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THE NUMBER OF EDGES OF RADIUS-INVARIANT GRAPHS

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ABSTRACT. The eccentricity $e(v)$ of vertex v is defined as a distance to a farthest vertex from v. The radius of a graph G is defined as $r(G) = \min_{u \in V(G)} \{e(u)\}$. We consider properties of unchanging the radius of a graph under two different situations: deleting an arbitrary edge and deleting an arbitrary vertex. This paper gives the upper bounds for the number of edges in such graphs.

1. Introduction

One of the interesting questions arising in extremal graph theory is the effect upon radius of a graph when an edge or vertex is removed from such graph. This type of knowledge can be viewed as a measure of stability of a graph — especially when radius does not change. Some properties of such graphs were examined in papers [1] and [3], [6], [8]. The present work concentrates on the maximum number of edges of such graphs.

All graphs considered in this paper are undirected, finite, without loops or multiple edges. Let G be a graph. Then $V(G)$ denotes the vertex set of G; $E(G)$ the edge set of G; $\deg_G(v)$ (or simply $\deg(v)$) the degree of vertex v in G; $\Delta(G)$ the maximum degree of G; $d_G(u,v)$ the distance between two vertices u, v in G; $e_G(v)$ the eccentricity of v; $N(v)$ the neighbourhood of v; $N_i(v)$ the ith neighbourhood of v (i.e., the set $N_i(v) = \{u_1, \dots, u_k\}$ of all vertices such that $d_G(v, u_j) = i$).

Radius $r(G)$ is the minimum eccentricity, while $d(G)$ denotes the diameter of G — the maximum eccentricity. The centre $C(G)$ is the set of vertices with

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minimum eccentricity. A graph G is said to be *self-centered* if $V(G) = C(G)$. The notions and notations not defined here are used according to the book [2].

Definition. A graph G is:
radius-edge-invariant if $r(G-e) = r(G)$ for every $e \in E(G)$;
radius-vertex-invariant if $r(G-v) = r(G)$ for every $v \in V(G)$.

The purpose of this paper is to prove the upper bounds for the number of edges of radius-edge-invariant and radius-vertex-invariant graphs with given radius. We prove that every radius-edge-invariant graph with n vertices and radius r has at most $\frac{n(n-1)}{2}$ edges if $r=1$, $\left\lfloor \frac{n(n-2)}{2} \right\rfloor$ edges if $r=2$ and $\frac{n^2-4nr+5n+4r^2-6r}{2}$ edges if $r \geq 3$. We also show that every radius-vertex-invariant graph with n vertices and radius r has at most $\frac{n(n-1)}{2}$ edges if $r=1$, $\frac{n(n-3)}{2}$ edges if $r=2$ and $\frac{n^2-4nr+3n+4r^2-2r-2}{2}$ edges if $r \geq 3$. All these bounds are sharp.

In Section 2, we begin with some preliminary results which will be needed to prove our main theorems. These are proved in Section 3.

2. Preliminary results

A *k-depth spanning tree* of a graph G is a spanning tree of G of height k. Obviously $k \geq r$. If $k = r(G)$, such trees must be rooted at a central vertex. A breadth first search algorithm beginning with any vertex v such that $e(v) = k$ will always produce a k-depth spanning tree. Moreover, if $d(u,v) = i$ then u belongs to level i. In other words u belongs to level i iff $u \in N_i(v)$. We will consider only breadth first search depth spanning trees later in this paper.

Lemma 1. *Let G be a radius-vertex-invariant graph with n vertices and radius r. Then $\Delta(G) \leq n - 2r + 1$.*

Proof. Consider a k-depth spanning tree rooted at arbitrary vertex v. Since G is radius-vertex-invariant, there exist at least two vertices on level r or higher, and at least two vertices at every lower level because G has no cutvertices. As v could be adjacent only with vertices at level 1, the theorem holds. □

As a consequence we have that if G is radius-vertex-invariant, then $|V(G)| \geq 2r+1$. Note that in every graph G with radius r we have $\Delta(G) \leq n - 2r + 2$, see [7]. Proof of the following lemma was given by Vizing in [7], too.

Lemma 2. *Let G be a graph with n vertices and radius $r \geq 3$. Let x and y be vertices such that $d(x,y) \geq 3$. Then*

$$\deg(x) + \deg(y) \leq n - 2r + 4.$$

Let G and G' be disjoint graphs and let $u \in V(G')$. We say that a graph H is a *substitution of G into G'* in place of u, if the vertex set $V(H) = (V(G') - \{u\} \cup V(G))$ and the edge set $E(H)$ consists of all edges of the graphs $G' - \{u\}$ and G and, moreover, every vertex of G is joined to every vertex from the neighbourhood of u in G'.

Let $n \geq 2r \geq 2$. We denote $f_e(n,r)$ the maximum number of edges which could appear in radius-edge-invariant graph, $f_v(n,r)$ the maximum number of edges which could appear in radius-vertex-invariant graph and $f(n,r)$ the maximum number of edges in arbitrary graph with n vertices and radius r. A graph with n vertices, radius r and $f(n,r)$ edges will be denoted as $C(n,r)$. Similarly, maximal radius-edge-invariant and radius-vertex-invariant graphs will be denoted as $C_e(n,r)$ and $C_v(n,r)$, respectively.

We will need the following theorem of Vizing [7]:

Theorem 1.

$$f(n,1) = \frac{n(n-1)}{2},$$
$$f(n,2) = \left\lfloor \frac{n(n-2)}{2} \right\rfloor,$$
$$f(n,r) = \frac{n^2 - 4nr + 5n + 4r^2 - 6r}{2} \qquad \text{if} \quad r \geq 3.$$

Lemma 3. $f_v(n+1,r) > f_v(n,r)$.

Proof. It is obvious for $r = 1$. Consider the graph $C_v(n,r)$ and the graph G obtained from $C_v(n,r)$ by substituting the complete graph K_2 for an arbitrary vertex $v \in C_v(n,r)$. Observe that $r(G) = r$ and $|V(G)| = n+1$. If $u \in K_2$, then $G - u \cong C_v(n,r)$, so that $r(G-u) = r(G)$. Now consider $x \in C_v(n,r) - v$. Then $e_{G-x}(w) = e_{C_v(n,r)-x}(w)$ for every $w \in C_v(n,r) - \{v,x\}$ and $e_{G-x}(w) = e_{C_v(n,r)-x}(v)$ for $w \in K_2$. Hence, G is radius-vertex-invariant having $\deg(v)+1$ more edges than $C_v(n,r)$. □

Lemma 4. $f_v(n,r+1) < f_v(n,r)$.

