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THE CONFIGURATION POLYTOPE OF ℓ-LINE CONFIGURATIONS IN STEINER TRIPLE SYSTEMS

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To Alex Rosa on the Occasion of his Seventieth Birthday

(*Communicated by Peter Horák*)

ABSTRACT. It has been shown that the number of occurrences of any ℓ-line configuration in a Steiner triple system can be written as a linear combination of the numbers of full m-line configurations for $1 \leq m \leq \ell$; full means that every point has degree at least two. More precisely, the coefficients of the linear combination are ratios of polynomials in v, the order of the Steiner triple system. Moreover, the counts of full configurations, together with v, form a linear basis for all of the configuration counts when $\ell \leq 7$. By relaxing the linear integer equalities to fractional inequalities, a configuration polytope is defined that captures all feasible assignments of counts to the full configurations. An effective procedure for determining this polytope is developed and applied when $\ell = 6$. Using this, minimum and maximum counts of each configuration are examined, and consequences for the simultaneous avoidance of sets of configurations explored.

1. Introduction

A *partial triple system* PTS(v, λ) is a set V of v elements and a collection $\mathscr{B}$ of triples, so that each unordered pair of elements occurs in at most λ triples of $\mathscr{B}$. Its *leave* is the multigraph on vertex set V in which the edge $\{x, y\}$ appears $\lambda - s$ times when there are precisely s triples of $\mathscr{B}$ containing $\{x, y\}$. When every pair occurs in exactly λ triples, the system is a *triple system*, TS(v, λ). When in addition $\lambda = 1$, it is a *Steiner triple system*, STS(v). By a configuration we mean a PTS(k, ℓ), $(K, \mathscr{L})$, with $|K| = k$ and $|\mathscr{L}| = \ell$, typically with ℓ a "small" fixed integer. The term "configuration" is applied in the literature much more

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generally to permit blocks of larger sizes, but we restrict to block size three. The triples are sometimes called *lines* here to conform with geometric terminology (in the same vein, elements are sometimes called *points* here). The *degree* of a point is the number of lines containing the point. We refer to [4, Chapter 13] for background.

Evidently, there are configurations that must occur in every nontrivial triple system, while others may be avoided altogether. This leads naturally to questions about ubiquity, as well as questions about avoidance and decompositions. A configuration whose number of occurrences in a $\mathrm{TS}(v, \lambda)$ depends only upon v and λ is *constant* (for these parameters). Otherwise, it is *variable*.

We restrict to configurations in Steiner triple systems here. A (k, ℓ)-*configuration* (in an $\mathrm{STS}(v)$) is a set of ℓ lines whose union contains precisely k points, so that no two points lie on more than one line. An $(\ell+2, \ell)$-configuration that contains no $(m+2, m)$-configuration for $1 < m < \ell$ is an *Erdős configuration*. A configuration in which every point has degree at least two is a *full configuration*; it is *minimal full* if it contains no full configuration on fewer lines. An Erdős configuration must be full, but need not be minimal full. A configuration in which every point has even degree is an *even configuration*. In Table 1 the numbers of configurations for $\ell \leq 8$ lines is given (for related enumeration results, see [8]). All ℓ-line configurations with $\ell \leq 4$ are shown in Figure 1. The *Pasch configuration*, shown as #9, is the smallest full configuration, the smallest Erdős configuration, and the smallest even configuration.

Table 1. Counts of configurations

ℓ	Configurations	Full	Minimal Full	Even	Erdős
1	1	0	0	0	0
2	2	0	0	0	0
3	5	0	0	0	0
4	16	1	1	1	1
5	56	1	1	0	1
6	282	5	4	2	2
7	1865	19	11	0	8
8	17100	153	78	12	64

Each configuration with at most three lines is constant. For each $\ell \geq 4$, there exist both variable and constant ℓ-line configurations. Grannell, Griggs, and Mendelsohn [13] show that of the 16 4-line configurations (Figure 1), five are constant and 11 are variable. One open problem concerns the characterization of constant configurations. Let S_ℓ, T_ℓ, U_ℓ, V_ℓ, W_ℓ be five ℓ-line configurations obtained from the $(\ell-1)$-star by adding a line. Figure 2 shows

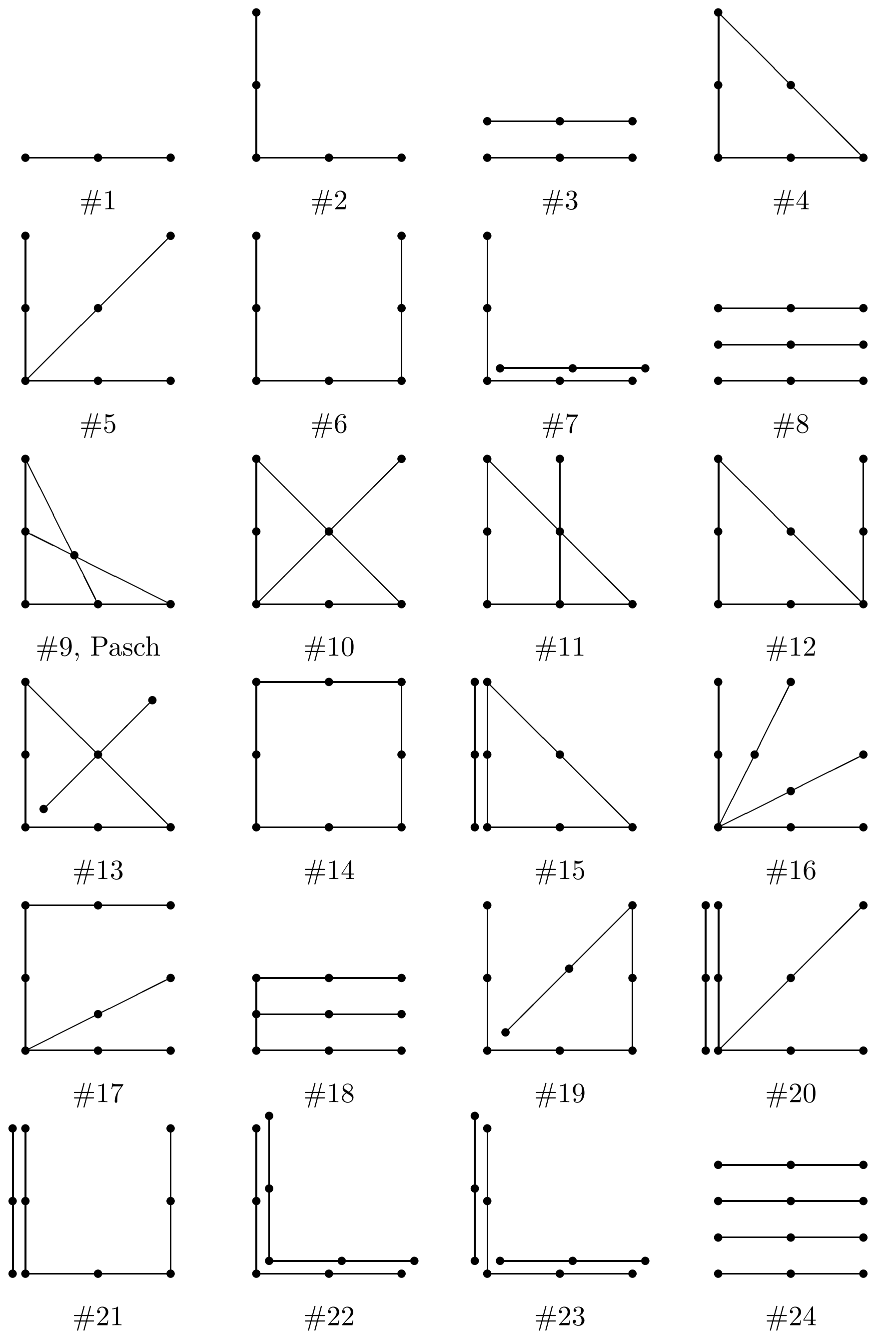

FIGURE 1. Small configurations

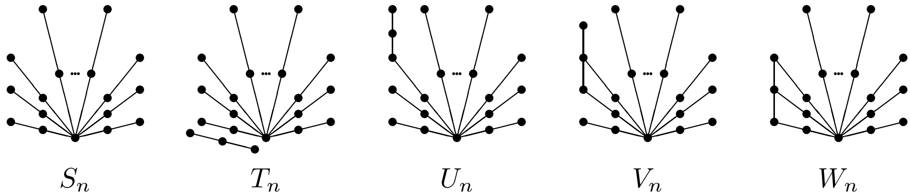

FIGURE 2. Adding a line to a star

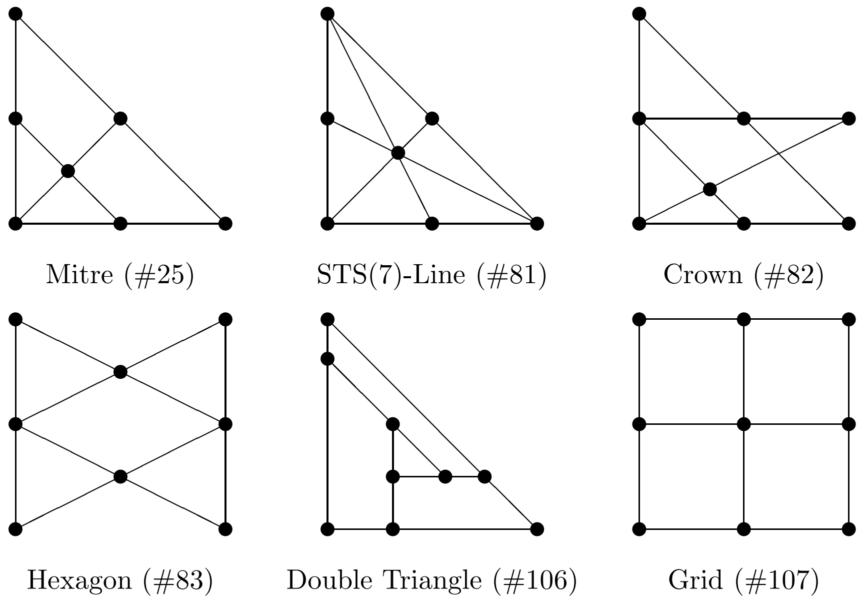

FIGURE 3. Full configurations

the star S_ℓ in which ℓ lines meet at a single point, obtained by adding a line to the $(\ell-1)$-star which meets the remaining lines at their common point. Also shown are T_ℓ, U_ℓ, V_ℓ, W_ℓ obtained by adding a line which meets 0, 1, 2, or 3 other points of the $(\ell-1)$-star.

THEOREM 1.1. ([19]) *For each $\ell \geq 4$, the configurations S_ℓ, T_ℓ, U_ℓ, V_ℓ, and W_ℓ are constant.*

Let $\Delta = \frac{1}{24}v(v-1)(v-3)$ henceforth. The number of occurrences of these constant configurations can be computed explicitly for $\ell \geq 4$:

$$\begin{aligned}
s_\ell &= 3\Delta(v-5)(v-7)\cdots(v-2\ell+1)/(2^{\ell-3}\ell!)\\
t_\ell &= \Delta(v-7)(v-9)\cdots(v-2\ell-3)/(2^{\ell-3}(\ell-1)!)\\
u_\ell &= 3\Delta(v-7)(v-9)\cdots(v-2\ell-1)/(2^{\ell-4}(\ell-2)!)\\
v_\ell &= 3\Delta(v-7)(v-9)\cdots(v-2\ell+1)/(2^{\ell-4}(\ell-3)!)\\
w_\ell &= \Delta(v-7)(v-9)\cdots(v-2\ell+3)/(2^{\ell-6}(\ell-4)!)
\end{aligned}$$

Conjecture 1.2. ([19]) *The five ℓ-line configurations S_ℓ, T_ℓ, U_ℓ, V_ℓ, and W_ℓ are the only constant configurations in Steiner triple systems.*

The *Mitre* configuration is shown in Figure 3; it is also an Erdős configuration, and is the only full 5-line configuration. Among the five full 6-line configurations also shown, Hexagon and Crown are Erdős, while Double Triangle and Grid are even. Avoidance of configurations has been extensively studied. For every $v \equiv 1,3 \pmod 6$, $v \notin \{7,13\}$, there is an *anti-Pasch* Steiner triple system, an STS(v) in which no four triples are isomorphic to the *Pasch configuration* ([15], [21]). There is an *anti-mitre* STS(v) (one that contains no configuration isomorphic to the Mitre configuration) if and only if $v \equiv 1,3 \pmod 6$ and $v \neq 9$ ([3], [10], [28]). An STS(v) is *r-sparse* if it contains no Erdős configuration on $2 \le \ell \le r$ lines. In 1976, Erdős [6] conjectured that an ℓ-sparse STS(v) exists for every integer $\ell \ge 2$. Every STS(v) is 3-sparse. An STS is 4-sparse exactly when it is anti-Pasch. It is 5-sparse when it is both anti-Pasch and anti-mitre. A 5-sparse STS(v) is known to exist when $v \equiv 3 \pmod 6$ and $v \ge 21$, and for many other orders ([11], [28]). However a complete characterization is not known. Forbes, Grannell, and Griggs [9] construct 29 6-sparse systems in the residue class 7 modulo 12, with orders ranging from 139 to 4447. They also present a recursive construction that establishes the existence of 6-sparse systems for an infinite set of orders. No ℓ-sparse STS(v) is known for any $\ell \ge 7$.

For avoidance, simultaneous avoidance, and decomposition into configurations, see [16], [17], [18], [20], [22], [25].

