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AN INTRODUCTION TO BALANCED SEQUENTIAL ARRAYS ON THE SQUARE GRID

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To Alex Rosa, on the occasion of his seventieth birthday

(*Communicated by Peter Horák*)

ABSTRACT. Plants grown in large numbers for commercial purposes are usually set out on a regular grid, triangular, square or hexagonal, or possibly a rectangular grid. We need to understand their behaviour when they compete with each other for light, water and nutrients. In a greenhouse, there may be a lamp over each plant, half the plants having their lamps on and half off. The intensity of light falling on any particular plant is determined *mainly* by whether its own lamp is on, *next* by the number of its nearest neighbours whose lamps are on, and *perhaps also* by the number of its second-nearest neighbours whose lamps are on.

Such arrays are also used to study competition among different types of plants and, with some extra restrictions, to design field layouts as well. The problem of constructing suitable arrays was introduced in [CORMACK, R. M.: *Spatial aspects of competition beween individuals.* In: Spatial and Temporal Analysis in Ecology (R. M. Cormack and J. K. Ord, eds.), International Co-operative Publishing House, Fairland, Maryland, USA, 1979] and discussed in [GATES, D. J.: *Competition between two types of plants with specified neighbour configurations*, Math. Biosci. **48** (1980), 195–209]. Here we consider only the square grid.

1. Binary arrays on the square grid

In the square grid, each plant has four nearest neighbours, that is, those in the same row or column, and four second-nearest neighbours, that is, those on the same diagonals. Initially we consider only nearest neighbours; see Freeman [8, p. 118], for conditions frequently imposed on such arrays.

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An array has *linear balance* if one in every k plants has i of its nearest neighbours receiving the same treatment as itself, where k must be constant over all possible values of i, and in the square grid i must be $0, 1, 2, 3$ or 4, and k must be 5.

We denote a plant whose lamp is on by `0` and one whose lamp is off by `1`, or, in a diagram, by a white or dark square, respectively. We label configurations according to the number of nearest neighbours which match the central element. Since each plant has four immediate neighbours, it can lie at the centre of any of $2^4 = 16$ possible neighbourhoods. These fall into five equivalence classes depending on how many of the neighbouring plants have their lamps in the same state as that of the central lamp. Thus, for instance, `1` is in the α configuration when surrounded by `1`s, and in the ε configuration when surrounded by `0`s.

Table 1 shows the 16 possible nearest neighbourhoods of `0`, a plant whose lamp is on, with neighbourhoods grouped into their five equivalence classes. For convenience, we subdivide the class of γ configurations into γ_1 and γ_2, according to the way that `1`s adjacent to the central plant are placed relative to each other.

An array on the square grid is *sequential* if each row and each column of the grid is occupied by the same sequence, which may run in either direction. A balanced sequential array of minimum period has length 20 ([24]). To see this, note that any `0` in the sequence occupies one of three positions: between two `0`s, between a `0` and a `1`, or between two `1`s, denoted by M, E, or I for middle, end or isolate, respectively. Table 2 shows the average contribution of each type of `0` to a sequence by each of the configurations shown in Table 1.

A sequence containing five `0`s, one each of α, β, γ_1, δ and ε per period would have (on average) $3/2$ type M, 2 type E and $3/2$ type I. To have whole numbers we must therefore have two of each type, giving ten `0`s in all. A sequence containing five `0`s, one each of α, β, γ_2, δ and ε per period would have (on average) 2 type M, 1 type E and 2 type I. There cannot be just one end per period, so again we must double the length of the sequence, giving ten `0`s in all. We refer to these sequences as types $(3, 4, 3)$ and $(4, 2, 4)$ respectively.

Similar arguments apply to the `1`s, so the minimum period of the sequence must be 20. Note that the sequence could be type $(3, 4, 3)$ in one symbol and type $(4, 2, 4)$ in the other. Longer sequences have period $20n$ and type $a(3, 4, 3) + b(4, 2, 4)$ in either symbol, where $a + b = n$.

A finite sequence is *primitive* if it is not the concatenation of k identical sequences each of length n/k for some k which divides n. For instance, `(0,1,0,0,2)` is primitive but `(1,1)` and `(0,1,0,1)` are not primitive. If a periodic sequence has period n, then it also has period jn where j is any positive integer. A periodic sequence is *primitive* if the period n is the least positive integer m such that $s_i = s_{i+m}$ for all i.