Proof. It is obvious for $r = 1$. It is also clear that $f(n,r+1) \geq f_v(n,r+1)$. Consider a graph G (see Figure 1) which arises by substituting the complete graph K_{n-2r} for one vertex of a cycle C_{2r+1}.

G is radius-vertex-invariant of radius r and

$$|E(G)| = \frac{n^2 - 4nr + 3n + 4r^2 - 2r - 2}{2}$$
$$> \frac{n^2 - 4nr + 3n + 4r^2 - 2r - 2 - (2n - 4r)}{2} = f(n,r+1).$$

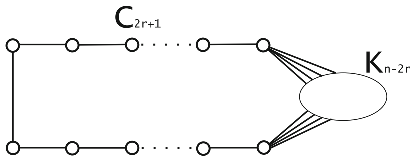

FIGURE 1

Hence,

$$f_v(n, r+1) \leq f(n, r+1) < |E(G)| \leq f_v(n, r).$$

□

We will use denotation

$$g(n, r) = \frac{n^2 - 4nr + 3n + 4r^2 - 2r - 2}{2} \qquad (n \geq 2r + 1 \geq 7)$$

later in this paper.

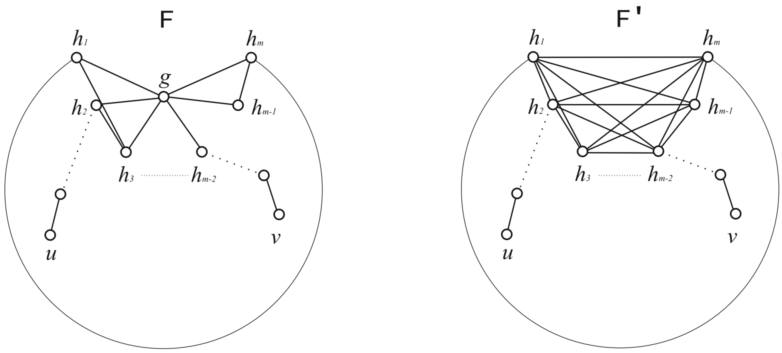

FIGURE 2

Let F be a graph and let g be a vertex of F with the neighbourhood $N(g) = \{h_1, \dots, h_m\} \in V(F)$. We will say that the vertex g is *omitted* from F (denotation $F@g$, see Figure 2) if we construct a graph $F' = F@g$ in the following way:

$$V(F') = V(F) - g,$$
$$E(F') = \big[E(F) - \{gh_i :\ h_i \in N(g)\}\big]$$
$$\cup \big\{h_i h_j :\ h_i, h_j \in N(g),\ i \neq j,\ h_i h_j \notin E(F)\big\}.$$

A similar operation called *smoothing* is used regularly and can be defined likewise but for vertices of degree 2 only (see [5]).

It is clear that if some vertex g is omitted from the graph F, then for all $u, v \in V(F')$ we have $d_F(u, v) \geq d_{F'}(u, v) \geq d_F(u, v) - 1$. Moreover, $d_{F'}(u, v) =$

$d_F(u,v)-1$ if and only if g lies on a u-v geodesic. Thus $r(F) \geq r(F') \geq r(F)-1$. For all $g,h \in V(G)$ we have $(G@g)@h \cong (G@h)@g$. We will briefly denote $(G@g)@h$ as $G@g,h$.

Lemma 5. *Let G be a graph of radius r and let $g,h \in V(G)$. Then $r(G@g,h) > r(G)-2$. Moreover, if G is radius-vertex-invariant, then for every $w \in V(G)-g-h$ it must be $r(G@g,h-w) > r(G)-2$.*

Proof. We will prove this lemma by a contradiction. Let $G' = G@g,h$; $r(G') = r(G)-2$. Then there exists a central vertex c of G' such that $e_{G'}(c) = r-2$, $e_G(c) = r$. Consider the set $N_r(c) = \{u_1,\ldots,u_s\}$ in the graph G. We have $d_{G'}(c,u_i) = r-2$ and thus g and h belong to a c-u_i geodesic (in G). But then there exists an r-depth spanning tree T of G rooted at the central vertex c containing g and h on the different levels $l_1 < l_2$. Moreover, for every u_i the vertices g,h lie on the u_i-c path in T (see Figure 3).

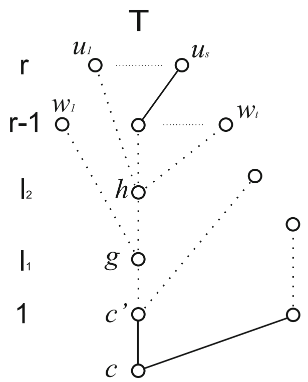

Figure 3

Similarly, if $w_1,\ldots,w_t$ are the vertices of $N_{r-1}(c)$, then the c-w_i geodesic must contain at least one of the vertices g or h. But from the structure of T, it follows that if there exists a c-w_i geodesic containing h, then there exists a c-w_i geodesic containing g.

Let c' be a vertex on the c-g geodesic such that $d(c,c') = 1$. Then $e_G(c') \leq e_T(c') = r-1$, a contradiction.

If G is radius-vertex-invariant, then we could use the same arguments as above for the graph $G@g,h-w$. If $r(G@g,h-w) = r-2$, then $e_{G-w}(c') \leq r-1$, again a contradiction. □

Lemma 6.

$$f_v(2r+1, r) = 2r+1.$$

Proof. If G is a graph with $n = 2r+1$ vertices and with at least $2r+2$ edges, then it contains at least one vertex of degree at least 3. But for every radius-vertex-invariant graph G and every vertex $v \in V(G)$ we have

$$\deg(v) \leq n - 2r + 1 = (2r+1) - 2r + 1 = 2.$$

Thus if $|V(G)| = 2r+1$ and $|E(G)| > 2r+1$, then G is not radius-vertex-invariant. If G is radius-vertex-invariant, then it has no cutvertices and therefore $\deg(v) \geq 2$ for all $v \in V(G)$. But then $\deg(v) = 2$ for all $v \in V(G)$ and thus $|E(G)| = n = 2r+1$. □

