmod{M_p} \iff \$_{2^p,-1}(s_{p-2}) = M_p. \tag{6}$$

2.2. More Mersenne primes

The Lucas-Lehmer primality test (6) comes from a full recursion $b_i(r)$ with $r = \sqrt{2}$, which has a corresponding $a_i(r)$ sequence of Fibonacci type. Described in [2], these full recursions have the form

$$b_i = rb_{i-1} + b_{i-2}, \qquad b_0 = 2, \ \ b_1 = r, \tag{7}$$

$$a_i = ra_{i-1} + a_{i-2}, \qquad a_0 = 0, \ \ a_1 = 1. \tag{8}$$

	Sequences at $r = \sqrt{2}$								
	$i = 0$	1	2	3	4	5	6	7	8
b_i	2	$\sqrt{2}$	4	$5\sqrt{2}$	14	$19\sqrt{2}$	52	$71\sqrt{2}$	194
a_i	0	1	$\sqrt{2}$	3	$4\sqrt{2}$	11	$15\sqrt{2}$	41	$56\sqrt{2}$

In [6], L e h m e r extended Lucas theory [12] by allowing for $r = \sqrt{R}$ with any integer R and proved the primality test (6) for Mersenne primes using $R = 2$. The interested reader is referred to [6] for solutions with other allowed recursion starting points other than $s_0 = 4$.

2.3. Exclusion principle

We want to check the primality of any general number $p = B + \Delta$. The congruences in the Mersenne number case with base $B = 2^t$ for $t \geq 2$ gave primality solutions when $\Delta = -1$. But before we start, we will first stipulate a prime number exclusion rule, and thus a requisite for p to be composite, based on the value selected for Δ.

Primality test exclusion requirements on $p = 2^t + \Delta$ for any odd Δ occur if $\Delta \equiv -1 \pmod 6$ when $2 \mid t$ or if $\Delta \equiv +1 \pmod 6$ when $2 \nmid t$, excepting at $p = 3$ if $\Delta = -2^k + 3$ for $k \geq 2$. This means $p = 2^t + \Delta$ is never prime in these exclusion cases, so we exclude them from our primality tests. This exclusion rule covers the fact that the exponent of 2 for Mersenne numbers must be odd; and for Fermat numbers $2^t + 1$, the exponent t must be even.

2.4. Testing for primes of general form

Using the recursion factor $r = \sqrt{2}$, we can write the generating function for sequence b_n from (7) and obtain integers β_n after extracting the square root factors, according to

$$\frac{1 + x^2}{1 - \sqrt{2}x - x^2} = \sum_{n \geq 0} b_n x^n \qquad \text{with} \quad \beta_n = \begin{cases} b_n & \text{if } 2 \mid n, \\ b_n/\sqrt{2} & \text{if } 2 \nmid n. \end{cases}$$

From experimental evidence, we offer the following conjecture:

Conjecture ($\boldsymbol{\beta}$). *For $p = 2^t + \Delta$ with odd Δ, a number p is prime if and only if congruence* (9) *is true for*

$$\boxed{\beta_n \equiv 1 \pmod p \quad at \quad t > 2} \qquad with \quad n = p + \begin{cases} \delta_0 & if \quad 2 \mid t, \\ \delta_1 & if \quad 2 \nmid t \end{cases} \tag{9}$$

$$where \quad \delta_0 = \begin{cases} -2 & if \ \Delta \equiv \{3, 5\} \pmod{24} \\ +2 & if \ \Delta \equiv \{-5, -3, 11, 13\} \pmod{24} \\ 0 & otherwise, \end{cases}$$

$$and \quad \delta_1 = \begin{cases} -2 & if \ \Delta \equiv \{-5, -3\} \pmod{24} \\ +2 & if \ \Delta \equiv \{3, 5, 11, 13\} \pmod{24} \\ 0 & otherwise. \end{cases}$$

At $t = 2$ the congruence is $\beta_n \equiv -1 \pmod p$. No pseudoprimes are known.