Two sequences are said to be *dihedrally equivalent* if one can be obtained from the other by some combination of cyclic shifts and reversals, that is, by the action of the appropriate dihedral group.

TABLE 1. The neighbourhoods of 0

			0												
		0	0	0											α
			0												
	0				1				0			0			
0	0	1		0	0	0		1	0	0	0	0	0		β
	0				0				0			1			
	0				0				1			1			
0	0	1		1	0	0		1	0	0	0	0	1	γ_1	
	1				1				0			0			γ
			0				1								
		1	0	1		0	0	0						γ_2	
			0				1								
	0				1				1			1			
1	0	1		0	0	1		1	0	1	1	0	0		δ
	1				1				0			1			
			1												
		1	0	1											ε
			1												

TABLE 2. Average contribution to sequences of types of 0s

	M	E	I
α	1	0	0
β	1/2	1/2	0
γ_1	0	1	0
γ_2	1/2	0	1/2
δ	0	1/2	1/2
ε	0	0	1

Possible arrangements of symbols in minimum period sequences were discussed in [24]. Such sequences can be enumerated by the methods of H u t c h i n s o n [16] but so few of the sequences lead to balanced arrays that her methods seem inappropriate here. Related results on periodic sequences and circulant arrays appear in F i n e [7], G i l b e r t and R i o r d a n [10], N e s t e r [18] and D a v i s [4].

2. Examples of balanced binary arrays

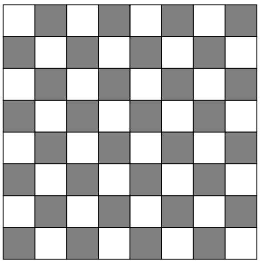
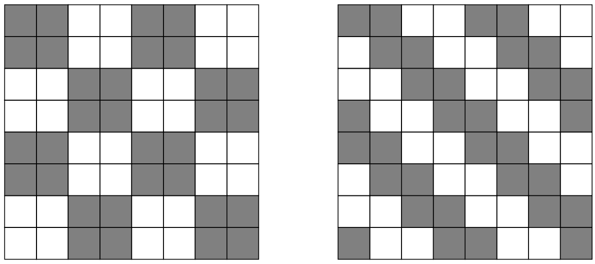


Figure 1



(a) (b)

Figure 2

Figure 1 shows the smallest binary sequential array, with sequence `01` of period two. Figure 2 shows the two sequential arrays with sequence `0011` of period four. Every element of Figure 1 is in the ε configuration, and every element of Figures 2(a) and 2(b) is in the γ_1 configuration. Note that each of these sequences has the form of a subsequence of length n followed by its complement, also of length n, where $n = 1$ for the sequence of Figure 1, and $n = 2$ for the sequence of Figures 2(a) and 2(b).

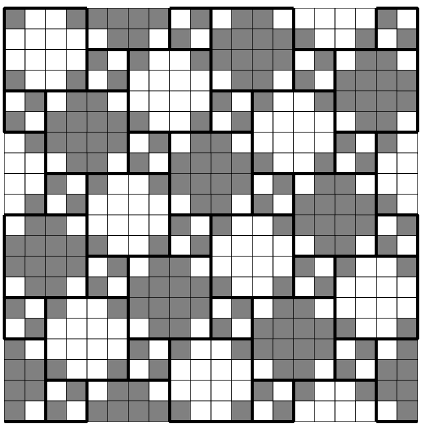


Figure 3

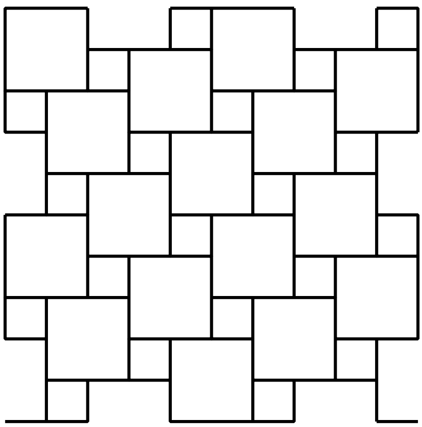


Figure 4

Figure 3 shows one of the smallest (20×20) blocks that is a balanced binary sequential array. Figure 4 shows the overlaid open tiling related to Figure 3.

The array of Figure 3 was found originally in [24]. Note that again the sequence of this array consists of a subsequence, followed by its complement, thus 00000101001111101011, where this time $n = 10$. The same sequence also forms the balanced binary arrays shown in Figures 5 and 6, and, surprisingly, the array in Figure 7, where every element is in a γ configuration.