2. Generating sets and bases

In general, a set M of configurations, each with $1 \le m \le \ell$ lines, is a *generating set* for ℓ-line configurations if, for each admissible order v, the number of occurrences of any ℓ-line configuration can be expressed as a linear combination of the number of occurrences of the configurations in M. To treat all values of v simultaneously, these numbers of occurrences are expressed as polynomials in v. A minimal generating set is a *basis*. As defined, a generating set is *linear*, and so therefore is a basis. One could define analogous notions of generating sets and bases using polynomial equalities rather than just linear ones; as we see later, such a polynomial basis can be smaller than a linear basis.

For 4-line configurations, any constant configuration together with the Pasch configuration forms a basis ([13]). One could simply take the constant configuration to be the unique 1-line configuration, whose count is the number of lines in the STS, thereby determining v. We state results to permit any constant configuration, but the single line typically is chosen.

G r a n n e l l, G r i g g s, and M e n d e l s o h n [13] conjecture that the set of Erdős configurations on at most ℓ lines, together with a constant configuration, forms a basis in general. This holds for $\ell = 5$ ([19]). (Explicit formulas for the numbers of each of the 56 5-line configurations appear in [5], and for the 6-line configurations in [7].) H o r á k, P h i l l i p s, W a l l i s, and Y u c a s prove a general theorem:

Theorem 2.1. ([19]) *Any constant configuration, together with all full configurations on at most ℓ lines, forms a generating set for the ℓ-line configurations.*

Indeed for $\ell = 6$, this generating set is a basis [19]; the seven full configurations are all needed, refuting the conjecture that the Erdős configurations suffice. The generating set is again a basis for $\ell = 7$ ([27]).

Conjecture 2.2. ([19]) *Any constant configuration, together with all full configurations on at most ℓ lines, forms a basis for the ℓ-line configurations.*

We employ the mechanics of the proof in [19] that the full configurations provide a generating set. Indeed we explicitly calculate the numbers of all 6-line configurations here in terms of this generating set. (We have completed this computation for 7-line and 8-line configurations as well, but do not attempt to tabulate information for the 1865 7-line and 17100 8-line configurations here.) We first outline the algorithm used to realize the constructive proof in [19].

Consider a configuration $C = (K, \mathscr{L})$. A *pointed configuration* is a triple $(K, \mathscr{L}, M)$, often written as (C, M) where $C = (K, \mathscr{L})$, is a (k, ℓ)-configuration and $M \subseteq K$. Members of M are marked points. A pointed configuration with $|M| \leq 3$ in which no two points in M are collinear is marked. Two pointed configurations (C, M) and (C', M') (with $C = (K, \mathscr{L})$ and $C' = (K', \mathscr{L}')$) are *isomorphic* if there is a bijection $\phi\colon K \to K'$ for which $\phi(M) = M'$ and $\phi(\mathscr{L}) = \mathscr{L}'$. Partition the powerset 2^K into equivalence classes defined by isomorphism, and let $\tau(C, M)$ be the cardinality of the equivalence class containing (C, M). Intuitively, $\tau(C, M)$ is the number of ways to mark points in K to obtain the pointed configuration (C, M). Each occurrence of C leads to $\tau(C, M)$ different occurrences of the pointed configuration, and hence counts of C and counts of any pointed configuration based on C are related by a factor depending only on C and M.

Now let $\gamma(C)$ denote the number of occurrences of a configuration C in a Steiner triple system. The proof in [19] derives equalities among these counts for different configurations; we derive their result in a similar manner here. Consider a $(k', \ell+1)$-configuration $C_1 = (K_1, \mathscr{L}_1)$ that is *not* full. Then C_1 contains a point $x \in K$ that has degree 1, and hence a unique line $L \in \mathscr{L}_1$ that contains x. Let P consist of the points on L that have degree 1 in C_1 and write $|P| = p$ and $k = k' - p$. Set $K = K_1 \setminus P$ and $\mathscr{L} = \mathscr{L}_1 \setminus \{L\}$. Then $C = (K, \mathscr{L})$ is a (k, ℓ)-configuration. Indeed it corresponds precisely to the *marked* configuration (C, M), with $M = L \setminus P$.

An *extension* of a marked (k, ℓ)-configuration to an $(\ell+1)$-line configuration is one obtained by adding any line that contains all of the marked points (and perhaps others, to achieve a line with three points). An extension is *proper* if the adjoined line uses at most one unmarked point of K. A proper extension is the *standard extension* if the adjoined line contains only marked points of K. (Up to isomorphism, the standard extension is unique.) For example, C_1 is the standard extension of (C, M), as it uses no such unmarked points. However, (C, M) may have other proper extensions, in which the adjoined line contains one of the unmarked points in K. Indeed all other proper extensions are obtained as standard extensions of $(C, M \cup \{y\})$ for $y \in K \setminus M$ with y not collinear in $\mathscr{L}$ with any point of M.

We count the distinct proper extensions of (C, M) on a total of v points, together with $2 - |M|$ points not in K (*anchors*), in two ways. First, the anchors can be chosen in $\binom{v-k}{2-|M|}$ ways. In a Steiner triple system, exactly one line employs all of the marked points *and* anchors (there are two points in total!), so adjoining the corresponding line gives a proper extension of (C, M), and each so constructed is distinct. This gives $\binom{v-k}{2-|M|}\tau(C, M)\gamma(C)$ ways to form the marked configurations with anchors.

Secondly, we count by classifying the proper extensions. Among them are all occurrences of C_1, the standard extension. For each occurrence of C_1, removing a line containing a point of degree one, removing all degree one points on that line, and marking the rest on the line, produces a marked configuration that may be isomorphic to (C, M); let $\mu(C, M)$ be the number of ways that an isomorph of (C, M) arises from C_1 in this way. Because $3 - |M|$ points not in K are used, yet only $2 - |M|$ are anchors, this yields in total $\mu(C, M)(3 - |M|)$ different ways to produce the marked configuration with anchors. Next consider the other (non-isomorphic) proper extensions $C_2, \ldots, C_e$ that are not standard; by our earlier remarks, for $2 \leq i \leq e$, C_i is the standard extension of $(C, M \cup \{y_i\})$ for some $y_i \in K \setminus M$. Let $\iota(C, M, \{y_i\})$ be the number of times the marked configuration (C, M) appears in $(C, M \cup \{y_i\})$; each marked configuration $(C, M \cup \{y_i\})$ accounts for $\iota(C, M, \{y_i\})$ marked configurations isomorphic to (C, M). Now the standard extension of $(C, M \cup \{y_i\})$ may contain the marked configuration more than once; indeed the removal of any line from this standard extension, marking the points of the line removed, yields a marked configuration that may be isomorphic to $(C, M \cup \{y_i\})$ or to another marked configuration. Let $\kappa(C, M, \{y_i\})$ be the number of times the standard extension of $(C, M \cup \{y_i\})$ contains an isomorph of $(C, M \cup \{y_i\})$ in this way. Then each occurrence of the proper extension C_i of (C, M) contains $\iota(C, M, \{y_i\})\kappa(C, M, \{y_i\})$ occurrences of (C, M).

This accounts for all occurrences of the marked configuration (C, M). Putting the pieces together, we have established that

$$\begin{aligned}&\binom{v-k}{2-|M|}\tau(C,M)\gamma(C)\\ &= \mu(C,M)(3-|M|)\gamma(C_1) + \sum_{i=2}^{e} \iota(C,M,\{y_i\})\kappa(C,M,\{y_i\})\gamma(C_i).\end{aligned} \tag{1}$$

The left hand side arises from the number of marked configurations (C, M) together with $2 - |M|$ anchors. The right hand side accounts for each such choice exactly once, as above. In order to make the calculation explicit,

$$\begin{aligned}\gamma(C_1) = \frac{1}{\mu(C,M)(3-|M|)}\Bigg[&\binom{v-k}{2-|M|}\tau(C,M)\gamma(C)\\ &-\sum_{i=2}^{e} \iota(C,M,\{y_i\})\kappa(C,M,\{y_i\})\gamma(C_i)\Bigg].\end{aligned} \tag{2}$$

We work one example here. Let C be the triangle $(K, \mathscr{L})$ with $K = \{1, 2, 3, 4, 5, 6\}$ and $\mathscr{L} = \big\{\{1, 2, 4\}, \{1, 3, 5\}, \{4, 5, 6\}\big\}$. Mark C using $M = \{2, 3\}$. There are three ways to mark C to obtain an isomorph of (C, M), namely marking $\{2, 3\}$, $\{2, 6\}$, or $\{3, 6\}$, so $\tau(\{2, 3\}, C) = 3$. The standard extension of this marked configuration is $C_1 = \big(K \cup \{7\}, \mathscr{L} \cup \{\{2, 3, 7\}\}\big)$. The configuration C_1 contains an isomorph of (C, M) twice, once removing line $\{4, 5, 6\}$ and once removing line $\{2, 3, 7\}$. So $\mu(C, M) = 2$. Now (C, M) has only one other proper extension. Indeed to mark another point within K, the only choice is point 6 while maintaining noncollinearity. This corresponds to the marked configuration $(C, M \cup \{6\})$, whose standard extension is $C_2 = (K, \mathscr{L} \cup \{\{2, 3, 6\}\})$. In this example, C_2 is isomorphic to the Pasch configuration. The Pasch configuration contains four marked configurations isomorphic to $(C, M \cup \{6\})$, each obtained by the removal of a line. Thus $\kappa(C, M, \{6\}) = 4$. Furthermore, $\big(C, M \cup \{6\}\big)$ contains three isomorphic copies of (C, M), each obtained by "unmarking" one of the points in $\{2, 3, 6\}$. So $\iota(C, M, \{6\}) = 3$. Simplifying all of this, we get the equation

$$3\gamma(C) = 2\gamma(C_1) + 12\gamma(C_2). \tag{3}$$

Of course this example is selected to be small enough for easy hand computation, and so involves few extensions, and simple integer coefficients.

In [19], the same development is done without marking the configurations; we have marked them here in order to make the coefficients explicit and therefore more easily calculated. The essential observation is that the right hand side in (2) contains only configurations with one fewer line or one fewer point than C_1 has. In [19] this is used to support a double induction, first on the number of lines, and second on the number of points, to establish that the count of any configuration that is not full can be written in terms of v and counts of full configurations with fewer points (and perhaps fewer lines).

As we see it, the advantage to marking is that equations arise from marked configurations; indeed there is a one-to-one correspondence between marked configurations with two or fewer marked points and the equations. Our interest is in explicit calculation of the equations produced.

We begin by constructive enumeration of all ℓ-line configurations with $\ell \leq 8$. This is easily accomplished by adjoining one line at a time in all possible ways, and using the canonical form routine of `nauty` ([23]) to preserve one representative of each isomorphism class. Then we mark each in all possible ways, marking at most three points and ensuring that no two of the marked points are collinear. Table 2 gives the number of isomorphism classes of marked configurations classified by the number of lines and the number of marked points.

TABLE 2. Counts of marked configurations

	Number of Lines							
Marked	1	2	3	4	5	6	7	8
0	1	2	5	16	56	282	1865	17100
1	1	3	11	48	258	1766	15708	
2		2	10	64	455	4088	45335	
3			5	38	364	4159	57218	

When treating counts of configurations on $\ell + 1$ lines, typically there are more equations arising from the marked configurations on ℓ lines than there are $(\ell + 1)$-line configurations. For example, for the 282 6-line configurations, $769 = 56 + 258 + 455$ equations are generated. This occurs because while each such marked configuration on 5 lines has a unique standard extension, that standard extension may be the same as one from another marked configuration. Duplication among the equations can be used in part to verify the computation, as each equation should determine the same relationship (using a different set of extensions).

With a list of all marked configurations in hand, it is an easy matter to calculate the quantities τ, ι, and κ used above for each of the marked configurations;

each changes the marking, or removes or adds a line, to produce another marked configuration, and `nauty` is again used to determine isomorphism.

The precomputation of marked configurations along with τ, ι, and κ permits us to determine all of the equations representing counts of all configurations in terms of counts of full configurations (and the variable v), for any maximum number of lines. We have carried out this computation completely for counts of the configurations on eight or fewer lines. As expected, the additional equations that arise from the multiplicity of marked configurations yield duplicate equations and no inconsistency results.

3. Configuration polytopes

As we have seen, substantial effort has been invested in determining counts of configurations; sometimes the maximum count for a specific configuration is of interest, as with Pasch configurations ([14], [26]) or Mitre configurations ([4, Chapter 13]). Sometimes the minimum is of interest. Indeed the *avoidance* problem asks whether the minimum count for a configuration can be zero. Simultaneous avoidance of multiple configurations asks whether all of their counts can simultaneously be zero. For example, the 6-sparse problem considers when the counts of Pasch, Mitre, Crown, and Hexagon, can each be zero. In [12], a problem in codes for computer-aided circuit design asks when the counts of Pasch, Double Triangle, and Grid can each be zero (these are the even configurations).

In order to treat all such questions in a standard way, denote by η_ℓ the number of nonisomorphic configurations on ℓ or fewer lines, and denote by φ_ℓ the number of these that are full. We sometimes abbreviate these to η and φ, assuming ℓ from the context. We consider the set of equations expressing the numbers of each of the η_ℓ configurations in terms of the numbers of the φ_ℓ full configurations and the variable v. For concreteness, $\eta_6 = 362$, and so there are 363 variables when v is not fixed. The form of the equations ensures that each expresses a configuration count as a linear combination of φ_ℓ configuration counts, with coefficients that can be ratios of polynomials in v. For fixed v, then, configuration counts are expressed as linear combinations of other configuration counts.