Lemma 7.

$$f_v(n, 3) = \frac{n^2 - 9n + 28}{2} = g(n, 3).$$

Proof. We first recall [7] that in every graph of radius 3 we have at least 3 disjoint pairs of vertices $\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}$ such that $d(a_i, b_i) = 3$. Consider a graph $G = C_v(n, 3)$. We distinguish the following cases depending on the maximum degree in G:

1) $\Delta(G) < n - 6$:

Suppose $\Delta(G) = n - 6 - i$, $i \in \mathbb{N}$. We have at least 3 pairs of vertices a_i, b_i in G such that

$$\deg(a_i) + \deg(b_i) \leq n - 2r + 4 = n - 6 + 4 = n - 2$$

by Lemma 2. There also are $n - 6$ additional vertices in G and thus

$$|E(G)| \leq \frac{3(n-2) + (n-6)(n-6-i)}{2} = \frac{n^2 - 9n + 28}{2} + \frac{6i - in + 2}{2}.$$

If $i = 1$ and $n = 7$, then $\Delta(G) = 0$ and $|E(G)| = 0$. In all other cases $\frac{6i-in+2}{2} = \frac{(6-n)i+2}{2} \leq \frac{-2+2}{2} = 0$. Thus

$$f_v(n, 3) = |E(G)| \leq \frac{n^2 - 9n + 28}{2} = g(n, 3).$$

2) $\Delta(G) = n - 6$:

According to Lemma 2 we have at least one vertex of degree 4 or less. Suppose that $v \in V(G)$, $\deg(v) = n - 6$ and there is no vertex $w \in V(G)$ such that $\deg(w) \leq 3$. We have either $|N_3(v)| = 3$ and $|N_2(v)| = 2$ or $|N_3(v)| = 2$ and $|N_2(v)| = 3$.

Consider the first case. Given assumption, $N_3(v) = \{a_1, a_2, a_3\}$, $N_2(v) = \{b_1, b_2\}$ and $\deg(a_i) = 4$ for all i. Thus b_j is adjacent to every a_i. We can take $c \in N(v)$ such that $cv, cb_1 \in E(G)$. But then $r(G - b_2) \leq e_{G-b_2}(c) = 2$, a contradiction.

In the second case we have $N_3(v) = \{a_1, a_2\}$, $N_2(v) = \{b_1, b_2, b_3\}$. Similarly as in the previous case every a_i must be adjacent to every b_j. This implies that there is no pair b_j, b_k of adjacent vertices of $N_2(v)$ and there is also no $c \in N(v)$ adjacent to two vertices of $N_2(v)$. Otherwise for the remaining vertex b_l we have again $r(G - b_l) < 3$. Since $\deg(b_i) \geq 4$, every b_i is adjacent to at least two vertices of $N(v)$ and thus $|N(v)| = n - 6 \geq 6$ and $n \geq 12$.

We have either $w \in V(G)$, $\deg(w) \leq 3$ or $n \geq 12$. Consider the graph $G - w$, $\deg(w) = 3 + i$, $i \in \{0, 1\}$. Since $r(G - w) = 3$, similarly as in the previous case we get

$$|E(G)| \leq |E(G - w)| + (3 + i) \leq \frac{3(n-3) + (n-7)(n-6)}{2} + (3 + i)$$
$$= \frac{n^2 - 9n + 28}{2} + \frac{11 + 2i - n}{2},$$

where $i = 0$, or both $i = 1$ and $n \geq 12$. If $i = 0$, $n \geq 10$ or if $i = 1$, $n \geq 12$ then we have $|E(G)| \leq \frac{n^2-9n+28}{2} + \frac{1}{2}$ which implies $|E(G)| \leq \frac{n^2-9n+28}{2}$.

Let $n \in \{7, 8, 9\}$. If $n = 7$, then $\Delta(G) = 1$ and thus G is not connected, a contradiction. For $n = 8$ we have $\Delta(G) = 2$. Thus

$$|E(G)| \leq \frac{2 \cdot 8}{2} < \frac{n^2 - 9n + 28}{2} = 10.$$

In such manner if $n = 9$ we have $\Delta(G) = 3$ and

$$|E(G)| \leq \frac{3 \cdot 9}{2} < \frac{n^2 - 9n + 28}{2} = 14.$$

3) $\Delta(G) = n - 5$:
We first describe some properties of such graphs: Let v be a vertex such that $\deg(v) = n - 5$. It is obvious that we have $n - 5$ vertices at distance 1 from v and, as G is radius-vertex-invariant, two vertices a_1, a_2 such that $d(v, a_i) = 2$ and two other vertices b_1, b_2 such that $d(v, b_j) = 3$.

Suppose $a_1 b_1 \in E(G)$. Then $a_1 b_2 \notin E(G)$. Otherwise there exists $c_i \in V(G)$ adjacent to v and a_1 such that $e_{G-a_2}(c_i) = 2$ (see Figure 4, a-edge 1), a contradiction. With the same argument we can show that $a_1 a_2 \notin E(G)$. Otherwise $e_{G-b_2}(c_i) = 2$ (see 4, a-edge 2). There is also no vertex c_i such that $c_i a_1, c_i a_2 \in E(G)$ (otherwise $e(c_i) = 2$, see 4, a-edge 3). Furthermore if for $c_i, c_j \in V(G)$ we have $vc_i, c_i a_1, a_1 b_1, vc_j, c_j a_2, a_2 b_2 \in E(G)$, then $c_j c_i \notin E(G)$. Otherwise $e_{G-b_2}(c_i) = e_{G-b_1}(c_j) = 2$, (see 4, a-edge 4). It is obvious that if $a_2 b_2 \in E(G)$, then $a_2 b_1 \notin E(G)$, too.