3. Primality confirmation by triangles

3.1. Generalized Delannoy recursion

We now define a more generalized recursion with arbitrary coefficient weights $\{\mu, \nu, \omega\}$ given as

$$d_{n,h} = \mu d_{n-1,h} + \nu d_{n-1,h-1} + \omega d_{n-2,h-1}, \tag{10}$$

with initial conditions $d_{0,0} = 1$, $d_{1,0} = \mu$, $d_{1,1} = \nu$ for the Delannoy triangle ([3], [5]), which comes from lattice path theory ([4], [10]). We will create a Lucas version of the recursive triangles with a value ω added to row $n = 2$, as inferred by the numerator of (11). Subsequent row elements $d_{n,h}$ for $n > 2$ follow recursion rules (10). The generating function for the triangle has row sums $A_n(z)$ given in terms of the $d_{n,h}$ row elements, defined in [4] as

$$\frac{1 + \omega x^2}{1 - (\mu + \nu)zx - \omega x^2} = \sum_{n \geq 0} A_n(z) x^n, \tag{11}$$

where

$$A_n(z) = \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} d_{n,h} z^{n-2h} = \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-h} \binom{n-h}{h} (u+v)^{n-2h} w^h z^{n-2h}.$$

3.2. Delannoy-Lucas triangle

When $z = 1$, we note that formula (11) in the case $\mu + \nu = 1$, $\omega = 1$ gives $A_n(1)$ as Lucas numbers, by definition of their generating function and binomial summation formulas. But we are interested in the recursive case at $z = 1$ with $\mu = 2$, $\nu = 0$, $\omega = 2$ and $d_{2,1} = 4$. This $d_{n,h}$ triangle gives even indexed b_i integers of the Lucas-Lehmer recursion (7) as reduced row sums $\Omega_n(\beta) = (\frac{1}{2})^{\lceil \frac{n}{2} \rceil} A_n$ as shown in Table D1. By observing row elements in this triangle, we make a conjecture about the primality of the row number.

D1	Delannoy Triangle $d_{n,h}$									$\Omega_n(\beta)$
n	$h = 0$	1	2	3	4	5	6	7	A_n	$(\frac{1}{2})^{\lceil \frac{n}{2} \rceil}$
0	1								1	1
1	2	0							2	1
2	4	4	0						8	4
3	8	12	0	0					20	5
4	16	32	8	0	0				56	14
5	32	80	40	0	0	0			152	19
6	64	192	144	16	0	0	0		416	52
7	128	448	448	112	0	0	0	0	1136	71

Primality Conjecture –DL. *For $d_{n,h}$ terms in nth row, n is prime if and only if*

$$\boxed{A_n \equiv 2 \pmod{n}} \qquad \textit{and} \tag{12}$$

$$\boxed{d_{n,h} \equiv 0 \pmod{n} \;\; \textit{for all} \;\; h \;\; \textit{at} \;\; 1 \leq h \leq n/2} \qquad \textit{and} \tag{13}$$

$$\boxed{d_{n,0} \equiv 2 \pmod{n}} \qquad \textit{for all} \quad n \geq 1, \tag{14}$$

where

$$d_{n,h} = \frac{n}{n-h}\binom{n-h}{h}2^{n-h}. \tag{15}$$

Taken individually, the congruences (12, 14) provide a quick check for compositeness if their residues fail to equal the values shown for primality. The condition (14) for $d_{n,0} \equiv 2 \pmod{n}$ is just Fermat's Little Theorem in base 2, since it says at $h = 0$ that $d_{n,0} = 2^n \equiv 2 \pmod{n}$ for prime n. We found empirically that condition (12) for $A_n \equiv 2 \pmod{n}$ holds up well if we require n to be square-free. And we comment that requirement (13) for $d_{n,h} \equiv 0 \pmod{n}$ for $1 \leq h \leq n/2$ can allow the factor 2^{n-h} to be dropped from each $d_{n,h}$ term for odd n, so that we can use

$$\boxed{d^\star_{n,h} \equiv 0 \pmod{n} \;\; \text{for all} \;\; h \;\; \text{at} \;\; 1 \leq h \leq n/2} \tag{16}$$

where

$$d^\star_{n,h} = \frac{n}{n-h}\binom{n-h}{h}.$$

So here, with the nth row sum of $d^\star_{n,h}$ terms being Lucas numbers, the congruence requirements in (16) can eliminate Lucas pseudoprimes ([8]).

If a number n fails the primality tests (12), (13), (14) and (16), the number n is composite. Moreover based on the residues and the column locations h where the congruences are nonzero, we can say something about the prime factors of n from $n = \prod f_j^{e_j}$.