We adopt a different viewpoint. Begin with real space of dimension $\eta + 1$ ($\mathbb{R}^{363}$ when $\ell = 6$). All simultaneous selections of counts for the configurations and for the order v of a Steiner triple system reside in the positive orthant $\mathbb{R}_+^{\eta+1}$. The polyhedron defined by the admissible counts and value of v does not have dimension $\eta + 1$, however! Instead the equations established earlier limit the dimension to at most $\varphi_\ell + 1$. So far this is a simple translation.

Now rather than using the full configurations to determine counts of the rest, we use counts of the rest to constrain the counts of the full configurations. When (C, M) is a marked configuration with at most two marked points, we combine two pieces of information. First, the number of times (C, M) occurs in a specific system is a nonnegative integer. Second, it is *equal* to a known linear combination of counts of full configurations. Combine these to observe that linear combination must be nonnegative.

To illustrate this, return to the example leading to (3). In that case, C is a constant configuration, a triangle, appearing precisely 4Δ times in every STS(v). However C_1 and the Pasch configuration C_2 are variable. Nevertheless we can bound the number of Pasch configurations, as follows. By (3), $\gamma(C_2) = \frac{1}{12}(3\gamma(C) - 2\gamma(C_1))$. Then as $\gamma(C_1) \geq 0$, we find that $\gamma(C_2) \leq \frac{1}{4}\gamma(C) = \Delta$. By similar arguments, an upper bound in terms of v on each of the full configurations can be established. We simply state them here:

Pasch	Mitre	STS(7) -Line	Crown	Hexagon	Double Triangle	Grid
Δ	2Δ	Δ	6Δ	2Δ	$\Delta(2v-3)$	$\Delta\frac{v-3}{3}$

In passing we remark that the maximum count of Pasch configurations $\Delta = \frac{1}{24}v(v-1)(v-3)$ is misstated in [4, Chapter 13]. More seriously, the maximum count of Mitre configurations, 2Δ, is more seriously misstated there and the mistake repeated in [1].

This crude argument does not take into account interactions among the counts of the full configurations, so may not yield a tight bound. Indeed it does not provide any information about simultaneous occurrence of two or more configurations; nor does it bound the counts of configurations that are not full.

In order to address these questions (and more), let $\gamma_1, \ldots, \gamma_\varphi$ be the counts for the full configurations (for $\ell = 6$, index them in the order shown as $\gamma_1, \ldots, \gamma_7$). Now consider again equation (2). As the count $\gamma(C_1)$ is nonnegative, so also is the linear combination on the right hand side. Now this linear combination can be written in terms of $\{\gamma_1, \ldots, \gamma_\varphi\}$, producing a linear inequality involving the φ variables. To be precise, the coefficient of γ_i for $1 \leq i \leq \varphi$ and the "constant" term are ratios of polynomials in v; so the linear inequalities are in $\mathbb{R}^\varphi$ for fixed v. Consider the set $\mathscr{I}_\ell$ (or simply $\mathscr{I}$) of *all* such inequalities arising from (2) from every marked configuration on $\ell + 1$ lines.

Since γ_i is bounded below and above by fixed functions of v (and all such inequalities appear in $\mathscr{I}_\ell$), the inequalities define a finite polyhedron $\mathscr{P}_\ell$ in $\mathbb{R}^{\varphi_\ell}$. We call this the *configuration polytope* for ℓ-line configurations, although there is actually one polytope $\mathscr{P}_\ell(v)$ for each choice of v. Refer to [24] for polyhedral theory and terms not defined here.

Why define a configuration polytope? The boundary and interior of this polytope defines regions in which *simultaneous* assignments to the counts of full configurations can be made that are valid under $\mathscr{I}$, and hence possible in principle as counts of full configurations in Steiner triple systems of order v. The exterior of the polytope certainly consists of assignments that are infeasible.

More importantly, according to (2) the configuration count of any ℓ-line configuration is a linear combination of $\{\gamma_1, \ldots, \gamma_\varphi\}$. Maximizing or minimizing the count of any configuration subject to $\mathscr{I}_\ell$, not enforcing integrality of the counts, is an optimization problem whose feasible region consists of the configuration polytope — and hence the optimum occurs at an extreme point [24, Theorem I.4.5].

For the purposes of computation we restrict to $\ell = 6$, so that $\eta = 362$ and $\varphi = 7$. We form the family of configuration polytopes $\mathscr{P}_6(v)$ parameterized by v, and by enumerative techniques we list all extreme points in Figure 3. We describe the method by which this is done in a moment, but remark on some interesting results first.

Our crude upper bound for γ_6 was $\Delta(2v-3)$ and that for γ_7 was $\Delta\frac{v-3}{3}$. Yet the extreme points never permit $\gamma_6 > 2\Delta(v-7)$ or $\gamma_7 > \frac{1}{3}\Delta(v-7)$, so in these two cases the extreme points provide a more accurate bound. Simultaneous occurrences can also be examined. For example, in an STS(v) that has no STS(7)-Line, the maximum number of Pasch configurations is at most $\frac{1}{3}\Delta$, only one-third of the maximum permitted when STS(7)-Line can occur.

To determine the extreme points, we start with the 362 constraints produced for 6-line configurations in (2). Some are vacuous because they do not involve any of $\{\gamma_1, \ldots, \gamma_7\}$. This occurs for all constant configurations, of which there are 23 in total. Then eliminating constraints that are easily seen to be dominated by another, only 110 remain. Treating cases when an inequality is dominated by a linear combination of two others does not appear to be effective in reducing this much further. Instead we use the crude upper bounds developed before as follows. Inequalities in $\mathscr{I}$ are written in the form $\sum_{i=1}^{7} \beta_i\gamma_i \leq \beta_0$. Let $\overline{\gamma}_i$ be the crude upper bound determined earlier, and $\underline{\gamma}_i = 0$. Then replace γ_i by $\overline{\gamma}_i$ if $\beta_i < 0$, or by $\underline{\gamma}_i$ if $\beta_i \geq 0$ and evaluate the sum. If it is always at most β_0, then the inequality is dominated by the constraints giving the lower and upper bounds on $\{\gamma_1, \ldots, \gamma_7\}$, and can be eliminated without changing the polytope.

This is very effective: Only 36 inequalities remain. At an extreme point, seven linearly independent inequalities hold as equalities (and others may also hold). Hence we can enumerate the choices of seven putative equalities systematically, avoiding linear dependences (which are revealed as further dominated inequalities); once a suitable set of 7 is found, the values $\{\gamma_1, \ldots, \gamma_7\}$ can be calculated

Table 3. Extreme points

Pasch	Mitre	STS(7) -Line	Crown	Hexagon	Double Triangle	Grid
0	0	0	0	0	0	0
0	0	0	0	0	0	$\frac{1}{3}\Delta(v-12)$
0	0	0	0	0	$2\Delta(v-14)$	0
0	0	0	0	0	$2\Delta(v-14)$	$\frac{1}{3}\Delta(v-12)$
0	0	0	0	2Δ	0	0
0	0	0	0	2Δ	0	$\frac{1}{3}\Delta(v-12)$
0	0	0	0	2Δ	$2\Delta(v-13)$	0
0	0	0	0	2Δ	$2\Delta(v-13)$	$\frac{1}{3}\Delta(v-12)$
0	0	0	6Δ	0	0	0
0	0	0	6Δ	0	0	$\frac{1}{3}\Delta(v-12)$
0	0	0	6Δ	0	$2\Delta(v-12)$	0
0	0	0	6Δ	0	$2\Delta(v-12)$	$\frac{1}{3}\Delta(v-12)$
0	2Δ	0	0	0	0	0
0	2Δ	0	0	0	0	$\frac{1}{3}\Delta(v-8)$
0	2Δ	0	0	0	$2\Delta(v-10)$	0
0	2Δ	0	0	0	$2\Delta(v-10)$	$\frac{1}{3}\Delta(v-8)$
0	2Δ	0	0	2Δ	0	0
0	2Δ	0	0	2Δ	0	$\frac{1}{3}\Delta(v-8)$
0	2Δ	0	0	2Δ	$2\Delta(v-9)$	0
0	2Δ	0	0	2Δ	$2\Delta(v-9)$	$\frac{1}{3}\Delta(v-8)$
$\frac{1}{3}\Delta$	0	0	0	0	0	0
$\frac{1}{3}\Delta$	0	0	0	0	0	$\frac{1}{9}\Delta(3v-31)$
$\frac{1}{3}\Delta$	0	0	0	0	$\frac{2}{3}\Delta(3v-35)$	0
$\frac{1}{3}\Delta$	0	0	0	0	$\frac{2}{3}\Delta(3v-35)$	$\frac{1}{9}\Delta(3v-31)$
$\frac{1}{3}\Delta$	0	0	0	$\frac{4}{3}\Delta$	0	0
$\frac{1}{3}\Delta$	0	0	0	$\frac{4}{3}\Delta$	0	$\frac{1}{9}\Delta(3v-31)$
$\frac{1}{3}\Delta$	0	0	0	$\frac{4}{3}\Delta$	$2\Delta(v-11)$	0
$\frac{1}{3}\Delta$	0	0	0	$\frac{4}{3}\Delta$	$2\Delta(v-11)$	$\frac{1}{9}\Delta(3v-31)$
$\frac{1}{3}\Delta$	0	0	4Δ	0	0	0
$\frac{1}{3}\Delta$	0	0	4Δ	0	0	$\frac{1}{9}\Delta(3v-31)$
$\frac{1}{3}\Delta$	0	0	4Δ	0	$\frac{2}{3}\Delta(3v-31)$	0
$\frac{1}{3}\Delta$	0	0	4Δ	0	$\frac{2}{3}\Delta(3v-31)$	$\frac{1}{9}\Delta(3v-31)$
Δ	0	Δ	0	0	0	0
Δ	0	Δ	0	0	0	$\frac{1}{3}\Delta(v-7)$
Δ	0	Δ	0	0	$2\Delta(v-7)$	0
Δ	0	Δ	0	0	$2\Delta(v-7)$	$\frac{1}{3}\Delta(v-7)$

Continued on next page

Pasch	Mitre	STS(7) -Line	Crown	Hexagon	Double Triangle	Grid
$\frac{1}{3}\Delta$	$\frac{2}{3}\Delta$	0	0	0	0	0
$\frac{1}{3}\Delta$	$\frac{2}{3}\Delta$	0	0	0	0	$\frac{1}{3}\Delta(v-9)$
$\frac{1}{3}\Delta$	$\frac{2}{3}\Delta$	0	0	0	$\frac{2}{3}\Delta(3v-31)$	0
$\frac{1}{3}\Delta$	$\frac{2}{3}\Delta$	0	0	0	$\frac{2}{3}\Delta(3v-31)$	$\frac{1}{3}\Delta(v-9)$
$\frac{1}{3}\Delta$	$\frac{2}{3}\Delta$	0	0	$\frac{4}{3}\Delta$	0	0
$\frac{1}{3}\Delta$	$\frac{2}{3}\Delta$	0	0	$\frac{4}{3}\Delta$	0	$\frac{1}{3}\Delta(v-9)$
$\frac{1}{3}\Delta$	$\frac{2}{3}\Delta$	0	0	$\frac{4}{3}\Delta$	$\frac{2}{3}\Delta(3v-29)$	0
$\frac{1}{3}\Delta$	$\frac{2}{3}\Delta$	0	0	$\frac{4}{3}\Delta$	$\frac{2}{3}\Delta(3v-29)$	$\frac{1}{3}\Delta(v-9)$

from the seven independent equalities, and these values form the extreme point. Some duplication arises in our enumeration, which we suppress in Table 3.

By choosing extreme points to maximize or minimize the linear combination of $\gamma_1, \ldots, \gamma_7$ specified by any configuration count, we can determine lower and upper bounds on the number of occurrences of that configuration. This is done in Table 4 for all configurations on 2, 3, and 4 lines.

Each line gives a configuration number, then the number of lines and number of points, and then one set of blocks isomorphic to this configuration. Seven columns then indicate the dependence of the count on $\gamma_1, \ldots, \gamma_7$; $\oplus$ indicates that the configuration is the full configuration, while $\odot$ indicates that the count of this configuration depends on the count of that full configuration. Finally lower and upper bounds, in terms of v, are given for the configuration count. A nonzero lower bound ensures that the configuration cannot be avoided. When the lower bound is 0, it might be avoidable.

There are some limitations to this analysis. Inclusion in the configuration polytope may not ensure that an integer point can be realized in a Steiner triple system. Also the extreme points may be fractional. For example, when $\gamma_1 = \cdots = \gamma_6 = 0$ and $\gamma_7 = \frac{1}{3}\Delta(v-12)$, the value of γ_7 is not integral when $v \equiv 7, 13 \pmod{18}$ but is integral otherwise.

Nevertheless the bounds produced in this manner are valid and suggest what the extreme values can be expected to be. Questions about simultaneous avoidance are more difficult to tabulate, but are easily addressed as follows. At each extreme point, one can tabulate all configurations that have lower bound 0. Then each such set corresponds to a maximal set whose counts can be simultaneously 0.