Hence, there is a set $K \subseteq V(G)$ of k vertices adjacent to v and not adjacent to a_1 nor a_2 and two nonempty sets $L, M \subseteq V(G)$ with l (m) vertices adjacent to v and a_1 (v and a_2). We have $k + l + m = n - 5$ and we know that vertices from L are not adjacent to those in M. Thus a subgraph S_1 generated by the

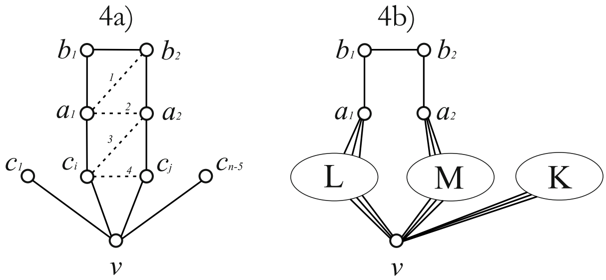

FIGURE 4

set of vertices $V(S_1) = K \cup L \cup M \cup \{v\}$ has at most $\left[\binom{n-4}{2} - lm\right]$ edges. G has also some additional edges: l edges joining L and a_1, m edges joining M and a_2 and at most 3 edges between a_1, a_2, b_1, b_2. No other edges appear in G. But then

$$|E(G)| \leq \binom{n-4}{2} - ml + m + l + 3$$
$$= \frac{n^2 - 9n + 20}{2} + \frac{6}{2} + (m + l - ml) = \frac{n^2 - 9n + 26}{2} + (m + l - ml)$$

where $(m + l - ml) \leq 1$ for any $m, l, n \in \mathbb{N}_0$. Thus

$$|E(G)| \leq \frac{n^2 - 9n + 26}{2} + 1 = \frac{n^2 - 9n + 28}{2} = g(n, 3).$$

To obtain a radius-vertex-invariant graph of radius 3 with $g(n, 3)$ edges it is sufficient to take C_7 and K_{n-6} in the graph depicted in Figure 1. This completes the proof. $\square$

Lemma 8. *Let G be a radius-vertex-invariant graph with n vertices and radius $r > 3$ such that $|N_r(v)| \geq 2$, $|N(v)| \geq 2$ and $|N_i(v)| > 2$ for all $v \in V(G)$, $i \in \{2, \ldots, r-1\}$. Then for $u \in V(G)$ such that $d(u, v) \geq r$*

$$\deg(u) + \deg(v) \leq n - 3r + 7. \tag{L8a}$$

If $r = 4$, then

$$|E(G)| \leq \frac{n^2 - 13n + 54}{2} = g(n, 4). \tag{L8b}$$

Moreover, if there is a pairing $\{p_i, q_i\}$, $i = \{1, \ldots, n\}$, of vertices of G such that $d(p_i, q_i) \geq r$, $\{p_i, q_i\} \neq \{p_j, q_j\}$ if $i \neq j$ and every $v \in V(G)$ lies in exactly two such pairs then

$$|E(G)| \leq \frac{n^2 - 4nr + 3n + 4r^2 - 2r - 2}{2} = g(n, r) \qquad \text{if} \quad r \geq 5. \tag{L8c}$$

P r o o f.

(L8a): Since for $u, v \in V(G), d(u,v) \geq r$

$$\deg(v) = |N(v)|, \qquad \deg(u) \leq n - 2 - \sum_{i=1}^{r-2} |N_i(v)|,$$

we have (as $\sum_{i=2}^{r-2} |N_i(v)| \geq 3(r-3)$, see Figure 5)

$$\deg(u) + \deg(v) \leq n - 2 - \sum_{i=2}^{r-2} |N_i(v)| \leq n - 2 - 3(r-3) = n - 3r + 7.$$

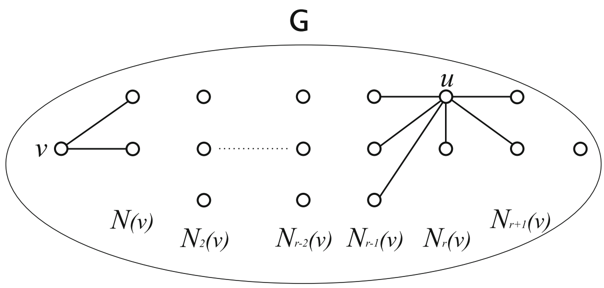

FIGURE 5

(L8b) & (L8c): It is obvious that $n \geq 3r-1$. First consider the case $n = 3r-1$. Then

$$\sum_{i=2}^{r} |N_i(v)| = 3(r-2) + 2 = 3r - 4.$$

Thus for every $v \in V(G)$ we have $\deg(v) = |N(v)| = 3r - 1 - (3r-4) - 1 = 2$. But then G is a cycle so it does not fulfil the conditions of Lemma 8.

Next suppose that $r = 4$ and $n > 3r - 1 = 11$. For any vertices $y, z \in V(G)$ at distance at least 4 there exists a vertex x not adjacent to y or z. Since $d_{G-x}(y,z) \geq 4$, we have $\deg(y) + \deg(z) \leq (n-1) - 2r + 4 = n - 5$ according to Lemma 2. For any other vertices x' and y' such that $d(x', y') = 3$ we have $\deg(x') + \deg(y') \leq n - 2r + 4 = n - 4$. Let $\deg(v) = \Delta(G) \leq n - 2 - 3(r-2) - 1 = n - 9$. We have $|N_4(v)| \geq 2$ and $|N_4(v)| + |N_3(v)| \geq 5$. Suppose $a_1, a_2 \in N_4(v)$ (see Figure 6).

Obviously there is a vertex $w \in N(v)$ such that $d(w, a_1) = 3$. Since G is radius-vertex-invariant, $e_{G-a_2}(w) \geq 4$ and thus there exists another vertex $w' \neq a_2$, $w' \neq v$, $w' \notin N(v)$, $w' \notin N_2(v)$ such that $d(w, w') \geq 4$. All vertices from $N(v)$ and $N_2(v)$ (there are at least five such vertices) have degree at most

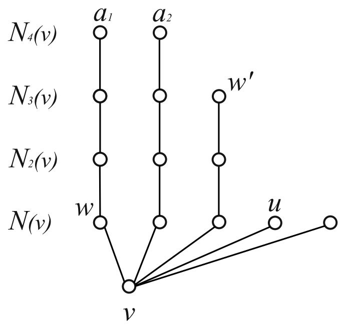

FIGURE 6

equal to $\deg(v)$. It is sufficient to take $u \in N_1(v)$, $u \neq w$ to obtain three pairs of vertices such that

$$(\deg(a_1) + \deg(v)) + (\deg(a_2) + \deg(u)) + (\deg(w) + \deg(w')) \leq 3(n-5).$$