Figure 8 shows an unexpected property of the polyomino which covers 16 small squares in the array of Figure 3, and how it can be tiled in three different ways by three different tetrominoes, T, L and skew; see Honsberger [13].

Related results on arrays on the square grid appear in Day and Street [5], Nester [17] and [18], Praeger and Nilrat [19], Praeger and Street [20], Robinson [21], and Street and Day [23].

Backtrack searches for suitable sequences, designed to work correctly and efficiently with both primitive and nonprimitive sequences, have been carried out on rows ([5]), and on columns ([18]). It makes no difference which is used, since the transpose of a sequential array is sequential. The search handles the complete list of distinct sequences dihedrally equivalent to any given sequence.

3. Further examples and their properties

Example 3.1. One of the requirements in [8] is that "all treatments shall be equally replicated and shall occur equally often in each row and column or, *if this is not possible, the numbers of occurrences in any row or column shall not differ by more than one*". This suggests that incidence sequences of quadratic residue difference sets may generate interesting sequential arrays; see Baumert [1]. For instance, the quadratic residues modulo 11 are $\{1, 3, 4, 5, 9\} \subset Z_{11}$. Their incidence sequence, starting from 1, is 10111000100. Figure 9(a)–(h) shows all the eight inequivalent sequential arrays obtainable from this sequence of quadratic residues.

The 0-*profile* of such an array is (a, b, c, d, e) where a is the number of 0s within the array in the α configuration, b the number of 0s within the array in the β configuration, and so on. Table 3 shows the profiles of the sequential arrays of Figure 9. In the last column of Table 3, the list of starting positions for the sequence 10111000100, forms L, the sequence of *labels* for the corresponding array. Note that Figure 9(g), (e), (h) are 1-, 2-, 4-step circulants respectively, and that all eight of the arrays in Figure 9 have the 11 starting positions of their rows (and columns) distinct. Obviously none of these arrays can be balanced, since the sequence has odd length.

Example 3.2. A serious drawback of a 20×20 array of trees is the area it occupies, typically of the order of half a hectare. It is often hard to find a large enough area

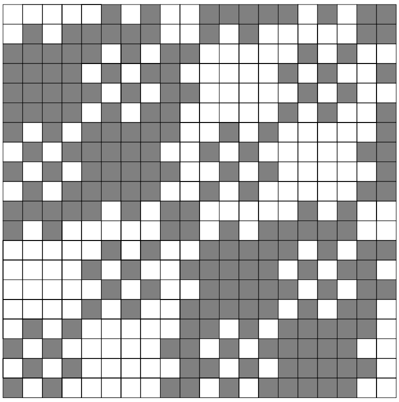


Figure 5

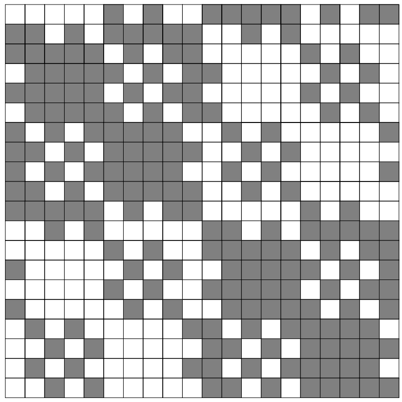


Figure 6

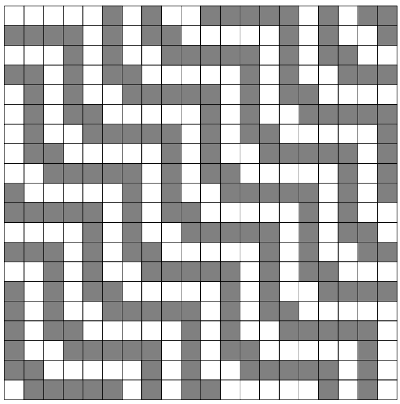


Figure 7

which is reasonably uniform. So the question arises: could we get an adequately balanced array of smaller period? This leads to the idea of *quadratic balance*, as opposed to the earlier idea of linear balance; see [18, p. 129].