To illustrate this, all 5-line configurations are given in Tables 5 and 6 in the same format as Table 4. Among the first 80 configurations, we can hope to avoid numbers 9, 11, 25–32, 36, 37, 38, and 45 in general individually (for small values of v, more may be avoidable). The maximal sets of configurations that can be

TABLE 4. Configurations with four and fewer lines

	k	ℓ	Configuration	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	Lower Bound	Upper Bound
1	1	3	abc								$\frac{1}{6}v(v\text{-}1)$	$\frac{1}{6}v(v\text{-}1)$
2	2	5	abc ade								3Δ	3Δ
3	2	6	abc def								$\frac{1}{3}\Delta(v\text{-}7)$	$\frac{1}{3}\Delta(v\text{-}7)$
4	3	6	abc ade bdf								4Δ	4Δ
5	3	7	abc ade afg								$\frac{1}{2}\Delta(v\text{-}5)$	$\frac{1}{2}\Delta(v\text{-}5)$
6	3	7	abc ade bfg								$3\Delta(v\text{-}7)$	$3\Delta(v\text{-}7)$
7	3	8	abc ade fgh								$\frac{1}{2}\Delta(v\text{-}7)(v\text{-}9)$	$\frac{1}{2}\Delta(v\text{-}7)(v\text{-}9)$
8	3	9	abc def ghi								$\frac{1}{54}\Delta(v\text{-}7)(v^2\text{-}19v+96)$	$\frac{1}{54}\Delta(v\text{-}7)(v^2\text{-}19v+96)$
9	4	6	abc ade bdf cef	$\oplus$							0	Δ
10	4	7	abc ade bdf cdg								4Δ	4Δ
11	4	7	abc ade bdf ceg	$\odot$							0	6Δ
12	4	8	abc ade bdf agh								$6\Delta(v\text{-}7)$	$6\Delta(v\text{-}7)$
13	4	8	abc ade bdf cgh	$\odot$							$6\Delta(v\text{-}9)$	$6\Delta(v\text{-}7)$
14	4	8	abc ade bfg dfh	$\odot$							$3\Delta(v\text{-}8)$	$3\Delta(v\text{-}7)$
15	4	9	abc ade bdf ghi	$\odot$							$\frac{2}{3}\Delta(v\text{-}7)(v\text{-}12)$	$\frac{2}{3}\Delta(v\text{-}10)(v\text{-}9)$
16	4	9	abc ade afg ahi								$\frac{1}{16}\Delta(v\text{-}5)(v\text{-}7)$	$\frac{1}{16}\Delta(v\text{-}5)(v\text{-}7)$
17	4	9	abc ade afg bhi								$\frac{3}{2}\Delta(v\text{-}7)(v\text{-}9)$	$\frac{3}{2}\Delta(v\text{-}7)(v\text{-}9)$
18	4	9	abc ade bfg chi	$\odot$							$\frac{1}{2}\Delta(v\text{-}7)(v\text{-}11)$	$\frac{1}{2}\Delta(v^2\text{-}18v+85)$
19	4	9	abc ade bfg dhi	$\odot$							$3\Delta(v\text{-}7)(v\text{-}11)$	$3\Delta(v\text{-}9)^2$
20	4	10	abc ade afg hij								$\frac{1}{12}\Delta(v\text{-}7)(v\text{-}11)(v\text{-}9)$	$\frac{1}{12}\Delta(v\text{-}7)(v\text{-}11)(v\text{-}9)$
21	4	10	abc ade bfg hij	$\odot$							$\frac{1}{2}\Delta(v\text{-}9)(v^2\text{-}20v+103)$	$\frac{1}{2}\Delta(v\text{-}7)(v^2\text{-}22v+129)$
22	4	10	abc ade fgh fij	$\odot$							$\frac{3}{16}\Delta(v\text{-}11)(v\text{-}9)^2$	$\frac{3}{16}\Delta(v\text{-}7)(v^2\text{-}22v+125)$
23	4	11	abc ade fgh ijk	$\odot$							$\frac{1}{24}\Delta(v\text{-}7)(v\text{-}13)(v^2\text{-}21v+126)$	$\frac{1}{24}\Delta(v\text{-}10)(v\text{-}9)(v^2\text{-}22v+129)$
24	4	12	abc def ghi jkl	$\odot$							$\frac{1}{1296}\Delta(v\text{-}13)(v\text{-}9)(v\text{-}10)(v^2\text{-}22v+141)$	$\frac{1}{1296}\Delta(v\text{-}7)(v^4\text{-}47v^3+853v^2\text{-}7125v+23382)$

simultaneously avoided at an extreme point of $\mathscr{P}_6$ are $A_1 = \{9, 25, 26, 27, 32\}$; $A_2 = \{9, 25, 27, 29, 31, 32\}$; and $A_3 = \{11, 26, 28, 29, 30, 31, 36, 37, 38, 45\}$.

By Minkowski's Theorem ([24, Theorem I.4.8]), every point of the configuration polytope can be written as a convex combination of the extreme points. It follows that whenever a configuration count is 0 for any point of the polytope, it must be 0 at some extreme point. While this ensures that we have captured all of the configurations that can in principle be avoided, it should not be concluded that other (smaller) sets of configurations cannot be the actual set avoided. Indeed at some of the extreme points of $\mathscr{P}_6$, *every* configuration has nonzero count. Once v is large enough, this is expected at some point in the polytope, but it is perhaps surprising that it holds at an extreme point.

In the Appendix, we tabulate all 282 6-line configurations. Again we can ask about simultaneous avoidance. Let

$$B_1 = A_1 \cup \{81, 85, 86, 108, 109, 110, 111, 112, 113, 114, 115, 166, 167, 168, 169, 170, 236, 237, 293\};$$

$$B_2 = A_2 \cup \{81, 82, 84, 85, 89, 91, 92, 93, 94, 95, 96, 97, 98, 102, 104, 105, 109, 110, 111, 112, 121, 122, 123, 124, 125, 126, 138, 139, 140, 141, 166, 167, 168, 169, 170, 172, 186, 236, 237, 293\};$$

$$B_3 = A_3 \cup \{82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 171, 172, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 242, 243, 244, 245, 295\};$$

$$B_4 = \{26\} \cup \{81, 83, 86, 88, 113, 114, 115\}; \qquad \text{and}$$

$$B_5 = \emptyset \cup \{81, 82, 88, 89, 92, 94\}.$$

Then the maximal sets of configurations that can all be set to 0 at an extreme point of $\mathscr{P}_6$ are listed in Table 7.

Again, it seems difficult to determine which of these sets can be avoided in an STS(v), but certainly no sets not contained in one of these can be avoided for "large" values of v.

4. The seven and eight line cases

Much of this effort can be carried through for the seven line case, and some through the eight line case as well. The number of relevant inequalities has

TABLE 5. Configurations with five lines I

	k	ℓ	Configuration	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	Lower Bound	Upper Bound
25	5	7	abc ade bdf cef cdg	$\odot$							0	3Δ
26	5	7	abc ade bdf cdg efg		$\oplus$						0	2Δ
27	5	8	abc ade bdf cef agh	$\odot$							0	$3\Delta(v\text{-}7)$
28	5	8	abc ade bdf cdg beh	$\odot$							0	12Δ
29	5	8	abc ade bdf cdg efh	$\odot$	$\odot$						0	12Δ
30	5	8	abc ade bdf ceg cfh	$\odot$							0	12Δ
31	5	8	abc ade bdf ceg fgh	$\odot$	$\odot$						0	6Δ
32	5	9	abc ade bdf cef ghi	$\odot$							0	$\frac{1}{6}\Delta(v\text{-}7)(v\text{-}12)$
33	5	9	abc ade bdf cdg ahi	$\odot$							$6\Delta(v\text{-}9)$	$6\Delta(v\text{-}7)$
34	5	9	abc ade bdf cdg dhi								$2\Delta(v\text{-}7)$	$2\Delta(v\text{-}7)$
35	5	9	abc ade bdf cdg ehi	$\odot$	$\odot$						$6\Delta(v\text{-}11)$	$6\Delta(v\text{-}7)$
36	5	9	abc ade bdf ceg ahi	$\odot$	$\odot$						0	$3\Delta(v\text{-}7)$
37	5	9	abc ade bdf ceg bhi	$\odot$							0	$12\Delta(v\text{-}9)$
38	5	9	abc ade bdf ceg fhi	$\odot$	$\odot$						0	$6\Delta(v\text{-}9)$
39	5	9	abc ade bdf agh bgi	$\odot$							$6\Delta(v\text{-}8)$	$6\Delta(v\text{-}7)$
40	5	9	abc ade bdf agh cgi	$\odot$	$\odot$						$12\Delta(v\text{-}10)$	$12\Delta(v\text{-}7)$
41	5	9	abc ade bdf agh fgi	$\odot$							$12\Delta(v\text{-}9)$	$12\Delta(v\text{-}7)$
42	5	9	abc ade bdf cgh egi	$\odot$	$\odot$						$12\Delta(v\text{-}11)$	$12\Delta(v\text{-}7)$
43	5	9	abc ade bfg dfh egi	$\odot$	$\odot$						$2\Delta(v\text{-}10)$	$2\Delta(v\text{-}7)$
44	5	10	abc ade bdf cdg hij	$\odot$	$\odot$						$\frac{2}{3}\Delta(v\text{-}15)(v\text{-}7)$	$\frac{2}{3}\Delta(v^2\text{-}22v+123)$
45	5	10	abc ade bdf ceg hij	$\odot$	$\odot$						0	$\Delta(v^2\text{-}22v+123)$
46	5	10	abc ade bdf agh aij								$\frac{3}{2}\Delta(v\text{-}7)(v\text{-}9)$	$\frac{3}{2}\Delta(v\text{-}7)(v\text{-}9)$
47	5	10	abc ade bdf agh bij	$\odot$							$3\Delta(v\text{-}7)(v\text{-}11)$	$3\Delta(v\text{-}9)^2$
48	5	10	abc ade bdf agh cij	$\odot$	$\odot$						$6\Delta(v\text{-}11)(v\text{-}9)$	$6\Delta(v\text{-}7)(v\text{-}11)$
49	5	10	abc ade bdf agh fij	$\odot$							$3\Delta(v\text{-}11)(v\text{-}9)$	$3\Delta(v\text{-}7)(v\text{-}11)$
50	5	10	abc ade bdf agh gij	$\odot$	$\odot$						$6\Delta(v\text{-}7)(v\text{-}13)$	$6\Delta(v^2\text{-}20v+101)$
51	5	10	abc ade bdf cgh cij	$\odot$							$\frac{3}{2}\Delta(v\text{-}11)(v\text{-}9)$	$\frac{3}{2}\Delta(v\text{-}7)(v\text{-}9)$
52	5	10	abc ade bdf cgh eij	$\odot$	$\odot$						$3\Delta(v\text{-}13)(v\text{-}9)$	$3\Delta(v\text{-}7)(v\text{-}11)$
53	5	10	abc ade bdf cgh gij	$\odot$	$\odot$						$6\Delta(v\text{-}13)(v\text{-}9)$	$6\Delta(v\text{-}7)(v\text{-}13)$
54	5	10	abc ade afg bhi dhj	$\odot$							$6\Delta(v\text{-}10)(v\text{-}9)$	$6\Delta(v\text{-}7)(v\text{-}11)$
55	5	10	abc ade bfg dfh cij	$\odot$	$\odot$						$6\Delta(v\text{-}9)(v\text{-}12)$	$6\Delta(v\text{-}7)(v\text{-}13)$
56	5	10	abc ade bfg dhi fhj	$\odot$	$\odot$						$\frac{5}{12}\Delta(v\text{-}7)(v\text{-}14)$	$\frac{5}{12}\Delta(v^2\text{-}21v+113)$