For all other vertices $b_l \in N_3(v)$, $b_l \neq w'$ we have $\deg(b_l) + \deg(v) \leq n - 4$ and thus $\deg(b_l) + \deg(f_k) \leq n - 4$ for all $f_k \in \{v\} \cup N(v) \cup N_2(v)$. Since $|N_3(v) - \{w'\}| \geq 2$ and $|N(v)| + |N_2(v)| \geq 5$ we can obtain two additional pairs of vertices $\{b_1, f_1\}$, $\{b_2, f_2\}$ such that $\deg(b_1) + \deg(f_1) \leq n - 4$ and $\deg(b_2) + \deg(f_2) \leq n - 4$ ($f_1, f_2, b_1, b_2 \notin \{w, w', u, v\}$). All other vertices have degree at most $n - 9$ and thus

$$\begin{aligned} |E(G)| &\leq \left\lfloor \frac{3(n-5) + 2(n-4) + (n-10)(n-9)}{2} \right\rfloor \\ &= g(n,4) + \left\lfloor \frac{13-n}{2} \right\rfloor \leq g(n,4) \qquad \text{since} \quad n > 11. \end{aligned}$$

At last let $r > 4$, $n = 3r + i$, $i \in \mathbb{N}_0$. Consider n given different pairs $\{p_i, q_i\}$ of vertices such that $d(p_i, q_i) \geq r$. Every v belongs to exactly two pairs, each of these pairs have at most $n - 3r + 7$ edges and thus

$$|E(G)| \leq \frac{n(n-3r+7)}{4}.$$

We have

$$\begin{aligned} &\frac{n(n-3r+7)}{4} = \frac{n^2 - 3nr + 7n}{4} \\ &= \frac{2n^2 - 8nr + 6n + 8r^2 - 4r - 4}{4} + \frac{-n^2 + 5nr + n - 8r^2 + 4r + 4}{4} \\ &= g(n,r) + \frac{-(3r+i)^2 + 5(3r+i)r + (3r+i) - 8r^2 + 4r + 4}{4} \\ &= g(n,r) + \frac{r(7-2r-i) + 4 - i^2 + i}{4}. \end{aligned}$$

Since $r \geq 5, 7 - 2r - i \leq -3$ and obviously $i^2 \geq i$ we have

$$|E(G)| \leq g(n, r).$$

□

According to the proof of the part (L8b) we can claim the following observation:

Lemma 9. *Let G be a radius-vertex-invariant graph with n vertices and radius $r = 4$ such that $|N_i(v)| > 2$, $|N(v)| \geq 2$ and $|N_r(v)| \geq 2$ for some $v \in C(G)$, $i \in \{2, 3\}$. Let moreover $\Delta(G) \leq n - 9$. Then*

$$|E(G)| \leq \frac{n^2 - 13n + 54}{2} = g(n, 4).$$

At last we will need the following well-known theorem of Hall (see [5]):

Theorem 2 (Hall's Theorem). *There exists a system of distinct representatives for a family of sets $S_1, S_2, \ldots, S_m$ iff the union of any k of these sets contains at least k elements for all $k = 1, \ldots, m$.*

3. The bounds

Theorem 3.

$$f_e(n,1) = \frac{n(n-1)}{2},$$
$$f_e(n,2) = \left\lfloor \frac{n(n-2)}{2} \right\rfloor,$$
$$f_e(n,r) = \frac{n^2 - 4nr + 5n + 4r^2 - 6r}{2} \qquad \textit{if} \quad r \geq 3.$$

P r o o f. The bounds are the same as Vizing's.

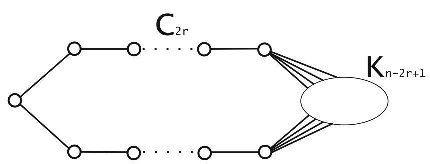

Figure 7

The radius-edge-invariant graphs of radius 1, 2 and r of the upper bound are K_n for $r = 1$, a graph with all vertices of degree $n - 2$ for $r = 2$, $n = 2k$, a graph with $n - 1$ vertices of degree $n - 2$ and one vertex of degree $n - 3$

for $r = 2$, $n = 2k + 1$ and a graph which arises by substituting the complete graph K_{n-2r+1} for one vertex of a cycle C_{2r} (see Figure 7). Thus we have the demanded equality. □

The bounds for radius-vertex-invariant graphs are somewhat different.

Theorem 4.

$$f_v(n,1) = \frac{n(n-1)}{2},$$
$$f_v(n,2) = \frac{n(n-3)}{2},$$
$$f_v(n,r) = \frac{n^2 - 4nr + 3n + 4r^2 - 2r - 2}{2} \qquad \textit{if} \quad r \geq 3.$$

P r o o f. The first case is obvious. The second is an immediate consequence of the fact that a radius-vertex-invariant graph of radius 2 has no vertex of degree $|V(G) - 2|$ or $|V(G) - 1|$.

Let $r > 2$. It is obvious that

$$\frac{n^2 - 4nr + 3n + 4r^2 - 2r - 2}{2} = g(n,r) \leq f_v(n,r)$$

as it was shown in the proof of Lemma 4 (see the graph in Figure 1). We will prove the opposite inequality $f_v(n,r) \leq g(n,r)$ by the double induction on r and n.

Base of induction:
According to Lemma 6

$$f_v(2r+1, r) = 2r + 1 = g(2r+1, r) \qquad \text{for all} \quad r \geq 3.$$

According to Lemma 7

$$f_v(n,3) = \frac{n^2 - 9n + 28}{2} = g(n,3) \qquad \text{for all} \quad n \geq 7.$$

Induction step:
Now show that if the inequality $f_v(n,r) \leq g(n,r)$ holds for all radius-vertex-invariant graphs of radius $r - 1$ and for all radius-vertex-invariant graphs with fewer than n vertices and radius r, then it holds also for any radius-vertex-invariant graph G with n vertices and radius r. We consider the following cases depending on the structure of G:

(A) There exists $v \in V(G)$ such that $G - v$ is radius-vertex-invariant.

(B) There exists $v \in V(G)$ and $u \in V(G-v)$ such that $\infty > r(G-v-u) > r(G)$.

Suppose none of the previous holds-let for all $v \in V(G)$ the graph $G - v$ is not radius-vertex-invariant and let there is no $u \in V(G - v)$ such that $\infty > r(G - v - u) > r(G)$. Let moreover:

(C) For $v \in V(G)$ there exists a vertex $u \in V(G-v)$ such that u is a cutvertex of $G-v$.