Composite Factorization Conjecture –CL. *When the $d_{n,h}$ terms in the nth row fail the* Primality Conjecture –DL*, the nonzero congruence residues $\{r_i\}$ in* (13), (16) *occur at locations $\{h_i\}$. Then we say that each r_i and h_i contains at least one prime factor of n from prime factor set $\{f_j\}$.*

3.3. Delannoy-Fibonacci triangles

As a complement to (11), we define a generalized Fibonacci polynomial $A(z)$ in variable z. The generating function for its recursive triangle has row sums

$A_n(z)$ expressed in terms of row elements $d_{n,h}$, given in [4] by

$$\frac{1}{1-(\mu+\nu)zx-\omega x^2} \quad = \quad \sum_{n\geq 0} A_n(z)x^n, \tag{17}$$

where

$$A_n(z) = \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} d_{n,h} z^{n-2h} \quad = \quad \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-h}{h} (u+v)^{n-2h} w^h z^{n-2h}.$$

The triangle terms $d_{n,h}$ can be generated by the Delannoy recursion (10), with initial conditions $d_{0,0} = 1$, $d_{1,0} = \mu$ and $d_{1,1} = \nu$. The recursion is applied for all $n \geq 0$, without augmentation at any row level since the numerator in (17) is 1.

When $z = 1$, Delannoy triangles with $\mu = 1, \nu = 0, \omega = 1$ have row sums $A_n = A_n(1)$ as Fibonacci numbers and triangle terms $d_{n,h}$ given by their corresponding binomial coefficients.

$$F_{n+1} = A_n(1) = \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} d_{n,h} = \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-h}{h}.$$

Examining the terms $d_{n,h}$ of this Delannoy triangle shown in the Appendix, we make a primality conjecture for all primes, which we illustrate in the Prime Triangle Table –T1.

Primality Conjecture –T1. *For nth row $d_{n,h}$ terms in Delannoy prime triangle, the number $p = n+2$ is prime if and only if*

$$\boxed{d_{n,h} \equiv 0 \pmod{h+1}} \quad \textit{for every } h \textit{ over range } 0 \leq h \leq n/2,$$

where

$$d_{n,h} = \binom{n-h}{h}.$$

T1	Prime Triangle $d_{n,h}$							prime
n	$h=0$	1	2	3	4	5	A_n	p
0	1						1	2
1	1						1	3
3	1	2					3	5
5	1	4	3				8	7
9	1	8	21	20	5		55	11
11	1	10	36	56	35	6	144	13

3.4. Prime constellations

For natural numbers t_j, we define the J-Tuple set $T_J = \{t_1, t_2, \ldots, t_J\}$ by constellations of the form $\{p + k_0, p + k_1, \ldots, p + k_{J-1}\}$ with $p = t_1$ and the separation set $K = \{k_j\}$ specified with $k_j = t_{j+1} - p$ for $0 \le j \le J-1$. If all the integers in the constellation are prime numbers, we call them Prime constellations of the Prime J-Tuples. We may also write a concise notation pk_j, with the letter p and number k_j, to describe generalized triangles $E(p, pk_1, \ldots, pk_{J-1})$ built from J-Tuple constellations.

Based on values of the separation set $K = \{k_j\}$, we add together rows of the Delannoy triangle (see the Appendix) to create constellation triangles. Analyzing elements $d_{n,h}$ of the new triangle, we construct conjectures for the primality of the constellation. With the $d_{n,h}$ terms as the sum of binomial coefficients, we illustrate the primality conjectures with select rows of the constellation triangles for twin primes and for prime triplets.

Twin Primality Conjecture –T2. *For nth row $d_{n,h}$ terms in triangle $E(p, p2)$, the numbers $\{p, p2\} = \{n+2, n+4\}$ are twin primes if and only if*

$$\boxed{d_{n,h} \equiv 0 \pmod{h+1}} \quad \textit{for every } h \textit{ over range } 0 \le h \le \left\lfloor \tfrac{n+2}{2} \right\rfloor$$

where

$$d_{n,h} = \binom{n-h}{h} + \binom{n+2-h}{h}.$$

T2	Twin Prime Triangle $d_{n,h}$							primes
n	$h=0$	1	2	3	4	5	A_n	$p, p2$
0	2	1					3	
1	2	2					4	3, 5
2	2	4	1				7	
3	2	6	3				11	5, 7
4	2	8	7	1			18	
9	2	18	57	76	40	6	199	11, 13

The row sums A_n give the ordinary Lucas sequence: $A_n = L_{n+2}$.