A binary array has *quadratic balance* if it contains equal numbers of 0s and 1s in each of the configurations α, γ and ε, but no 0s or 1s in configurations β or δ. An argument similar to that of Section 1, where arrays have equal numbers of each of five configurations for each symbol, leads us to a sequence of length 12, hence to a 12×12 array, far more convenient. Figure 10 shows nine inequivalent sequential arrays of period 12, based on six different sequences: 000100111011

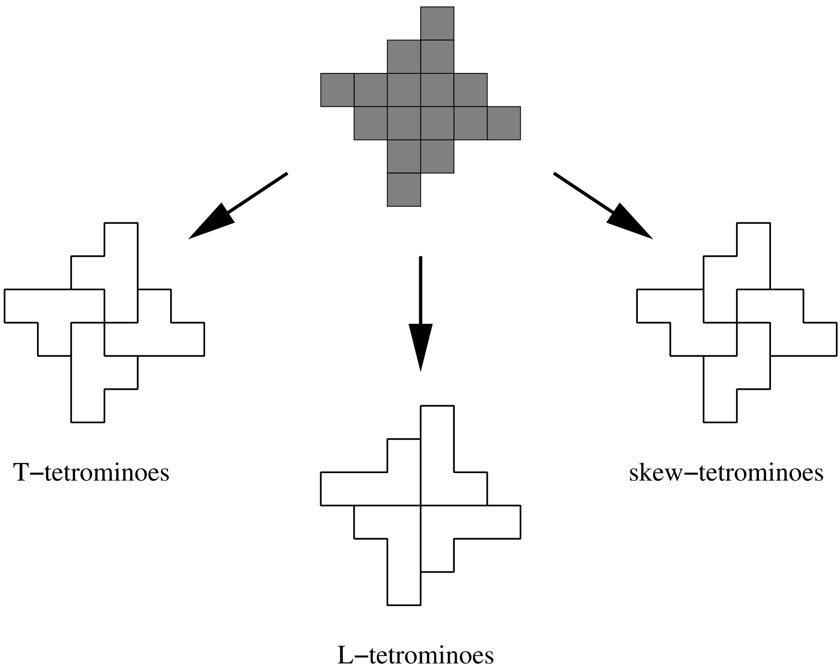
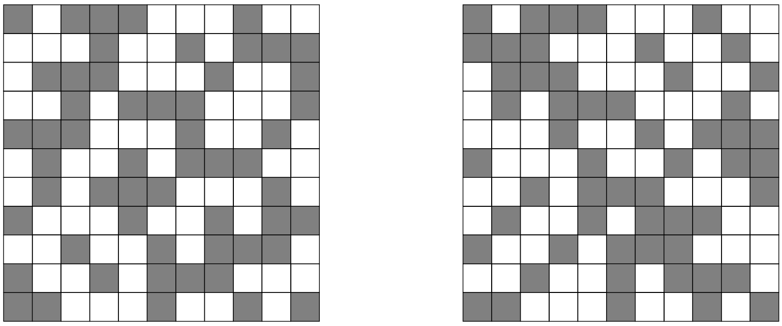


Figure 8

TABLE 3. Profiles of the eight sequential arrays with sequence 10111000100

Figure	0-profile	1-profile	starting positions
9(a)	$(2, 10, 40, 14, 0)$	$(2, 6, 20, 22, 5)$	1,7,0,3,A,5,2,8,6,4,9
9(b)	$(0, 20, 28, 16, 2)$	$(2, 12, 14, 16, 11)$	1,A,0,2,7,8,3,5,4,6,9
9(c)	$(4, 14, 27, 20, 1)$	$(4, 10, 9, 24, 8)$	1,5,6,0,A,2,7,4,3,9,8
9(d)	$(2, 14, 36, 10, 4)$	$(3, 4, 24, 16, 8)$	1,3,0,7,4,5,6,9,2,A,8
9(e)	$(0, 11, 44, 11, 0)$	$(0, 11, 22, 11, 11)$	1,3,5,7,9,0,2,4,6,8,A
9(f)	$(1, 16, 33, 14, 2)$	$(0, 12, 21, 10, 12)$	1,5,6,8,7,0,2,9,4,A,3
9(g)	$(11, 0, 44, 0, 11)$	$(11, 0, 22, 0, 22)$	1,2,3,4,5,6,7,8,9,A,0
9(h)	$(0, 22, 22, 22, 0)$	$(0, 11, 11, 33, 0)$	1,5,9,2,6,A,3,7,0,4,8

for 10(a), (f) and (g); 000110011011 for 10(b); 000010111101 for 10(c) and (d); 000010101111 for 10(e); 000101100111 for 10(h); 000011010111 for 10(i). A starting position such as -8 means that the sequence starts in column 8 and runs right to left. Table 4 shows the profiles of the sequential arrays of Figure 10. Again the list of starting positions for these sequences, given in column four of Table 4, forms L, the sequence of labels for the corresponding array.