At last suppose that for all $v \in V(G)$ there is no vertex $u_1 \in V(G-v)$ such that $\infty > r(G-v-u_1) > r$, no vertex u_2 such that u_2 is a cutvertex in $G-v$ and $G-v$ is not radius-vertex-invariant graph. Then:

(D) For all $v \in V(G)$ there exists at least one vertex u such that $r(G-v-u) = r-1$ (otherwise $G-v$ is radius-vertex-invariant).

(A): There exists a vertex v such that $G-v$ is radius-vertex-invariant. Then $|E(G-v)| \le f_v(n-1,r)$. As we already know from Lemma 2 $\deg(v) \le n-2r+1$ and thus

$$|E(G)| \le f_v(n-1,r) + n - 2r + 1 = g(n,r).$$

(B): There exists a vertex u in $G-v$ such that

$$\infty > r(G-v-u) \ge r+i \ge r+1 > r.$$

As it was shown by Vizing (Theorem 1), for every graph H with $n-2$ vertices and radius $r+i$ we have $|E(H)| \le f(n-2,r+i)$. Thus

$$\begin{aligned}|E(G-u-v)| &\le f(n-2,r+i) \le f(n-2,r+1)\\ &= \frac{(n-2)^2 - 4(r+1)(n-2) + 5(n-2) + 4(r+1)^2 - 6(r+1)}{2}\\ &= \frac{n^2 - 4nr - 3n + 4r^2 + 10r}{2}.\end{aligned}$$

Moreover, since $r(G-u-v) > r$ we have $|V(G-u-v)| \ge 2(r+1)$ and thus $|V(G)| \ge 2r+4$. But then

$$\begin{aligned}|E(G)| &\le f(n-2,r+1) + 2(n-2r+1)\\ &\le \frac{n^2 - 4nr + n + 4r^2 + 2r + 4}{2}\\ &= \frac{n^2 - 4nr + 3n + 4r^2 - 2r - 2}{2} + \frac{-2n + 4r + 6}{2}\\ &= g(n,r) - (n - (2r+3)) < g(n,r).\end{aligned}$$

(C): Let a_1 and a_2 be two vertices such that $G - a_1 - a_2$ is not connected. This is the most complicated case and we will divide it into five subcases as follows:

(**C1**) $d(a_1,a_2) > 2$ for some a_1, a_2,

(C2) $d(a_1,a_2) \in \{1,2\}$ for all such pairs a_1, a_2 and

- (**C2a**) G is self-centered having $|N_i(v)| \ge 3$ for all $v \in V(G)$, $1 < i < r$,
- (C2b) G is not self-centered having $|N_i(v)| \ge 3$ for all $v \in C(G)$, $1 < i < r$, and

(**C2ba**) $r(G-u-v)=r$ for some $u,v\in V(G)$, $d(u,v)>r$, or
(**C2bb**) $r(G-u-v)=r-1$ for all $u,v\in V(G)$, $d(u,v)>r$,
(**C2c**) $d(a_1,a_2)\in\{1,2\}$ and there is $v\in C(G)$ having $|N_i(v)|=2$ for some $1<i<r$.

(**C1**): $d(a_1,a_2)>2$, i.e., a_1 and a_2 have no common neighbours.

Since a_1 is a cutvertex of $G-a_2$ we have at least two sets A_{11}, A_{12} of vertices such that $A_{11}\cup A_{12}=N(a_1)$, $A_{11}\cap A_{12}=\{\emptyset\}$, $A_{11}\neq\{\emptyset\}$, $A_{12}\neq\{\emptyset\}$ and no vertex of A_{11} is adjacent to a vertex of A_{12}. Similarly we can form two sets A_{21}, A_{22} for the vertex a_2. As $d(a_1,a_2)>2$ we have $N(a_1)\cap N(a_2)=\{\emptyset\}$.

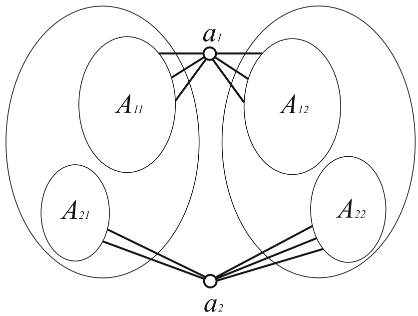

FIGURE 8

Thus $|E(G@a_1)|\geq|E(G)|-|A_{11}|-|A_{12}|+|A_{11}|\cdot|A_{12}|\geq|E(G)|-1$ ($|E(G@a_2)|\geq|E(G)|-1$) and, obviously $|E(G@a_1,a_2)|\geq|E(G)|-2$. If $r(G@a_1,a_2)=r(G)$, then

$$|E(G)|\leq f(n-2,r)+2<g(n,r).$$

Otherwise we have that $r(G@a_1,a_2)=r(G)-1$ by Lemma 5. There is no vertex $u\in G@a_1,a_2$ such that u is a cutvertex of the graph $G@a_1,a_2$. Otherwise u is a cutvertex of G. If there exists a vertex $w\in G@a_1,a_2$ such that $r(G@a_1,a_2-w)\geq r(G)$, then $|V(G)|\geq 2r+3$ and thus

$$\begin{aligned}|E(G)|&\leq f(n-3,r)+2+(n-2r+2)\\&=\frac{n^2-4nr+3n+4r^2-2r-2}{2}-(n-2r-2)<g(n,r).\end{aligned}$$

Otherwise $G@a_1,a_2$ is radius-vertex-invariant of radius $r-1$. Together with induction assumption we have that

$$|E(G)|\leq|E(G@a_1,a_2)|+2\leq f_v(n-2,r-1)+2=g(n,r).$$

(C2): $d(a_1, a_2) \in \{1, 2\}$.

(**C2a**): G is self-centered having $|N_i(v)| \geq 3$ for all $v \in V(G), 1 < i < r$.