Triple Primality Conjecture –T3. *For nth row $d_{n,h}$ terms in triangle $E(p, p2, p6)$, the numbers $\{p, p2, p6\} = \{n+2, n+4, n+8\}$ are prime triplets if and only if*

$$\boxed{d_{n,h} \equiv 0 \pmod{h+1}} \quad \textit{for every } h \textit{ over range } 0 \le h \le \left\lfloor \tfrac{n+6}{2} \right\rfloor$$

where

$$d_{n,h} = \binom{n-h}{h} + \binom{n+2-h}{h} + \binom{n+6-h}{h}.$$

T3	Triple Prime Triangle $d_{n,h}$									primes
n	$h=0$	1	2	3	4	5	6	7	A_n	p, $p2$, $p6$
0	3	6	6	1					16	
1	3	8	10	4					25	
2	3	11	16	10	1				41	
3	3	14	24	20	5				66	5, 7, 11
9	3	32	135	296	370	258	84	8	1186	11, 13, 17

Incidentally, if all separation values k_j are positive integers, the row sums A_n will follow the second order recursion $A_{n+2} = A_{n+1} + A_n$ for $n \geq 0$.

There is no requirement that the prime separation values k_j must be constant valued. So for example, Sophie Germain primes have separations defined in terms of the initial prime p with $k_1 = p + 1$. Thus we can also write a primality conjecture for that pair of constellation primes.

Sophie Germain Primality Conjecture –G1. *For nth row $d_{n,h}$ terms in triangle $E(p, 2p+1)$, the numbers $\{p, 2p+1\} = \{n+2, 2n+5\}$ are Sophie Germain primes if and only if*

$$\boxed{d_{n,h} \equiv 0 \pmod{h+1}} \quad \text{for every } h \text{ over range } 0 \leq h \leq n+1$$

where

$$d_{n,h} = \binom{n-h}{h} + \binom{2n+3-h}{h}.$$

G1	SG Prime Triangle $d_{n,h}$						primes
n	$h=0$	1	2	3	4	A_n	$p, 2p+1$
0	2	2				4	2, 5
1	2	4	3			9	3, 7
2	2	7	10	4		23	
3	2	10	21	20	5	58	5, 11

3.4.1. Primality tests for constellation triangles $E_{n,h}$

Elements $E_{n,h}$ of J-Tuple triangles at any $J \geq 1$ are defined variously by

$$E_{n,h} = d^*_{n+k_{J-1},h} = \sum_{j=0}^{J-1} \binom{n-h+k_j}{h} = \sum_{j=0}^{J-1} d_{n+k_j,h} \tag{18}$$

for $n \geq 0$ and $0 \leq h \leq \lfloor \frac{n+k_{J-1}}{2} \rfloor$. The terms $\binom{m}{h}$ are binomial coefficients and we define $\binom{m}{h} = 0$ when $m < 0$. Both coefficients $d^*_{m,h}$ and $d_{m,h}$ are derived from the same Delannoy recursion

$$d_{m,h} = d_{m-1,h} + d_{m-2,h-1}$$

with the same initial conditions $d_{0,0} = d_{1,0} = 1$, $d_{1,1} = 0$ and $d_{m,-1} = d_{-m,-1} = 0$ and $d_{m,g} = 0$ for all $g > \lfloor m/2 \rfloor$ and $m \geq 0$. The added condition for triangles with $d^*_{m,h}$ coefficients is that we specify element values $d^*_{k_{J-1}-k_j,0} = J - j$ with j in $0 \leq j \leq J - 1$, as we recursively develop each successive triangle row. Also the terms for $d^*_{n+k_{J-1},h}$ in (18) can be obtained as coefficients of t when $m = n + k_{J-1}$ in the expansion of the generating function:

$$\frac{\sum_{j=0}^{J-1} z^{k_{J-1}-k_j}}{1 - z - tz^2} = \sum_{m \geq 0} z^m \sum_{h=0}^{\lfloor \frac{m}{2} \rfloor} d^*_{m,h} t^h.$$