(a) (b)

Figure 9

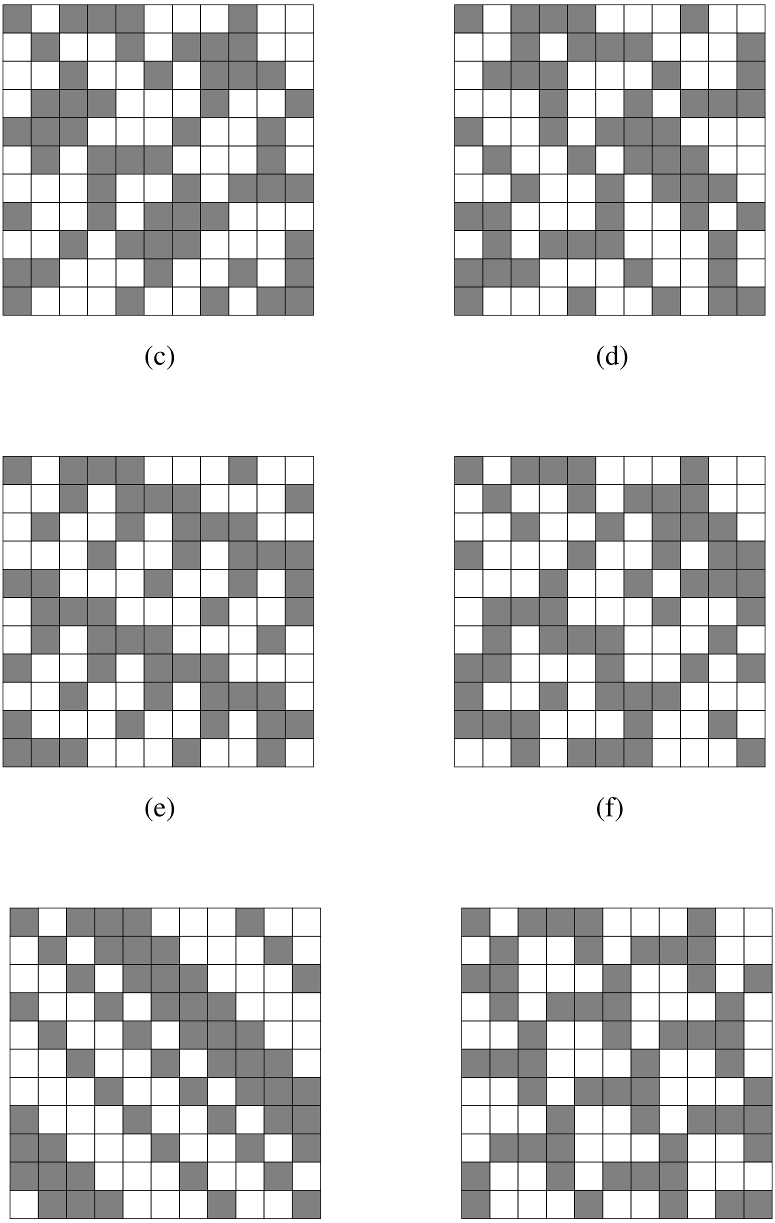
TABLE 4. Profiles of nine sequential arrays with sequences of length 12

Figure	0-profile	1-profile	starting positions
10(a)	(0,0,72,0,0)	(0,0,72,0,0)	1,-10,11,-8,9,-6,7,-4,5,-2,3,-12
10(b)	(0,0,72,0,0)	(0,0,72,0,0)	1,-10,11,-8,9,-6,7,-4,5,-2,3,-12
10(c)	(0,0,72,0,0)	(0,0,72,0,0)	1,8,3,10,5,12,7,2,9,4,11,6
10(d)	(24,0,24,0,24)	(24,0,24,0,24)	1,12,11,10,9,8,7,6,5,4,3,2
10(e)	(24,0,24,0,24)	(24,0,24,0,24)	1,12,11,10,9,8,7,6,5,4,3,2
10(f)	(0,24,24,24,0)	(0,24,24,24,0)	1,-2,10,-8,1,-11,7,-8,4,-2,7,-5
10(g)	(0,24,24,24,0)	(0,24,24,24,0)	1,12,-11,6,-5,-4,7,6,-5,12,-11,-10
10(h)	(0,24,24,24,0)	(0,24,24,24,0)	1,-10,11,-8,9,-6,7,-4,5,-2,3,-12
10(i)	(0,24,24,24,0)	(0,24,24,24,0)	1,-10,11,-8,9,-6,7,-4,5,-2,3,-12