Since G is radius-vertex-invariant we have $|N(v)| \geq 2$ and $|N_r(v)| \geq 2$. It follows from Lemma 8 (part L8b) that if $r = 4$, then

$$|E(G)| \leq g(n, 4).$$

Suppose $r > 4$. Let u and v be two vertices such that $d(u, v) = r$. We have

$$\deg(u) + \deg(v) \leq n - 3r + 7$$

(see Lemma 8, part (L8a)). Thus either (since u and v cannot be the cutvertices or vertices such that $r(G - u - v) > r(G)$)

$$\begin{aligned} |E(G)| &= |E(G - u - v)| + \deg(u) + \deg(v) \\ &\leq f(n-2, r) + n - 3r + 7 \\ &= f(n-2, r) + n - 2r + 2 + (5 - r) \\ &= g(n, r) + (5 - r) \leq g(n, r) \qquad \text{for} \quad r \geq 5 \end{aligned}$$

if $r(G - u - v) = r$ for some $u, v \in V(G), d(u, v) = r$ or $r(G - u - v) = r - 1$ for all u and v such that $d(u, v) = r$.

Consider the second case. We are now going to find the pairing of vertices demanded in part L8c of Lemma 8. Suppose that c is an arbitrary (central) vertex of G. We have $N_r(c) \geq 2$. Let $N_r(c) = \{v_1, v_2, \dots\}$. Then $r(G - c - v_i) = r - 1$ since $d(c, v_i) = r$. Furthermore, if c' is another central vertex and there are vertices u_1, u_2 such that $r(G - u_1 - u_2) = e_{G-u_1-u_2}(c') = r - 1$, then $N_r(c') = \{u_1, u_2\}$ and $\{u_1, u_2\}$ is the unique pair of vertices such that its removal decreases the eccentricity of c'. Removal of any other pair will leave at least one vertex $u_i, d(c', u_i) \geq r$.

Thus for every $c \in V(G)$ we have at least two pairs $\{c, v_1\}$, $\{c, v_2\}$ of vertices containing c which removal will decrease the radius of G. It follows that we can form at least $m \geq n$ such pairs in G. Suppose that we assign every pair $\{u_1, u_2\}$ with the central vertex c' such that $e_{G-u_1-u_2}(c') = r - 1$.

We can assign every vertex c' of G with at most one of these pairs, but every pair must be assigned with at least one central vertex. Since there are $m \geq n$ pairs and n central vertices we have that $m = n$ and thus every vertex belongs to exactly two pairs.

We can denote the pairs of vertices which removal decreases the radius of G as $S_1, \dots, S_n$. Since all k sets $S_{i_1}, \dots, S_{i_k}$ taken from $S_1, \dots, S_n$ have 2 elements and every vertex belongs to at most two such sets, we have $|S_{i_1} \cup \dots \cup S_{i_k}| \geq k$. But then from Hall's theorem we can find a system of distinct representatives (i.e., for every set $S_i = \{p_{i_1}, p_{i_2}\}$ the vertex $p_i = p_{i_1}$ or $p_i = p_{i_2}$) and form another pairing $P = \{P_1, \dots, P_n\}$, $P_i = \{p_i, c_i\}$ by taking p_i and its appropriate central vertex c_i such that $d(c_i, p_i) = r$, $e_{G-S_i}(c_i) = r - 1$. Every vertex is in two pairs and thus from Lemma 8, $|E(G)| \leq g(n, r)$.

(C2b): G is not self-centered but for all r-depth spanning trees we have at least 3 vertices at each level $2, \ldots, r-1$. For each pair u, v of vertices of a radius-vertex-invariant graph such that $d(u, v) > r$ we have $\deg(u)+\deg(v) \leq n-2r+2$. $G - u - v$ is connected.

(**C2ba**): If $r(G - u - v) = r$, then

$$|E(G)| \leq f(n - 2, r) + n - 2r + 2 = g(n, r).$$

(**C2bb**): $r(G - u - v) = r(G) - 1$ for all vertices v, u at distance greater than r. Moreover, if v is a central vertex of G and u is a vertex such that $d(u, v) = r$, then by Lemma 8, part (L8a)

$$\deg(u) + \deg(v) \leq n - 3r + 7.$$

Thus if for any u and v such that $v \in C(G)$ we have $r(G - u - v) = r(G)$, then again

$$|E(G)| \leq f(n - 2, r) + n - 3r + 7 \leq g(n, r) \qquad \text{for} \quad r \geq 5.$$

First consider the case $r = 4$. If $\deg(v) \leq n - 9$ for all vertices $v \in V(G)$ the demanded result follows from Lemma 9. In another case there is a vertex v such that $e(v) > 4$ and $\deg(v) > n - 9$. Since G has no cutvertices we have $|N_i(v)| \geq 2$ for $i = 2, \ldots, 4$, $|N_5(v)| \geq 1$ and thus $|\{v\}| + \sum_{i=2}^{5} |N_i(v)| = 8$. But then we have $\deg(v) = n - 8$ and $e(v) = r + 1 = 5$.

Then the $(r + 1)$-depth spanning tree rooted at v has 2 vertices on levels $2, \ldots, 4$ and one vertex on level 5. Let f_5, f_{41}, f_{42}, f_{31}, f_{32}, f_{21} and f_{22} be the vertices on levels $5, 4, 3, 2$.

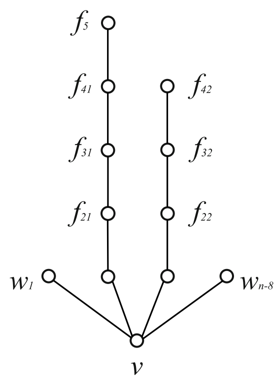

FIGURE 9

It is clear that $\deg(f_5) = 2, \deg(f_{4i}) \leq 3$ and $\deg(f_{3j}) \leq 4$. Moreover, if we have $\deg(f_{31}) = \deg(f_{32}) = 4$, then $d(f_{21}, f_{22}) \leq 3$, $d(f_{2i}, f_{3j}) \leq 2$,

$d(f_{2i}, f_{4j}) \leq 3$, $d(f_{2i}, f_5) \leq 3$ for some $i \in \{1, 2\}$ and all $j \in \{1, 2\}$. Since $d(f_{2i}, w_k) \leq 3$ for all vertices w_k on level 1, the vertex f_{2i} on level 2 has eccentricity 3, a contradiction. Finally, we have $\deg(f_5) + \deg(f_{41}) + \deg(f_{42}) + \deg(f_{31}) + \deg(f_{32}) \leq 2 + 3 \cdot 3 + 4$ and thus

$$|E(G)| \leq \left\lfloor \frac{2 + 3.3 + 4 + (n-5)(n-8)}{2} \right\rfloor = \frac{n^2 - 13n + 54}{2} = g(n, 4).$$

Suppose $r > 4$. Consider an arbitrary vertex v and a depth spanning tree rooted at v. It follows that there is a vertex u at distance at least r such that $r(G - u - v) = r(G) - 1$ and therefore a central vertex c having $N_r(c) = \{u, v\}$. Again there is either $|E(G)| \leq f(n-2, r) + n - 3r + 7 \leq g(n, r)$ or $r(G - v - c) = r(G) - 1$. For each vertex $c' \in C(G)$ there is an unique pair of vertices y, z such that $e_{G-y-z}(c') = r(G - y - z) = r(G) - 1$. Since each vertex of G is at least in two pairs whose removal decrease the radius of G, there must be n pairs of vertices and n corresponding central vertices. But then G is self-centered, a contradiction.