TABLE 6. Configurations with five lines II

57	5	11	abc ade bdf agh ijk	⊙	⊙						$\Delta(v-7)(v^2-25v+162)$	$\Delta(v-9)(v^2-23v+136)$
58	5	11	abc ade bdf cgh ijk	⊙	⊙						$\Delta(v-12)(v^2-22v+129)$	$\Delta(v-7)(v^2-25v+162)$
59	5	11	abc ade bdf ghi gjk	⊙	⊙						$\frac{1}{2}\Delta(v-7)(v^2-27v+186)$	$\frac{1}{2}\Delta(v-9)(v^2-25v+162)$
60	5	11	abc ade afg ahi ajk								$\frac{1}{160}\Delta(v-5)(v-7)(v-9)$	$\frac{1}{160}\Delta(v-5)(v-7)(v-9)$
61	5	11	abc ade afg ahi bjk								$\frac{1}{4}\Delta(v-7)(v-11)(v-9)$	$\frac{1}{4}\Delta(v-7)(v-11)(v-9)$
62	5	11	abc ade afg bhi bjk	⊙							$\frac{3}{16}\Delta(v-11)(v-9)^2$	$\frac{3}{16}\Delta(v-7)(v^2-22v+125)$
63	5	11	abc ade afg bhi cjk	⊙	⊙						$\frac{3}{8}\Delta(v-7)(v-11)(v-13)$	$\frac{3}{8}\Delta(v-9)(v^2-22v+129)$
64	5	11	abc ade afg bhi djk	⊙	⊙						$\frac{3}{2}\Delta(v-7)(v^2-24v+151)$	$\frac{3}{2}\Delta(v-9)(v^2-22v+125)$
65	5	11	abc ade afg bhi hjk	⊙							$\frac{3}{2}\Delta(v-7)(v-11)(v-13)$	$\frac{3}{2}\Delta(v-9)(v-11)^2$
66	5	11	abc ade bfg dfh ijk	⊙	⊙						$\frac{1}{2}\Delta(v-12)(v^2-21v+116)$	$\frac{1}{2}\Delta(v-7)(v^2-25v+168)$
67	5	11	abc ade bfg chi djk	⊙	⊙						$\frac{3}{2}\Delta(v-7)(v^2-26v+173)$	$\frac{3}{2}\Delta(v-9)(v^2-24v+155)$
68	5	11	abc ade bfg dhi fjk	⊙	⊙						$3\Delta(v-7)(v^2-26v+177)$	$3\Delta(v-9)(v^2-24v+151)$
69	5	12	abc ade bdf ghi jkl	⊙	⊙						$\frac{1}{18}\Delta(v-15)(v-7)(v^2-25v+174)$	$\frac{1}{18}\Delta(v^2-22v+129)(v^2-25v+162)$
70	5	12	abc ade afg ahi jkl								$\frac{1}{96}\Delta(v-7)(v-11)(v-13)(v-9)$	$\frac{1}{96}\Delta(v-7)(v-11)(v-13)(v-9)$
71	5	12	abc ade afg bhi jkl	⊙	⊙						$\frac{1}{4}\Delta(v-13)(v-9)(v^2-22v+129)$	$\frac{1}{4}\Delta(v-7)(v-13)(v^2-24v+159)$
72	5	12	abc ade afg hij hkl	⊙							$\frac{1}{16}\Delta(v-13)(v-9)(v-11)^2$	$\frac{1}{16}\Delta(v-7)(v-11)(v^2-26v+177)$
73	5	12	abc ade bfg chi jkl	⊙	⊙						$\frac{1}{12}\Delta(v-7)(v-13)(v^2-26v+189)$	$\frac{1}{12}\Delta(v^4-46v^3+808v^2-6442v+19743)$
74	5	12	abc ade bfg dhi jkl	⊙	⊙						$\frac{1}{2}\Delta(v-7)(v^3-39v^2+527v-2481)$	$\frac{1}{2}\Delta(v^4-46v^3+804v^2-6330v+18987)$
75	5	12	abc ade bfg hij hkl	⊙	⊙						$\frac{3}{8}\Delta(v-9)(v^3-37v^2+467v-2039)$	$\frac{3}{8}\Delta(v-7)(v^3-39v^2+527v-2473)$
76	5	13	abc ade afg hij klm	⊙	⊙						$\frac{1}{144}\Delta(v-7)(v-13)(v-15)(v^2-23v+156)$	$\frac{1}{144}\Delta(v-9)(v^2-22v+129)(v^2-27v+188)$
77	5	13	abc ade bfg hij klm	⊙	⊙						$\frac{1}{24}\Delta(v^5-60v^4+1458v^3-17944v^2+111909v-283428)$	$\frac{1}{24}\Delta(v-7)(v^4-53v^3+1087v^2-10287v+38124)$
78	5	13	abc ade fgh fij klm	⊙	⊙						$\frac{1}{32}\Delta(v^5-60v^4+1454v^3-17784v^2+109809v-274284)$	$\frac{1}{32}\Delta(v-7)(v^4-53v^3+1083v^2-10187v+37428)$
79	5	14	abc ade fgh ijk lmn	⊙	⊙						$\frac{1}{432}\Delta(v-7)(v-15)(v^2-22v+165)(v^2-31v+252)$	$\frac{1}{432}\Delta(v^6-75v^5+2370v^4-40402v^3+391905v^2-2051235v+4530492)$
80	5	15	abc def ghi jkl mno	⊙	⊙						$\frac{1}{38880}\Delta(v^7-91v^6+3588v^5-79510v^4+1069873v^3-8742231v^2+40167162v-80101224)$	$\frac{1}{38880}\Delta(v-7)(v^6-84v^5+3000v^4-58510v^3+660303v^2-4113630v+11131992)$

TABLE 7. Simultaneous avoidance for 6-line configurations

$B_1 \cup \{82, 83, 84, 106, 107\}$	$B_1 \cup \{82, 83, 84, 106, 165\}$
$B_1 \cup \{82, 83, 84, 107, 164\}$	$B_1 \cup \{82, 83, 84, 164, 165\}$
$B_1 \cup \{82, 84, 101, 103, 106, 107\}$	$B_1 \cup \{82, 84, 101, 103, 106, 165\}$
$B_1 \cup \{82, 84, 101, 103, 107, 163, 164\}$	$B_1 \cup \{82, 84, 101, 103, 163, 164, 165\}$
$B_1 \cup \{83, 84, 98, 103, 106, 107\}$	$B_1 \cup \{83, 84, 98, 103, 106, 165\}$
$B_1 \cup \{83, 84, 98, 103, 107, 164, 166\}$	$B_1 \cup \{83, 84, 98, 103, 164, 165, 166\}$
$B_2 \cup \{83, 106, 107, 108\}$	$B_2 \cup \{83, 106, 108, 165\}$
$B_2 \cup \{83, 107, 108, 164\}$	$B_2 \cup \{83, 108, 164, 165\}$
$B_2 \cup \{101, 103, 106, 107, 108\}$	$B_2 \cup \{101, 103, 106, 108, 165\}$
$B_2 \cup \{101, 103, 107, 108, 163, 164\}$	$B_2 \cup \{101, 103, 108, 163, 164, 165\}$
$B_3 \cup \{106, 107\}$	$B_3 \cup \{106, 165\}$
$B_3 \cup \{107, 163, 164\}$	$B_3 \cup \{163, 164, 165\}$
$B_4 \cup \{82, 106, 107\}$	$B_4 \cup \{82, 106, 165\}$
$B_4 \cup \{82, 107, 164\}$	$B_4 \cup \{82, 164, 165\}$
$B_4 \cup \{82, 101, 103, 106, 107\}$	$B_4 \cup \{82, 101, 103, 106, 165\}$
$B_4 \cup \{82, 101, 103, 107, 163, 164\}$	$B_4 \cup \{82, 101, 103, 163, 164, 165\}$
$B_4 \cup \{82, 92, 98, 103, 106, 107\}$	$B_4 \cup \{82, 92, 98, 103, 106, 165\}$
$B_4 \cup \{82, 92, 98, 103, 107, 164\}$	$B_4 \cup \{82, 92, 98, 103, 164, 165\}$
$B_5 \cup \{83, 106, 107\}$	$B_5 \cup \{83, 106, 165\}$
$B_5 \cup \{83, 107, 164\}$	$B_5 \cup \{83, 106, 165\}$
$B_5 \cup \{101, 103, 106, 107\}$	$B_5 \cup \{101, 103, 106, 165\}$
$B_5 \cup \{101, 103, 107, 163, 164\}$	$B_5 \cup \{101, 103, 163, 164, 165\}$

been dramatically reduced from 362 to 36 for 6-line configurations. Nonetheless this represents the primary obstacle to treating 7-line configurations in the same manner; the initial set has 2227 inequalities, but worse – there are 26 full configurations to treat, so the configuration polytope is in $\mathbb{R}^{26}$ (and, by [27], has full dimension).

The elimination of constraints easily seen to be dominated by another reduces the 2227 inequalities to 1373. Rather than using crude bounds, we use the 44 extreme points of $\mathscr{P}_6$ and the linear inequalities to determine maxima for the 19 counts of full 7-line configurations. These are given in Table 8. Enforcing these upper bounds to eliminate further inequalities leaves 545 inequalities to define $\mathscr{P}_7$. While in principle the extreme points of $\mathscr{P}_7$ could now be determined by finding sets of 26 independent inequalities among the 545 that are met with equality, we leave this for a (much) longer day.

It must be emphasized that the maxima presented in Table 8 may not be achievable for a particular choice of v even if the maximum can be achieved on occasion. For example, as the first five all contain a Pasch configuration, their

Table 8. Maximum counts of full 7-line configurations

Configuration	Maximum Count	Comment
abc ade bdf cef cdg beg afg	$\frac{1}{7}\Delta$	STS(7), has Pasch
abc ade bdf cef cdg beh agh	Δ	has Pasch
abc ade bdf cef agh bgi chi	$2\Delta(v-7)$	has Pasch
abc ade bdf cef agh bgi dhi	4Δ	has Pasch
abc ade bdf cef agh bgi ehi	12Δ	has Pasch
abc ade bdf cdg efg beh cfh	4Δ	has Mitre, Crown
abc ade bdf cdg beh afi ghi	4Δ	Erdős
abc ade bdf cdg beh cfi ghi	12Δ	Erdős
abc ade bdf cdg beh bgi fhi	3Δ	Erdős
abc ade bdf cdg beh egi fhi	12Δ	Erdős
abc ade bdf cdg efh egi ahi	6Δ	Erdős, has Double Triangle
abc ade bdf cdg efh egi bhi	12Δ	Erdős
abc ade bdf cdg ehi fhj gij	$2\Delta(v-7)$	
abc ade bdf ceg cfh bgi ahi	2Δ	Erdős, has Grid
abc ade bdf ceg cfh bgi dhi	6Δ	Erdős
abc ade bdf ceg ahi fhj gij	$6\Delta(v-9)$	
abc ade bdf ceg bhi fhj gij	$3\Delta(v-9)$	
abc ade bdf agh cgi ehj fij	$4\Delta(v-7)$	
abc ade bdf agh cgi fhj eij	$3\Delta(v-7)$	

maxima could be achieved only when the number of Pasch configurations is Δ, and hence the system is a projective triple system. Indeed there is no guarantee that the maxima given can ever be achieved, as the linear equalities capture some but not all of the combinatorial restrictions.

5. Conclusions

We have adopted a polyhedral view of configurations in Steiner triple systems. Equations from the generating set, relaxed to nonnegative fractional inequalities, define a family of polytopes. Every feasible assignment of counts to ℓ-line configurations is within the polytope $\mathscr{P}_\ell$. We have outlined an effective computation of $\mathscr{P}_\ell$ and applied it with $\ell = 6$. This procedure employs the strategy of the original proof, but using marked configurations. Determining $\mathscr{P}_\ell$ enables us to determine the maximum and minimum possible counts of each ℓ-line configuration easily, addressing not only avoidance but also simultaneous avoidance. In closing, we remark that every Steiner triple system with at least seven points contains a full configuration on seven or fewer blocks ([2], [12]); this is established using polynomial equalities among configuration counts that generalize the linear equalities explored here. These polynomial equalities establish that for $\ell \geq 7$, the origin of the polytope explored herein is infeasible. Hence it is of interest to explore polynomial bases for the configuration counts.

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Appendix: The 6-line configurations

TABLE 9. The six-line configurations I

81	6	7	abc ade bdf cef cdg beg			$\oplus$					0	Δ
82	6	8	abc ade bdf cdg beh fgh				$\oplus$				0	6Δ
83	6	8	abc ade bdf ceg cfh dgh					$\oplus$			0	2Δ
84	6	8	abc ade bdf cef cdg beh	$\odot$		$\odot$					0	Δ
85	6	8	abc ade bdf cef cdg agh	$\odot$		$\odot$					0	4Δ
86	6	8	abc ade bdf cdg efg beh		$\odot$						0	12Δ
87	6	9	abc ade bdf cdg beh cei	$\odot$		$\odot$					0	3Δ
88	6	9	abc ade bdf cdg beh afi	$\odot$		$\odot$					0	4Δ
89	6	9	abc ade bdf cdg beh cfi	$\odot$	$\odot$	$\odot$					0	24Δ
90	6	9	abc ade bdf cdg beh bgi	$\odot$							0	12Δ
91	6	9	abc ade bdf cdg beh egi	$\odot$	$\odot$						0	24Δ
92	6	9	abc ade bdf cdg beh fgi	$\odot$	$\odot$	$\odot$	$\odot$				0	24Δ
93	6	9	abc ade bdf cdg beh ghi	$\odot$	$\odot$		$\odot$				0	12Δ
94	6	9	abc ade bdf cdg efh egi	$\odot$	$\odot$	$\odot$					0	12Δ
95	6	9	abc ade bdf cdg efh ahi	$\odot$	$\odot$		$\odot$				0	24Δ
96	6	9	abc ade bdf cdg efh chi	$\odot$	$\odot$						0	12Δ
97	6	9	abc ade bdf cdg efh dhi	$\odot$	$\odot$						0	6Δ
98	6	9	abc ade bdf cdg efh ghi	$\odot$	$\odot$		$\odot$				0	12Δ
99	6	9	abc ade bdf ceg cfh efi	$\odot$							0	4Δ
100	6	9	abc ade bdf ceg cfh bgi	$\odot$			$\odot$				0	12Δ
101	6	9	abc ade bdf ceg cfh dgi	$\odot$				$\odot$			0	12Δ
102	6	9	abc ade bdf ceg cfh fgi	$\odot$	$\odot$						0	24Δ
103	6	9	abc ade bdf ceg cfh ghi	$\odot$			$\odot$	$\odot$			0	12Δ
104	6	9	abc ade bdf ceg fgh ahi	$\odot$	$\odot$						0	6Δ
105	6	9	abc ade bdf ceg fgh bhi	$\odot$	$\odot$		$\odot$				0	24Δ
106	6	9	abc ade bdf cgh egi fhi						$\oplus$		0	$2\Delta(v\text{-}7)$
107	6	9	abc ade bfg dfh egi chi							$\oplus$	0	$\frac{1}{3}\Delta(v\text{-}7)$
108	6	9	abc ade bdf cef cdg ahi	$\odot$		$\odot$					0	$6\Delta(v\text{-}7)$
109	6	9	abc ade bdf cef cdg chi	$\odot$							0	$3\Delta(v\text{-}7)$
110	6	9	abc ade bdf cef cdg ghi	$\odot$		$\odot$					0	$\frac{3}{2}\Delta(v\text{-}7)$
111	6	9	abc ade bdf cef agh bgi	$\odot$		$\odot$					0	$12\Delta(v\text{-}7)$
112	6	9	abc ade bdf cef agh fgi	$\odot$							0	$3\Delta(v\text{-}7)$
113	6	9	abc ade bdf cdg efg ahi		$\odot$						0	$6\Delta(v\text{-}9)$
114	6	9	abc ade bdf cdg efg dhi		$\odot$						0	$\Delta(v\text{-}7)$
115	6	10	abc ade bdf cdg efg hij		$\odot$						0	$\frac{1}{3}\Delta(v\text{-}9)(v\text{-}13)$