The arrays are ordered by their profiles: the first three have all symbols in the γ configuration; the next two have quadratic balance; the last four have equal numbers of each symbol in β, γ and δ configurations, but none in α or ε configurations. Also, only six different sets, L, of labels occur: one for 10(a) and (b); one for 10(d) and (e); one for 10(h) and (i); and different labels for each of the remaining arrays, 10(c), (f) and (g).

4. Some further issues

We state a few of the large number of unsolved problems that come to mind.
1. Figure 11 shows two diagrams of plants on a square grid, with the central (subject) plant in the ε configuration in both diagrams. However in the first diagram each nearest neighbour of the subject plant is in a β configuration,



(c) (d) (e) (f) (g) (h)

Figure 9 (continued)

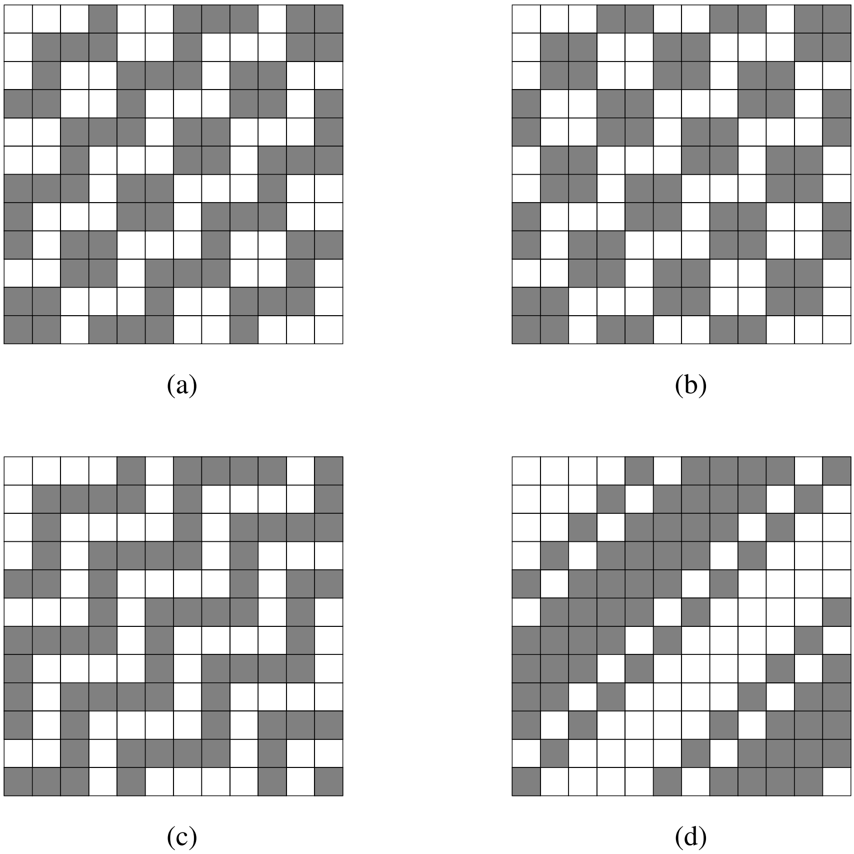
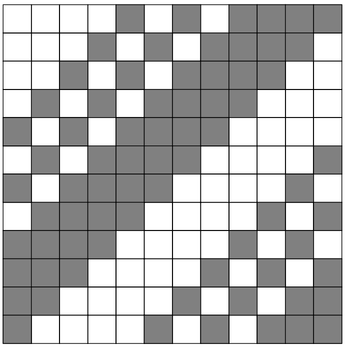


Figure 10

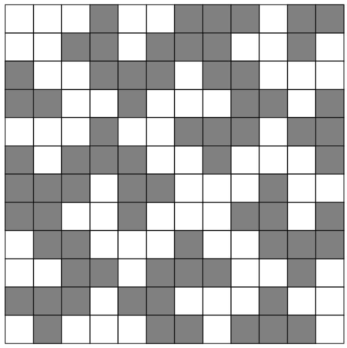
whereas in the second diagram each nearest neighbour is in an ε configuration. Thus nearest neighbours of the nearest neighbours may have a bearing on how well the nearest neighbours themselves compete with the subject plant. We would like arrays that allow us to study this interaction.