(**C2c**): Assume that there is an r-depth spanning tree rooted at the central vertex c such that $\{a_1, a_2\} = N_i(v)$ for some i, $r > i > 1$. It is clear that $G - a_1 - a_2$ is not connected. There is no vertex u at level $i - 1$ such that $ua_1, ua_2 \in E(G)$. Otherwise the vertex c' on level 1 such that $d(u, c') = i - 2$ has $e(c') = r - 1$.

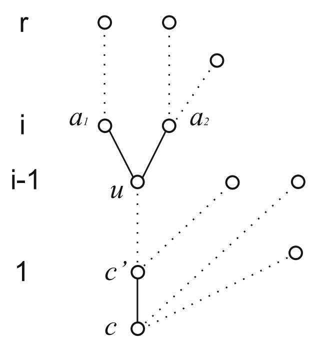

FIGURE 10

Using the same argument we can show that neither a_1, nor a_2 is adjacent to all vertices on level $i + 1$ but every vertex on level $i + 1$ is adjacent to a_1 or a_2. Let there be the set A of vertices on level $i - 1$ adjacent to a_1, the set B of vertices on level $i - 1$ adjacent to a_2 and sets C, D, E of vertices on level $i + 1$ adjacent to a_1, a_1 and a_2, a_2, respectively.

Compared to G the graph $G@a_1, a_2$ does not have the edges adjacent to a_1 and a_2 but it has some additional edges joining vertices of A and C, A and D, B and D, B and E, respectively. Moreover, if $a_1a_2 \in E(G)$, then $G@a_1, a_2$ contains

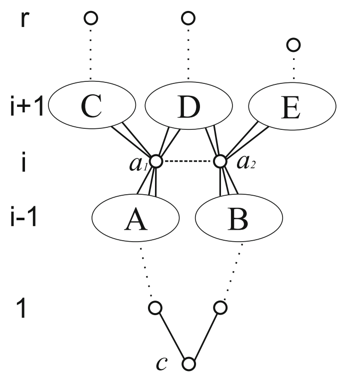

FIGURE 11

also edges joining the sets A and E, B and C. It is clear that $|A| \cdot |B| \cdot |C| \cdot |E| > 0$. Thus if $a_1 a_2 \notin E(G)$

$$\begin{aligned}&|E(G)| - |E(G@a_1, a_2)|\\ &\leq |A| + |B| + |C| + 2|D| + |E| - |A| \cdot |C| - |A| \cdot |D| - |B| \cdot |D| - |B| \cdot |E|\\ &= (|C| + |D| - 1) \cdot (1 - |A|) + (|D| + |E| - 1) \cdot (1 - |B|) + 2 \leq 2.\end{aligned}$$

Otherwise $a_1 a_2 \in E(G)$, and

$$\begin{aligned}&|E(G)| - |E(G@a_1, a_2)|\\ &\leq |A| + |B| + |C| + 2|D| + |E| + 1 - |A| \cdot |C| - |A| \cdot |D| - |B| \cdot |D|\\ &\qquad - |B| \cdot |E| - |A| \cdot |E| - |B| \cdot |C|\\ &= (|C| + |D| - 1) \cdot (1 - |A|) + (|D| + |E| - 1) \cdot (1 - |B|) + 2\\ &\qquad + (1 - |A| \cdot |E| - |B| \cdot |C|) \leq 2.\end{aligned}$$

Now we can follow arguments used in the section (C1) and as a result we get that

$$|E(G)| \leq |E(G@a_1, a_2)| + 2 \leq g(n, r).$$

(D): Given assumption, we have that for all $z \in V(G)$, $i \in \{1, \ldots, r-1\}$ there is $|N_i(z)| \geq 3$. Otherwise there are two vertices a_1, a_2 such that $\{a_1, a_2\} = N_i(z)$ and $G - a_1 - a_2$ is not connected. But this case is considered in the previous section. Thus if $r(G) = 4$, then $|E(G)| \leq g(n, 4)$ according to Lemma 8 (part (L8b)).

Now let $r(G) > 4$. Suppose $u, v \in V(G)$ are two vertices such that $r(G-u-v) = r - 1$. Let c be a central vertex of the graph $G - u - v$. We have $d(c, v) = r$. If $r(G - v - c) = r$, then

$$|E(G)| \leq f(n - 2, r) + n - 3r + 7 \leq g(n, r)$$

according to Lemma 8 (part (L8a)). Otherwise for each vertex v there are at least two vertices u, c such that $r(G-u-v) = r(G-v-c) = r-1$. Thus we have at least $m \geq n$ different pairs $S_i = \{p_{i_1}, p_{i_2}\}$ where $r(G-S_i) = r-1$ and every v is in at least two of them. Again for every vertex c_i there is at most one pair of vertices such that $r(G-p_{i_1}-p_{i_2}) = e_{G-p_{i_1}-p_{i_2}}(c_i) = r-1$. But then $m = n$ and every vertex belongs to exactly two pairs. Similarly as in part (C2a) we can form pairing of vertices at distance r having every vertex in two different pairs. But then from Lemma 8 (part (L8c)) we get $g(n,r) \geq E(G)$ and thus we proved the inequality

$$f_v(n,r) \leq g(n,r).$$

Recall that the graph in Figure 1 certifies that our bound is sharp. The proof is now complete. □

At the end we give the following problem: A graph is said to be *radius-adding-invariant* if for all $e \in E(\overline{G})$ we have $r(G+e) = r(G)$. Such graphs were studied together with radius-edge-invariant and radius-vertex-invariant graphs ([1], [3], [4]).

Problem 1. Find the upper bound for the number of edges of radius-adding-invariant graph.

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