Prime Constellation Conjecture $E_{n,h}$. *The nth row elements $E_{n,h}$ in the J-Tuple triangle $E(p, pk_1, \ldots, pk_{J-1})$ have congruences*

$$\boxed{E_{n,h} \equiv 0 \pmod{h+1}} \quad \textit{for every } h \textit{ over range } 0 \leq h \leq \left\lfloor \tfrac{n+k_{J-1}}{2} \right\rfloor \tag{19}$$

if and only if all terms t_j are prime in the J-Tuple set, with $p = t_1 = n + 2$ as the initial prime and the constellation values given by

$$\{t_1, t_2, \ldots, t_J\} = \{n + 2, n + 2 + k_1, \ldots, n + 2 + k_{J-1}\}. \tag{20}$$

So for example, if we want to know whether a J-Tuple constellation is prime starting with integer p, we just examine the congruences of the elements $E_{n,h}$ in the row $n = p - 2$. And when using the Prime Constellation Conjecture, the number of $E_{n,h}$ congruences (19) to test may be reduced to cover only locations h that give primes at $h + 1$ with values less than $\sqrt{n}$. That maximum number of tests is the prime counting number $\pi(\sqrt{n})$.

3.4.2. Factoring composites

The Prime Constellation Conjecture fails, when at least one congruence

$$\boxed{E_{n,h} \not\equiv 0 \pmod{h+1}} \quad \text{for any } h \text{ over range } 0 \leq h \leq \left\lfloor \tfrac{n+k_{J-1}}{2} \right\rfloor, \tag{21}$$

if and only if at least one constellation term $t_j = n + 2 + k_{j-1}$ is composite. Since J-Tuple congruences are additive (18), a nonzero residue in (21) for the constellation means at least one term t_j contributed a nonzero residue.

Combining all J-Tuple factors from the constellation as $F = \prod_{i \geq 1} f_i^{e_i}$, we can identify small factors f_i in a composite by the smallest locations $h > 0$ that give nonzero congruences (21); where the prime factors are $f_i = h+1$. Some residues of $E_{n,h}$ (mod $h+1$) apparently give the number of t_j terms (mod $h+1$) for which f_i divides the J-Tuple constellation. So for constellations of many terms, we should still check the locations at $h > J-1$ for nonzero congruences, even if all congruences are zero at lower h values. But in most cases, either the number of terms divisible by f_1 will not be 0 (mod $h+1$), or term count J will be less than the smallest factor f_1; in either case, the first nonzero congruence at $h > 0$ will correspond to the smallest factor $f_1 = h+1$. Note that these small factors f_i of F arise from the composite terms t_j having that same f_i factor.

Composite Constellation Factorization Conjecture –CE. *When the nth row terms $E_{n,h}$ fail the* Prime Constellation Conjecture, *the nonzero congruence residues in* (21) *occur at locations $\{h_i\}$. Then for each nonzero residue of an E_{n,h_i} congruence, there is at least one prime factor of at least one J-Tuple term in $\{t_j\}$ which divides the locational modulus h_i+1.*

3.4.3. Proof of Prime Constellation Conjecture

We want to know if all the terms t_j are primes in the constellation

$$\{t_1, t_2, \ldots, t_J\} = \{p, p+k_1, \ldots, p+k_{J-1}\}.$$

We use the following substitutions, defined for the conjecture (19) as

$$t_{j+1} = p + k_j, \qquad n = p-2, \qquad f_i = h_i + 1. \tag{22}$$

for $0 \le j \le J-1$. We define $\{f_i\}$ as the set of primes $\le t_J/2$, prescribed by the maximum location h in (19) as $\lfloor \frac{n+k_{J-1}}{2} \rfloor$. For convenience, the primes in $\{f_i\}$ are associated with locations h which have prime values of $h+1$. We simply use prime values $h+1 \in \{f_i\}$, since composite values $h+1$ would have factors no larger than $t_J/2$ and would already be included in $\{f_i\}$.

With the first two substitutions from (22), we can write the binomial coefficients of (18) with constellation terms t_j:

$$\binom{n-h+k_j}{h} = \binom{p-2+k_j-h}{h} = \binom{t_{j+1}-2-h}{h}.$$

The Prime Constellation Conjecture requires that we examine their congruences in (19) according to

$$E_{n,h} = \sum_{j=0}^{J-1} \binom{n-h+k_j}{h} = \sum_{j=0}^{J-1} \binom{t_{j+1}-2-h}{h} \equiv 0 \pmod{h+1}.$$

Since we want to prove the conjecture for all t_j being prime, it would suffice to use a variable t to represent any such prime. Then we can see if the residue is zero for each binomial coefficient term generalized as

$$\binom{t-2+h}{h} \equiv 0 \pmod{h+1}.$$

So if each term has a zero residue, the sum of their residues is also zero.