2. Many planting systems use a rectangular grid in which each subject plant has two nearest neighbours (first order) and two more distant neighbours placed at a right angle to the nearest neighbours (second order). Are there rectangular grid designs for two or more plant types which incorporate first and second order balance?

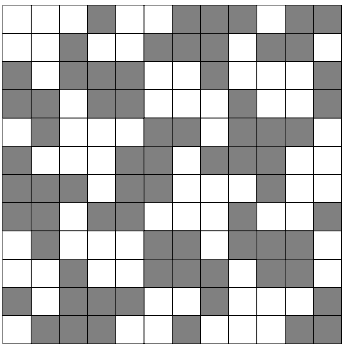
3. Is it possible to construct efficient designs for investigating interactions between three or more types of plants? Introducing additional types of plants will also mean introducing new definitions of balance.



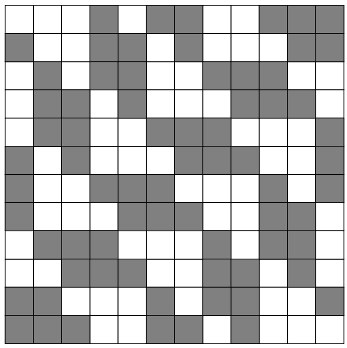
(e)



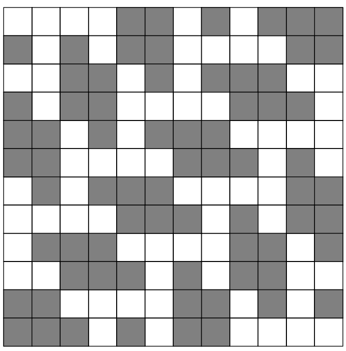
(f)



(g)



(h)



(i)

Figure 10 (continued)

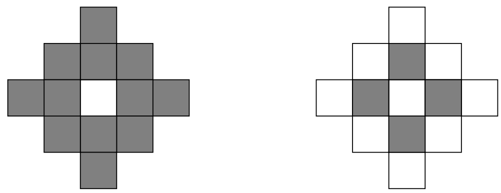


Figure 11

4. Nester [18] introduced the idea of *shadows* of sequences and of sequential arrays. For instance, if $S = (1, 2, 1, 1, 2, 3, 4)$, then the shadow of element 1 is $\sigma_1 = (1, 0, 1, 1, 0, 0, 0)$, the shadow of element 2 is $\sigma_2 = (0, 1, 0, 0, 1, 0, 0)$, and the shadow of the set $\{1, 2\}$ is $\sigma_{1,2} = (1, 1, 1, 1, 1, 0, 0)$.

[18, Theorem 3.6] states that:

> *If M is a sequential array derived from sequence S using labels L, then every possible shadow of M is also a sequential array based on the same labels.*

It is easy to construct an example which shows that even if all the unary shadows of a sequence form valid sequential arrays using the same labels, then it need not be true that the sequence itself will form a sequential array using the same labels; see Figure 12. But it seemed plausible to conjecture that:

> *If all possible nontrivial shadows (not just the unary ones) are valid sequential arrays using the same set of labels, then the whole array will be sequential.*

Alas, this is not the case. The array of Figure 13 is a smallest ternary counterexample to this conjecture. All its rows and all its even-numbered columns are occupied by the same sequence, but all its odd-numbered columns are occupied by a different sequence, even though all its shadows are sequential. Under what additional conditions, if any, might this conjecture be true?

This array can be constructed by taking a circulant array, based on the sequence 00110212, and swapping rows 4 and 8. Similarly, the circulant quaternary array, based on the sequence 001021031, but with rows 6 and 9 swapped, gives a smallest quaternary counterexample: all its rows and six of its columns are occupied by the same sequence, but columns 1, 4, and 7 are occupied by a different sequence. We have not yet managed to generalise this construction.

5. Suppose a set of equivalence classes is defined on the neighbourhoods of a subject plant in a square grid. For which equivalence relations will balanced sequential arrays exist? This question is of theoretical interest, but not only of theoretical interest. In field experiments, considering only *numbers* of matching neighbours of a subject plant may not lead to sensible definitions of equivalence.