Table 10. The six-line configurations II

116	6	10	abc ade bdf cdg beh aij	$\odot$	$\odot$	$\odot$					0	$6\Delta(v\text{-}9)$
117	6	10	abc ade bdf cdg beh bij	$\odot$							0	$12\Delta(v\text{-}9)$
118	6	10	abc ade bdf cdg beh cij	$\odot$	$\odot$	$\odot$					0	$12\Delta(v\text{-}9)$
119	6	10	abc ade bdf cdg beh fij	$\odot$	$\odot$	$\odot$	$\odot$				0	$6\Delta(v\text{-}9)$
120	6	10	abc ade bdf cdg beh gij	$\odot$	$\odot$	$\odot$	$\odot$				0	$12\Delta(v\text{-}9)$
121	6	10	abc ade bdf cdg efh aij	$\odot$	$\odot$	$\odot$	$\odot$				0	$12\Delta(v\text{-}10)$
122	6	10	abc ade bdf cdg efh cij	$\odot$	$\odot$						0	$6\Delta(v\text{-}11)$
123	6	10	abc ade bdf cdg efh dij	$\odot$	$\odot$						0	$6\Delta(v\text{-}9)$
124	6	10	abc ade bdf cdg efh eij	$\odot$	$\odot$	$\odot$					0	$12\Delta(v\text{-}11)$
125	6	10	abc ade bdf cdg efh gij	$\odot$	$\odot$	$\odot$	$\odot$				0	$6\Delta(v\text{-}11)$
126	6	10	abc ade bdf cdg efh hij	$\odot$	$\odot$		$\odot$				0	$6\Delta(v\text{-}11)$
127	6	10	abc ade bdf cdg ahi bhj	$\odot$	$\odot$	$\odot$					$12\Delta(v\text{-}11)$	$12\Delta(v\text{-}7)$
128	6	10	abc ade bdf cdg ahi dhj	$\odot$							$12\Delta(v\text{-}9)$	$12\Delta(v\text{-}7)$
129	6	10	abc ade bdf cdg ahi ehj	$\odot$	$\odot$	$\odot$	$\odot$				$12\Delta(v\text{-}13)$	$12\Delta(v\text{-}7)$
130	6	10	abc ade bdf cdg ahi fhj	$\odot$	$\odot$		$\odot$				$24\Delta(v\text{-}12)$	$24\Delta(v\text{-}7)$
131	6	10	abc ade bdf cdg dhi ehj	$\odot$	$\odot$						$12\Delta(v\text{-}11)$	$12\Delta(v\text{-}7)$
132	6	10	abc ade bdf cdg ehi fhj	$\odot$	$\odot$	$\odot$	$\odot$				$12\Delta(v\text{-}14)$	$12\Delta(v\text{-}7)$
133	6	10	abc ade bdf ceg cfh aij	$\odot$	$\odot$		$\odot$				0	$12\Delta(v\text{-}9)$
134	6	10	abc ade bdf ceg cfh cij	$\odot$							0	$6\Delta(v\text{-}9)$
135	6	10	abc ade bdf ceg cfh dij	$\odot$				$\odot$			0	$6\Delta(v\text{-}9)$
136	6	10	abc ade bdf ceg cfh eij	$\odot$	$\odot$						0	$12\Delta(v\text{-}9)$
137	6	10	abc ade bdf ceg cfh gij	$\odot$	$\odot$		$\odot$	$\odot$			0	$12\Delta(v\text{-}9)$
138	6	10	abc ade bdf ceg fgh aij	$\odot$	$\odot$						0	$3\Delta(v\text{-}11)$
139	6	10	abc ade bdf ceg fgh bij	$\odot$	$\odot$		$\odot$				0	$12\Delta(v\text{-}10)$
140	6	10	abc ade bdf ceg fgh fij	$\odot$	$\odot$						0	$6\Delta(v\text{-}11)$
141	6	10	abc ade bdf ceg fgh hij	$\odot$	$\odot$		$\odot$				0	$3\Delta(v\text{-}11)$
142	6	10	abc ade bdf ceg ahi bhj	$\odot$	$\odot$	$\odot$	$\odot$				0	$24\Delta(v\text{-}9)$
143	6	10	abc ade bdf ceg ahi fhj	$\odot$	$\odot$		$\odot$				0	$12\Delta(v\text{-}9)$
144	6	10	abc ade bdf ceg bhi chj	$\odot$	$\odot$	$\odot$	$\odot$				0	$12\Delta(v\text{-}9)$
145	6	10	abc ade bdf ceg bhi dhj	$\odot$	$\odot$	$\odot$					0	$12\Delta(v\text{-}9)$
146	6	10	abc ade bdf ceg bhi ehj	$\odot$				$\odot$			0	$12\Delta(v\text{-}9)$
147	6	10	abc ade bdf ceg bhi fhj	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			0	$24\Delta(v\text{-}9)$
148	6	10	abc ade bdf ceg bhi ghj	$\odot$	$\odot$	$\odot$	$\odot$				0	$24\Delta(v\text{-}9)$
149	6	10	abc ade bdf ceg fhi ghj	$\odot$	$\odot$		$\odot$	$\odot$			0	$6\Delta(v\text{-}9)$

TABLE 11. The six-line configurations III

150	6	10	abc ade bdf agh bgi dgj	$\odot$		$\odot$					$\Delta(v\text{-}9)$	$\Delta(v\text{-}7)$
151	6	10	abc ade bdf agh bgi egj	$\odot$	$\odot$						$12\Delta(v\text{-}10)$	$12\Delta(v\text{-}7)$
152	6	10	abc ade bdf agh bgi ehj	$\odot$	$\odot$	$\odot$					$12\Delta(v\text{-}12)$	$12\Delta(v\text{-}7)$
153	6	10	abc ade bdf agh bgi fhj	$\odot$	$\odot$		$\odot$				$12\Delta(v\text{-}11)$	$12\Delta(v\text{-}7)$
154	6	10	abc ade bdf agh cgi fgj	$\odot$	$\odot$						$24\Delta(v\text{-}11)$	$24\Delta(v\text{-}7)$
155	6	10	abc ade bdf agh cgi ehj	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			$4\Delta(v\text{-}14)$	$4\Delta(v\text{-}7)$
156	6	10	abc ade bdf agh cgi fhj	$\odot$	$\odot$		$\odot$	$\odot$			$24\Delta(v\text{-}13)$	$24\Delta(v\text{-}7)$
157	6	10	abc ade bdf agh cgi fij	$\odot$	$\odot$		$\odot$				$12\Delta(v\text{-}13)$	$12\Delta(v\text{-}7)$
158	6	10	abc ade bdf agh fgi fhj	$\odot$				$\odot$			$3\Delta(v\text{-}11)$	$3\Delta(v\text{-}7)$
159	6	10	abc ade bdf agh fgi aij	$\odot$							$3\Delta(v\text{-}9)$	$3\Delta(v\text{-}7)$
160	6	10	abc ade bdf agh fgi cij	$\odot$	$\odot$		$\odot$				$24\Delta(v\text{-}12)$	$24\Delta(v\text{-}7)$
161	6	10	abc ade bdf agh fgi hij	$\odot$	$\odot$		$\odot$				$12\Delta(v\text{-}13)$	$12\Delta(v\text{-}7)$
162	6	10	abc ade bdf cgh egi fgj	$\odot$	$\odot$	$\odot$					$4\Delta(v\text{-}12)$	$4\Delta(v\text{-}7)$
163	6	10	abc ade bdf cgh egi fhj	$\odot$	$\odot$		$\odot$		$\odot$		0	$12\Delta(v\text{-}7)$
164	6	10	abc ade bdf cgh egi hij	$\odot$	$\odot$		$\odot$	$\odot$	$\odot$		0	$6\Delta(v\text{-}7)$
165	6	10	abc ade bfg dfh egi chj	$\odot$	$\odot$					$\odot$	0	$3\Delta(v\text{-}7)$
166	6	10	abc ade bdf cef cdg hij	$\odot$		$\odot$					0	$\frac{1}{2}\Delta(v\text{-}7)(v\text{-}15)$
167	6	10	abc ade bdf cef agh aij	$\odot$							0	$\frac{3}{4}\Delta(v\text{-}7)(v\text{-}9)$
168	6	10	abc ade bdf cef agh bij	$\odot$		$\odot$					0	$3\Delta(v\text{-}7)(v\text{-}11)$
169	6	10	abc ade bdf cef agh fij	$\odot$							0	$\frac{3}{4}\Delta(v\text{-}7)(v\text{-}11)$
170	6	10	abc ade bdf cef agh gij	$\odot$		$\odot$					0	$3\Delta(v\text{-}7)(v\text{-}13)$
171	6	11	abc ade bdf cdg beh ijk	$\odot$	$\odot$	$\odot$	$\odot$				0	$2\Delta(v^2\text{-}25v+162)$
172	6	11	abc ade bdf cdg efh ijk	$\odot$	$\odot$	$\odot$	$\odot$				0	$2\Delta(v^2\text{-}25v+162)$
173	6	11	abc ade bdf cdg ahi ajk	$\odot$							$\frac{3}{2}\Delta(v\text{-}11)(v\text{-}9)$	$\frac{3}{2}\Delta(v\text{-}7)(v\text{-}9)$
174	6	11	abc ade bdf cdg ahi bjk	$\odot$	$\odot$	$\odot$					$3\Delta(v\text{-}13)(v\text{-}9)$	$3\Delta(v\text{-}7)(v\text{-}11)$
175	6	11	abc ade bdf cdg ahi djk	$\odot$							$3\Delta(v\text{-}11)(v\text{-}9)$	$3\Delta(v\text{-}7)(v\text{-}11)$
176	6	11	abc ade bdf cdg ahi ejk	$\odot$	$\odot$	$\odot$	$\odot$				$3\Delta(v^2\text{-}24v+147)$	$3\Delta(v\text{-}7)(v\text{-}11)$
177	6	11	abc ade bdf cdg ahi fjk	$\odot$	$\odot$	$\odot$	$\odot$				$6\Delta(v^2\text{-}24v+145)$	$6\Delta(v\text{-}7)(v\text{-}11)$
178	6	11	abc ade bdf cdg ahi hjk	$\odot$	$\odot$	$\odot$	$\odot$				$6\Delta(v\text{-}15)(v\text{-}9)$	$6\Delta(v\text{-}15)(v\text{-}7)$
179	6	11	abc ade bdf cdg dhi djk								$\frac{1}{2}\Delta(v\text{-}7)(v\text{-}9)$	$\frac{1}{2}\Delta(v\text{-}7)(v\text{-}9)$
180	6	11	abc ade bdf cdg dhi ejk	$\odot$	$\odot$						$3\Delta(v\text{-}11)^2$	$3\Delta(v\text{-}7)(v\text{-}11)$
181	6	11	abc ade bdf cdg dhi hjk	$\odot$	$\odot$						$2\Delta(v\text{-}15)(v\text{-}7)$	$2\Delta(v^2\text{-}22v+123)$
182	6	11	abc ade bdf cdg ehi ejk	$\odot$	$\odot$	$\odot$					$\frac{3}{2}\Delta(v\text{-}11)(v\text{-}13)$	$\frac{3}{2}\Delta(v\text{-}7)(v\text{-}9)$
183	6	11	abc ade bdf cdg ehi fjk	$\odot$	$\odot$	$\odot$	$\odot$				$3\Delta(v\text{-}13)^2$	$3\Delta(v\text{-}7)(v\text{-}11)$