We expand the binomial coefficient as

$$\binom{t-2+h}{h} = \frac{(t-2-h)\cdots(t+1))}{h!} = \frac{(t+h-2(h+1))\cdots(t+1)}{h!}. \tag{23}$$

We apply congruences (19) to numerator (23) with h consecutive factors, and make the final substitution from (22) with each of the primes f_i to get

$$\overbrace{(t+h)\cdots(t+1)}^{h\text{ factors}} \pmod{h+1} = \overbrace{(t+h)\cdots(t+1)}^{h\text{ factors}} \pmod{f_i}. \tag{24}$$

The denominator factors of $h!$ in (23) are individually less than the modulus $h+1$. So even if the numerator was reduced by some denominator factors, it would not affect whether or not the numerator was divisible by $h+1$.

Now we are left with the following two conditions. First consider going in the direction from the primality of constellation terms $\{t_j\}$ to the $E_{n,h}$ congruences. If we assume t is composite, then there is an f_i that divides a term t. Thus by (24), $\big(h(h-1)\cdots 1\big) \pmod{f_i} = h! \pmod{h+1} \not\equiv 0$, we get agreement with (21) and the Prime Constellation Conjecture (19).

If we assume t is a prime, another prime f_i cannot divide t. Also f_i cannot equal t, since per (19) the maximum h is $< t/2$. So by (24) with its residue of h consecutive factors, f_i would have to divide one of those factors; thus making its residue $\pmod{h+1} \equiv 0$ which agrees with Prime Constellation Conjecture (19).

The second condition goes in the direction from $E_{n,h}$ congruences to the primality of constellation terms $\{t_j\}$. If we assume at least one $E_{n,h}$ congruence has a nonzero residue according to (21), then there must be a prime factor f_i which does not divide any of the numerator factors in (24). Since that prime value $f_i = h+1$, this situation in (24) can happen if and only if the corresponding constellation term t were composite, which agrees with (21) and the Prime Constellation Conjecture (19). We mention that even if one of the factors of a composite t yields a $E_{n,h}$ congruence with zero residue, at least one of the other factors of t will yield the assumed nonzero residue; noting that all the factors of t must be included in the conjecture (19) since all prime factors of composites have $f_i \leq t/2$, per the largest modulus $h+1$ at maximum location h in (19).

Finally if the congruences (19) all have zero residues, then there must be a prime factor f_i which divides one of the numerator factors in (24). Since all factors $f_i = h+1$ are primes, this can happen in (24) if and only if the corresponding constellation term t is a prime, which is the statement of the Prime Constellation Conjecture (19). So in summation, the necessary and sufficient conditions for all possible primality and congruence scenarios are satisfied to complete the proof of the Prime Constellation Conjecture.

APPENDIX

Delannoy triangle

The recursion used to build a Delannoy triangle with $\mu = 1$, $\nu = 0$, $\omega = 1$ and boundary conditions $d_{0,0} = 1$, $d_{n,n} = 0$, $d_{n,0} = 1$ for $n > 1$ is given by

$$d_{n,h} = \mu d_{n-1,h} + \nu d_{n-1,h-1} + \omega d_{n-2,h-1}.$$

	Delannoy Triangle $d_{n,h}$								
n	$h = 0$	1	2	3	4	5	6	7	A_n
0	1								1
1	1								1
2	1	1							2
3	1	2							3
4	1	3	1						5
5	1	4	3						8
6	1	5	6	1					13
7	1	6	10	4					21
8	1	7	15	10	1				34
9	1	8	21	20	5				55
10	1	9	28	35	15	1			89
11	1	10	36	56	35	6			144
12	1	11	45	84	70	21	1		233
13	1	12	55	120	126	56	7		377
14	1	13	66	165	210	126	28	1	610
15	1	14	78	220	330	252	84	8	987

The row sums A_n give the ordinary Fibonacci sequence: $A_n = F_{n+1}$.

The generating function for these $d_{n,h}$ coefficients is

$$\frac{1}{1 - z - tz^2} = \sum_{n \geq 0} A_n z^n = \sum_{n \geq 0} z^n \sum_{h=0}^{\lfloor n/2 \rfloor} d_{n,h} t^h.$$

The term values are given by

$$d_{n,h} = \binom{n-h}{h}.$$

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