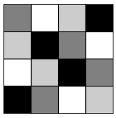


Figure 12

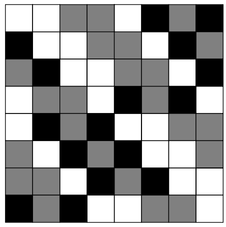


Figure 13

Tea plants, which must be grown on sloping ground, will very likely react differently to identical plants grown above them or below them. We need to consider *directional balance.*

6. Some plant species have maximum biomass production per plant if grown at very close spacing when young and wide spacings when old. Efficient designs are needed for determining the optimal timing and intensity of thinnings ([22]).

7. Circulant arrays are special cases of sequential arrays (SA), or to put it another way, sequential arrays are generalizations of circulant arrays. We can in fact generate a tower of generalizations as follows:

A: circulants, including stepcirculants and backcirculants;

B: SAs, with labels a permutation of $\{1,\ldots,n\}$;

C: SAs, with distinct starting labels which are a mixture of positive and negative;

D: SAs, with some repeated starting labels;

E: bisequential arrays, that is, arrays in which all row sequences are identical, all column sequences are identical, but row sequences are different from column sequences;

F: arrays for which the relative frequencies of symbols are constant for all rows and all columns, such as binary arrays which correspond to matrices with fixed row and column sums.

We have made some progress in addressing **B**, which may lead naturally to characterising **C** and **D**. We know very little about **E** except for a few examples; see Figure 14. There is a vast literature dealing with binary examples of **F** such as permutation matrices, incidence matrices of symmetric balanced incomplete block designs, and doubly stochastic matrices. See [12], for instance.

8. Table 5 gives a summary of searches based on nonzero quadratic residues of primes. For given values of v other than 7 and 11, all the equivalence classes are (step)circulant: for $v = 7$ or 11, 2 (respectively 3) of the 8 classes are (step)circulant, denoted by 8/2 and 8/3. Table 6 gives a summary of searches

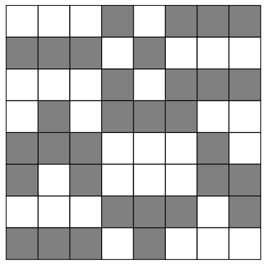


Figure 14

based on miscellaneous cyclic difference sets. The $(37, 9, 2)$ and $(35, 17, 8)$ difference sets lead to 5 (respectively 7) (step)circulant equivalence classes. All other difference sets listed here lead to many non-(step)circulant equivalence classes, indicated by 27/2 for the $(13, 4, 1)$ difference sets, where 2 of the 27 equivalence classes are (step)circulant, and similarly for the remaining difference sets.

TABLE 5. Searches based on nonzero quadratic residues of primes

v Prime	Incidence sequence (beginning with 1 at right)	Number of equivalence classes
3	001	1
5	01001	1
7	0001011	8/2
11	00100011101	8/3
13	0101100001101	2
17	01101000110001011	3
19	0011000010101111001	5
23	00000101001100110101111	6
29	01001111010001001000101111001	4
31	0001001000011101010001111011011	8
37	0101100101111000100001000111101001101	5
41	01101100111000001010110101000001110011011	6
43	0011010110001000001110100011111011100101001	11
47	00000100001101010001101100100111010100111101111	12

Finally, most woven fabrics can be considered as binary arrays. Traditionally, weavers represent warp threads, running along the fabric, as black, and weft threads, running across the fabric, as white. See Grunbaum and Shephard [11], Hoskins [14], and Hoskins and Hoskins [15]. An interesting question is whether fabric woven on a loom and then removed from the loom will remain in one layer or, as weavers might say, whether it "hangs together". This question has been considered independently by Clapham [2] and Delaney [6]. It follows from results in [6] that any balanced binary sequential array, considered as a fabric, can be woven so that it hangs together, but not necessarily with all warp threads black and all weft threads white.

TABLE 6. Searches based on miscellaneous cyclic difference sets

Cyclic difference set	Elements of difference set	Number of equivalence classes
(13,4,1)	0 1 3 9	27/2
(21,5,1)	3 6 7 12 14	175/4
(31,6,1)	1 5 11 24 25 27	245/2
(57,8,1)	1 6 7 9 19 38 42 49	1015/2
(37,9,2)	1 7 9 10 12 16 26 33 34	5
(40,13,4)	1 2 3 5 6 9 14 15 18 20 25 27 35	4039/3
(35,17,8)	0 1 3 4 7 9 11 12 13 14 16 17 21 27 28 29 33	7

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