TABLE 12. The six-line configurations IV

184	6	11	abc ade bdf cdg ehi hjk	$\odot$	$\odot$	$\odot$	$\odot$				$6\Delta(v\text{-}13)^2$	$6\Delta(v\text{-}15)(v\text{-}7)$
185	6	11	abc ade bdf ceg cfh ijk	$\odot$	$\odot$		$\odot$	$\odot$			0	$2\Delta(v^2\text{-}25v+162)$
186	6	11	abc ade bdf ceg fgh ijk	$\odot$	$\odot$		$\odot$				0	$\Delta(v^2\text{-}25v+162)$
187	6	11	abc ade bdf ceg ahi ajk	$\odot$	$\odot$						0	$\frac{3}{4}\Delta(v\text{-}7)(v\text{-}9)$
188	6	11	abc ade bdf ceg ahi bjk	$\odot$	$\odot$	$\odot$	$\odot$				0	$6\Delta(v\text{-}11)(v\text{-}9)$
189	6	11	abc ade bdf ceg ahi fjk	$\odot$	$\odot$		$\odot$				0	$3\Delta(v\text{-}11)(v\text{-}9)$
190	6	11	abc ade bdf ceg ahi hjk	$\odot$	$\odot$	$\odot$	$\odot$				0	$3\Delta(v\text{-}13)(v\text{-}9)$
191	6	11	abc ade bdf ceg bhi bjk	$\odot$							0	$3\Delta(v\text{-}11)(v\text{-}9)$
192	6	11	abc ade bdf ceg bhi cjk	$\odot$	$\odot$	$\odot$	$\odot$				0	$3\Delta(v^2\text{-}22v+125)$
193	6	11	abc ade bdf ceg bhi djk	$\odot$	$\odot$	$\odot$					0	$3\Delta(v^2\text{-}22v+125)$
194	6	11	abc ade bdf ceg bhi ejk	$\odot$				$\odot$			0	$3\Delta(v\text{-}11)^2$
195	6	11	abc ade bdf ceg bhi fjk	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			0	$6\Delta(v\text{-}11)^2$
196	6	11	abc ade bdf ceg bhi gjk	$\odot$	$\odot$	$\odot$	$\odot$				0	$6\Delta(v\text{-}13)(v\text{-}9)$
197	6	11	abc ade bdf ceg bhi hjk	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			0	$12\Delta(v^2\text{-}24v+149)$
198	6	11	abc ade bdf ceg fhi fjk	$\odot$	$\odot$						0	$\frac{3}{2}\Delta(v\text{-}11)(v\text{-}9)$
199	6	11	abc ade bdf ceg fhi gjk	$\odot$	$\odot$		$\odot$	$\odot$			0	$\frac{3}{2}\Delta(v^2\text{-}22v+125)$
200	6	11	abc ade bdf ceg fhi hjk	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			0	$6\Delta(v^2\text{-}24v+139)$
201	6	11	abc ade bdf agh bgi ajk	$\odot$							$6\Delta(v\text{-}10)(v\text{-}9)$	$6\Delta(v\text{-}7)(v\text{-}11)$
202	6	11	abc ade bdf agh bgi cjk	$\odot$	$\odot$	$\odot$	$\odot$				$3\Delta(v\text{-}9)(v\text{-}14)$	$3\Delta(v\text{-}7)(v\text{-}11)$
203	6	11	abc ade bdf agh bgi djk	$\odot$	$\odot$	$\odot$					$6\Delta(v\text{-}12)(v\text{-}9)$	$6\Delta(v\text{-}7)(v\text{-}13)$
204	6	11	abc ade bdf agh bgi ejk	$\odot$	$\odot$	$\odot$	$\odot$				$12\Delta(v\text{-}9)(v\text{-}14)$	$12\Delta(v\text{-}7)(v\text{-}13)$
205	6	11	abc ade bdf agh cgi ajk	$\odot$	$\odot$						$6\Delta(v^2\text{-}21v+112)$	$6\Delta(v\text{-}7)(v\text{-}11)$
206	6	11	abc ade bdf agh cgi bjk	$\odot$	$\odot$	$\odot$	$\odot$				$12\Delta(v^2\text{-}23v+134)$	$12\Delta(v\text{-}7)(v\text{-}11)$
207	6	11	abc ade bdf agh cgi djk	$\odot$	$\odot$	$\odot$					$12\Delta(v^2\text{-}23v+134)$	$12\Delta(v\text{-}7)(v\text{-}13)$
208	6	11	abc ade bdf agh cgi ejk	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			$12\Delta(v^2\text{-}25v+158)$	$12\Delta(v\text{-}7)(v\text{-}13)$
209	6	11	abc ade bdf agh cgi fjk	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			$12\Delta(v^2\text{-}25v+157)$	$12\Delta(v\text{-}7)(v\text{-}13)$
210	6	11	abc ade bdf agh fgi ajk	$\odot$							$6\Delta(v\text{-}11)(v\text{-}9)$	$6\Delta(v\text{-}7)(v\text{-}11)$
211	6	11	abc ade bdf agh fgi bjk	$\odot$	$\odot$		$\odot$				$12\Delta(v\text{-}13)(v\text{-}9)$	$12\Delta(v\text{-}7)(v\text{-}13)$
212	6	11	abc ade bdf agh fgi cjk	$\odot$	$\odot$		$\odot$	$\odot$			$12\Delta(v\text{-}15)(v\text{-}9)$	$12\Delta(v\text{-}7)(v\text{-}13)$
213	6	11	abc ade bdf agh fgi fjk	$\odot$				$\odot$			$6\Delta(v\text{-}13)(v\text{-}9)$	$6\Delta(v\text{-}7)(v\text{-}11)$
214	6	11	abc ade bdf agh fgi gjk	$\odot$	$\odot$						$6\Delta(v\text{-}13)(v\text{-}9)$	$6\Delta(v\text{-}7)(v\text{-}13)$
215	6	11	abc ade bdf agh fgi hjk	$\odot$	$\odot$		$\odot$	$\odot$			$6\Delta(v\text{-}15)(v\text{-}9)$	$6\Delta(v\text{-}15)(v\text{-}7)$
216	6	11	abc ade bdf agh fgi ijk	$\odot$	$\odot$		$\odot$				$6\Delta(v\text{-}15)(v\text{-}9)$	$6\Delta(v\text{-}15)(v\text{-}7)$
217	6	11	abc ade bdf agh aij gik	$\odot$	$\odot$						$3\Delta(v\text{-}7)(v\text{-}14)$	$3\Delta(v^2\text{-}21v+112)$

TABLE 13. The six-line configurations V

218	6	11	abc ade bdf agh bij gik	$\odot$	$\odot$	$\odot$	$\odot$				$12\Delta(v\text{-}13)(v\text{-}9)$	$12\Delta(v\text{-}7)(v\text{-}14)$
219	6	11	abc ade bdf agh cij eik	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			$6\Delta(v^2\text{-}24v+145)$	$6\Delta(v\text{-}7)(v\text{-}13)$
220	6	11	abc ade bdf agh cij fik	$\odot$	$\odot$		$\odot$				$12\Delta(v^2\text{-}24v+145)$	$12\Delta(v\text{-}7)(v\text{-}13)$
221	6	11	abc ade bdf agh cij gik	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			$24\Delta(v\text{-}15)(v\text{-}9)$	$24\Delta(v\text{-}7)(v\text{-}14)$
222	6	11	abc ade bdf agh fij gik	$\odot$	$\odot$	$\odot$	$\odot$				$12\Delta(v\text{-}15)(v\text{-}9)$	$12\Delta(v\text{-}15)(v\text{-}7)$
223	6	11	abc ade bdf agh gij hik	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			$6\Delta(v\text{-}16)(v\text{-}9)$	$6\Delta(v\text{-}7)(v\text{-}14)$
224	6	11	abc ade bdf cgh egi cjk	$\odot$	$\odot$	$\odot$	$\odot$				$12\Delta(v^2\text{-}24v+145)$	$12\Delta(v\text{-}7)(v\text{-}11)$
225	6	11	abc ade bdf cgh egi fjk	$\odot$	$\odot$	$\odot$	$\odot$		$\odot$		$6\Delta(v\text{-}13)^2$	$6\Delta(v\text{-}7)(v\text{-}11)$
226	6	11	abc ade bdf cgh egi gjk	$\odot$	$\odot$	$\odot$					$6\Delta(v^2\text{-}24v+145)$	$6\Delta(v\text{-}7)(v\text{-}13)$
227	6	11	abc ade bdf cgh egi hjk	$\odot$	$\odot$		$\odot$	$\odot$	$\odot$		$12\Delta(v\text{-}13)^2$	$12\Delta(v\text{-}7)(v\text{-}13)$
228	6	11	abc ade bdf cgh eij gik	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$		$12\Delta(v\text{-}17)(v\text{-}9)$	$12\Delta(v\text{-}7)(v\text{-}14)$
229	6	11	abc ade bdf cgh gij hik	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$		$3\Delta(v\text{-}18)(v\text{-}9)$	$3\Delta(v\text{-}7)(v\text{-}14)$
230	6	11	abc ade afg bhi dhj fhk	$\odot$				$\odot$			$2\Delta(v\text{-}12)(v\text{-}9)$	$2\Delta(v\text{-}7)(v\text{-}11)$
231	6	11	abc ade afg bhi dhj eik	$\odot$	$\odot$		$\odot$				$6\Delta(v^2\text{-}23v+134)$	$6\Delta(v\text{-}7)(v\text{-}13)$
232	6	11	abc ade afg bhi dhj fik	$\odot$	$\odot$	$\odot$	$\odot$				$12\Delta(v\text{-}9)(v\text{-}14)$	$12\Delta(v\text{-}7)(v\text{-}14)$
233	6	11	abc ade bfg dfh egi cjk	$\odot$	$\odot$		$\odot$			$\odot$	$3\Delta(v^2\text{-}25v+160)$	$3\Delta(v\text{-}7)(v\text{-}13)$
234	6	11	abc ade bfg dfh cij eik	$\odot$	$\odot$		$\odot$		$\odot$	$\odot$	$6\Delta(v^2\text{-}25v+143)$	$6\Delta(v\text{-}7)(v\text{-}14)$
235	6	11	abc ade bfg dfh cij hik	$\odot$	$\odot$		$\odot$	$\odot$	$\odot$		$6\Delta(v\text{-}16)(v\text{-}9)$	$6\Delta(v\text{-}15)(v\text{-}7)$
236	6	11	abc ade bdf cef agh ijk	$\odot$		$\odot$					0	$\frac{1}{2}\Delta(v\text{-}7)(v^2\text{-}25v+162)$
237	6	11	abc ade bdf cef ghi gjk	$\odot$		$\odot$					0	$\frac{1}{8}\Delta(v\text{-}7)(v^2\text{-}27v+186)$
238	6	12	abc ade bdf cdg ahi jkl	$\odot$	$\odot$	$\odot$	$\odot$				$\Delta(v\text{-}9)(v^2\text{-}28v+207)$	$\Delta(v\text{-}13)(v\text{-}15)(v\text{-}7)$
239	6	12	abc ade bdf cdg dhi jkl	$\odot$	$\odot$						$\frac{1}{3}\Delta(v\text{-}13)(v\text{-}15)(v\text{-}7)$	$\frac{1}{3}\Delta(v\text{-}13)(v^2\text{-}22v+123)$
240	6	12	abc ade bdf cdg ehi jkl	$\odot$	$\odot$	$\odot$	$\odot$				$\Delta(v^3\text{-}39v^2+521v\text{-}2403)$	$\Delta(v\text{-}13)(v\text{-}15)(v\text{-}7)$
241	6	12	abc ade bdf cdg hij hkl	$\odot$	$\odot$	$\odot$	$\odot$				$\frac{1}{2}\Delta(v\text{-}15)(v\text{-}17)(v\text{-}7)$	$\frac{1}{2}\Delta(v\text{-}13)(v^2\text{-}26v+177)$
242	6	12	abc ade bdf ceg ahi jkl	$\odot$	$\odot$	$\odot$	$\odot$				0	$\frac{1}{2}\Delta(v\text{-}9)(v\text{-}13)^2$
243	6	12	abc ade bdf ceg bhi jkl	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			0	$2\Delta(v^3\text{-}37v^2+465\text{v}$ $\text{-}1983)$
244	6	12	abc ade bdf ceg fhi jkl	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			0	$\Delta(v\text{-}9)(v^2\text{-}28v+207)$
245	6	12	abc ade bdf ceg hij hkl	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			0	$\frac{3}{4}\Delta(v^3\text{-}39v^2+515v$ $\text{-}2293)$
246	6	12	abc ade bdf agh bgi jkl	$\odot$	$\odot$	$\odot$	$\odot$				$\Delta(v\text{-}14)(v^2\text{-}22v+129)$	$\Delta(v\text{-}7)(v^2\text{-}28v+201)$
247	6	12	abc ade bdf agh cgi jkl	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			$2\Delta(v^3\text{-}38v^2+493v$ $\text{-}2208)$	$2\Delta(v\text{-}7)(v^2\text{-}28v+201)$
248	6	12	abc ade bdf agh fgi jkl	$\odot$	$\odot$		$\odot$	$\odot$			$2\Delta(v^3\text{-}37v^2+465v$ $\text{-}2007)$	$2\Delta(v\text{-}7)(v^2\text{-}28v + 207)$
249	6	12	abc ade bdf agh aij akl								$\frac{1}{4}\Delta(v\text{-}7)(v\text{-}11)(v\text{-}9)$	$\frac{1}{4}\Delta(v\text{-}7)(v\text{-}11)(v\text{-}9)$
250	6	12	abc ade bdf agh aij bkl	$\odot$							$\frac{3}{2}\Delta(v\text{-}7)(v\text{-}11)(v\text{-}13)$	$\frac{3}{2}\Delta(v\text{-}9)(v\text{-}11)^2$
251	6	12	abc ade bdf agh aij ckl	$\odot$	$\odot$						$\frac{3}{2}\Delta(v\text{-}11)(v\text{-}13)(v\text{-}9)$	$\frac{3}{2}\Delta(v\text{-}7)(v\text{-}11)(v\text{-}13)$
252	6	12	abc ade bdf agh aij fkl	$\odot$							$\frac{3}{4}\Delta(v\text{-}11)(v\text{-}13)(v\text{-}9)$	$\frac{3}{4}\Delta(v\text{-}7)(v\text{-}11)(v\text{-}13)$

TABLE 14. The six-line configurations VI

253	6	12	abc ade bdf agh aij gkl	⊙	⊙						$3\Delta(v\text{-}7)(v^2\text{-}26v+173)$	$3\Delta(v^3\text{-}33v^2+365v\text{-}1357)$
254	6	12	abc ade bdf agh bij ckl	⊙	⊙	⊙	⊙				$\frac{3}{2}\Delta(v\text{-}9)(v^2\text{-}26v+177)$	$\frac{3}{2}\Delta(v\text{-}11)(v\text{-}15)(v\text{-}7)$
255	6	12	abc ade bdf agh bij dkl	⊙	⊙	⊙					$\frac{1}{2}\Delta(v\text{-}7)(v^2\text{-}26v+173)$	$\frac{1}{2}\Delta(v\text{-}9)(v^2\text{-}24v+151)$
256	6	12	abc ade bdf agh bij ekl	⊙	⊙	⊙	⊙				$3\Delta(v\text{-}9)(v^2\text{-}26v+177)$	$3\Delta(v\text{-}7)(v^2\text{-}26v+173)$
257	6	12	abc ade bdf agh bij gkl	⊙	⊙	⊙	⊙				$6\Delta(v\text{-}7)(v^2\text{-}28v+199)$	$6\Delta(v\text{-}13)(v^2\text{-}22v+127)$
258	6	12	abc ade bdf agh cij ckl	⊙	⊙	⊙	⊙				$\frac{3}{2}\Delta(v\text{-}9)(v^2\text{-}26v+173)$	$\frac{3}{2}\Delta(v\text{-}7)(v\text{-}11)(v\text{-}13)$
259	6	12	abc ade bdf agh cij ekl	⊙	⊙	⊙	⊙	⊙			$\frac{3}{2}\Delta(v^3\text{-}37v^2+455v\text{-}1843)$	$\frac{3}{2}\Delta(v\text{-}7)(v^2\text{-}26v+173)$
260	6	12	abc ade bdf agh cij fkl	⊙	⊙	⊙	⊙	⊙			$3\Delta(v^3\text{-}37v^2+455v\text{-}1835)$	$3\Delta(v\text{-}7)(v^2\text{-}26v+173)$
261	6	12	abc ade bdf agh cij gkl	⊙	⊙	⊙	⊙	⊙			$6\Delta(v^3\text{-}37v^2+463v\text{-}1907)$	$6\Delta(v\text{-}7)(v^2\text{-}28v+199)$
262	6	12	abc ade bdf agh cij ikl	⊙	⊙	⊙	⊙	⊙			$6\Delta(v^3\text{-}37v^2+455v\text{-}1831)$	$6\Delta(v\text{-}7)(v^2\text{-}28v+199)$
263	6	12	abc ade bdf agh fij fkl	⊙				⊙			$\frac{3}{4}\Delta(v\text{-}13)(v\text{-}11)^2$	$\frac{3}{4}\Delta(v\text{-}7)(v\text{-}11)(v\text{-}13)$
264	6	12	abc ade bdf agh fij gkl	⊙	⊙	⊙	⊙	⊙			$3\Delta(v^3\text{-}37v^2+463v\text{-}1907)$	$3\Delta(v\text{-}7)(v^2\text{-}28v+203)$
265	6	12	abc ade bdf agh fij ikl	⊙	⊙	⊙	⊙				$3\Delta(v\text{-}9)(v^2\text{-}28v+203)$	$3\Delta(v\text{-}7)(v^2\text{-}28v+203)$
266	6	12	abc ade bdf agh gij gkl	⊙	⊙	⊙					$\frac{3}{2}\Delta(v\text{-}13)(v\text{-}15)(v\text{-}7)$	$\frac{3}{2}\Delta(v\text{-}13)(v^2\text{-}22v+125)$
267	6	12	abc ade bdf agh gij hkl	⊙	⊙	⊙	⊙	⊙			$\frac{3}{2}\Delta(v\text{-}7)(v\text{-}15)^2$	$\frac{3}{2}\Delta(v^3\text{-}37v^2+471v\text{-}2059)$
268	6	12	abc ade bdf agh gij ikl	⊙	⊙	⊙	⊙	⊙			$6\Delta(v\text{-}7)(v^2\text{-}30v+229)$	$6\Delta(v^3\text{-}37v^2+463v\text{-}1967)$
269	6	12	abc ade bdf cgh egi jkl	⊙	⊙	⊙	⊙	⊙	⊙		$2\Delta(v^3\text{-}39v^2+515v\text{-}2319)$	$2\Delta(v\text{-}7)(v^2\text{-}28v+207)$
270	6	12	abc ade bdf cgh cij ckl	⊙							$\frac{1}{4}\Delta(v\text{-}11)(v\text{-}13)(v\text{-}9)$	$\frac{1}{4}\Delta(v\text{-}7)(v\text{-}11)(v\text{-}9)$
271	6	12	abc ade bdf cgh cij ekl	⊙	⊙	⊙	⊙				$\frac{3}{2}\Delta(v\text{-}13)(v\text{-}15)(v\text{-}9)$	$\frac{3}{2}\Delta(v\text{-}7)(v\text{-}11)(v\text{-}13)$
272	6	12	abc ade bdf cgh cij gkl	⊙	⊙	⊙	⊙	⊙			$3\Delta(v^3\text{-}37v^2+459v\text{-}1871)$	$3\Delta(v\text{-}7)(v^2\text{-}26v+173)$
273	6	12	abc ade bdf cgh eij fkl	⊙	⊙	⊙	⊙		⊙		$\frac{1}{2}\Delta(v\text{-}13)(v\text{-}17)(v\text{-}9)$	$\frac{1}{2}\Delta(v\text{-}7)(v^2\text{-}26v+173)$
274	6	12	abc ade bdf cgh eij gkl	⊙	⊙	⊙	⊙	⊙	⊙		$6\Delta(v^3\text{-}39v^2+503v\text{-}2109)$	$6\Delta(v\text{-}7)(v^2\text{-}28v+203)$
275	6	12	abc ade bdf cgh gij gkl	⊙	⊙	⊙					$\frac{3}{2}\Delta(v\text{-}13)(v\text{-}15)(v\text{-}9)$	$\frac{3}{2}\Delta(v\text{-}13)(v\text{-}15)(v\text{-}7)$
276	6	12	abc ade bdf cgh gij hkl	⊙	⊙	⊙	⊙	⊙	⊙		$\frac{3}{2}\Delta(v^3\text{-}39v^2+503v\text{-}2105)$	$\frac{3}{2}\Delta(v\text{-}7)(v^2\text{-}30v+233)$
277	6	12	abc ade bdf cgh gij ikl	⊙	⊙	⊙	⊙	⊙	⊙		$6\Delta(v^3\text{-}39v^2+509v\text{-}2163)$	$6\Delta(v\text{-}7)(v^2\text{-}30v+233)$
278	6	12	abc ade bdf ghi gjk hjl	⊙	⊙	⊙	⊙	⊙	⊙		$\frac{1}{3}\Delta(v\text{-}7)(v\text{-}15)(v\text{-}18)$	$\frac{1}{3}\Delta(v\text{-}9)(v^2\text{-}31v+264)$
279	6	12	abc ade afg ahi bjk djl	⊙							$\frac{3}{2}\Delta(v\text{-}9)(v\text{-}11)(v\text{-}12)$	$\frac{3}{2}\Delta(v\text{-}7)(v\text{-}11)(v\text{-}13)$
280	6	12	abc ade afg bhi dhj bkl	⊙	⊙	⊙	⊙				$3\Delta(v\text{-}13)(v^2\text{-}21v+116)$	$3\Delta(v\text{-}7)(v^2\text{-}26v+177)$
281	6	12	abc ade afg bhi dhj ckl	⊙	⊙	⊙	⊙	⊙			$6\Delta(v^3\text{-}36v^2+437v\text{-}1754)$	$6\Delta(v\text{-}7)(v^2\text{-}28v+199)$
282	6	12	abc ade afg bhi dhj fkl	⊙	⊙	⊙	⊙	⊙			$6\Delta(v^3\text{-}36v^2+439v\text{-}1832)$	$6\Delta(v\text{-}7)(v^2\text{-}28v+205)$
283	6	12	abc ade afg bhi dhj hkl	⊙				⊙			$\frac{3}{2}\Delta(v\text{-}12)(v\text{-}11)^2$	$\frac{3}{2}\Delta(v\text{-}11)(v\text{-}15)(v\text{-}7)$
284	6	12	abc ade afg bhi dhj ikl	⊙	⊙	⊙	⊙				$6\Delta(v\text{-}9)(v^2\text{-}27v+190)$	$6\Delta(v\text{-}7)(v^2\text{-}28v+199)$
285	6	12	abc ade afg bhi cjk hjl	⊙	⊙		⊙				$\frac{3}{2}\Delta(v\text{-}9)(v^2\text{-}27v+184)$	$\frac{3}{2}\Delta(v\text{-}13)(v\text{-}15)(v\text{-}7)$

TABLE 15. The six-line configurations VII

286	6	12	abc ade afg bhi djk hjl	$\odot$	$\odot$	$\odot$	$\odot$				$6\Delta(v\text{-}7)(v^2\text{-}29v+212)$	$6\Delta(v\text{-}13)(v^2\text{-}23v+139)$
287	6	12	abc ade bfg dfh egi jkl	$\odot$	$\odot$		$\odot$			$\odot$	$\frac{1}{3}\Delta(v\text{-}15)(v^2\text{-}23v+142)$	$\frac{1}{3}\Delta(v\text{-}7)(v^2\text{-}28v+213)$
288	6	12	abc ade bfg dfh cij ekl	$\odot$	$\odot$		$\odot$	$\odot$	$\odot$	$\odot$	$3\Delta(v\text{-}9)(v^2\text{-}29v+216)$	$3\Delta(v\text{-}7)(v^2\text{-}30v+233)$
289	6	12	abc ade bfg dfh cij hkl	$\odot$	$\odot$		$\odot$	$\odot$	$\odot$		$\frac{3}{2}\Delta(v^3\text{-}38v^2+477v\text{-}1952)$	$\frac{3}{2}\Delta(v\text{-}7)(v^2\text{-}30v+237)$
290	6	12	abc ade bfg dfh cij ikl	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$6\Delta(v\text{-}9)(v^2\text{-}29v+220)$	$6\Delta(v\text{-}7)(v^2\text{-}30v+235)$
291	6	12	abc ade bfg chi djk fjl	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$		$6\Delta(v\text{-}7)(v^2\text{-}31v+242)$	$6\Delta(v^3\text{-}38v^2+496v\text{-}2217)$
292	6	12	abc ade bfg dhi fjk hjl	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$2\Delta(v\text{-}7)(v^2\text{-}31v+247)$	$2\Delta(v^3\text{-}38v^2+492v\text{-}2078)$
293	6	12	abc ade bdf cef ghi jkl	$\odot$		$\odot$					0	$\frac{1}{72}\Delta(v\text{-}15)(v\text{-}7)(v^2\text{-}25v+174)$
294	6	13	abc ade bdf cdg hij klm	$\odot$	$\odot$	$\odot$	$\odot$				$\frac{1}{18}\Delta(v\text{-}15)(v\text{-}7)(v^2\text{-}31v+252)$	$\frac{1}{18}\Delta(v^4\text{-}53v^3+1075v^2\text{-}9915v+35316)$
295	6	13	abc ade bdf ceg hij klm	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			0	$\frac{1}{12}\Delta(v^4\text{-}53v^3+1075v^2\text{-}9915v+35316)$
296	6	13	abc ade bdf agh aij klm	$\odot$	$\odot$						$\frac{1}{4}\Delta(v\text{-}7)(v\text{-}13)(v^2\text{-}27v+192)$	$\frac{1}{4}\Delta(v\text{-}13)(v\text{-}9)(v^2\text{-}25v+162)$
297	6	13	abc ade bdf agh bij klm	$\odot$	$\odot$	$\odot$	$\odot$				$\frac{1}{2}\Delta(v\text{-}7)(v^3\text{-}42v^2+605v\text{-}2976)$	$\frac{1}{2}\Delta(v\text{-}9)(v^3\text{-}40v^2+549v\text{-}2622)$
298	6	13	abc ade bdf agh cij klm	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			$\Delta(v\text{-}9)(v^3\text{-}42v^2+611v\text{-}3114)$	$\Delta(v\text{-}7)(v^3\text{-}42v^2+605v\text{-}2976)$
299	6	13	abc ade bdf agh fij klm	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			$\frac{1}{2}\Delta(v^4\text{-}51v^3+989v^2\text{-}8661v+28962)$	$\frac{1}{2}\Delta(v\text{-}15)(v\text{-}7)(v^2\text{-}27v+200)$
300	6	13	abc ade bdf agh gij klm	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			$\Delta(v\text{-}15)(v\text{-}7)(v^2\text{-}29v+232)$	$\Delta(v^4\text{-}51v^3+991v^2\text{-}8723v+29574)$
301	6	13	abc ade bdf agh ijk ilm	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			$\frac{3}{4}\Delta(v\text{-}7)(v^3\text{-}44v^2+663v\text{-}3412)$	$\frac{3}{4}\Delta(v^4\text{-}51v^3+983v^2\text{-}8469v+27232)$
302	6	13	abc ade bdf cgh cij klm	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			$\frac{1}{4}\Delta(v\text{-}13)(v\text{-}14)(v^2\text{-}24v+159)$	$\frac{1}{4}\Delta(v\text{-}7)(v\text{-}13)(v^2\text{-}27v+192)$
303	6	13	abc ade bdf cgh eij klm	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$		$\frac{1}{2}\Delta(v\text{-}9)(v^3\text{-}44v^2+673v\text{-}3606)$	$\frac{1}{2}\Delta(v\text{-}7)(v^3\text{-}42v^2+605v\text{-}2976)$
304	6	13	abc ade bdf cgh gij klm	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$		$\Delta(v\text{-}9)(v^3\text{-}44v^2+673v\text{-}3642)$	$\Delta(v\text{-}15)(v\text{-}7)(v^2\text{-}29v+232)$
305	6	13	abc ade bdf cgh ijk ilm	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$		$\frac{3}{4}\Delta(v^4\text{-}53v^3+1071v^2\text{-}9819v)+25872$	$\frac{3}{4}\Delta(v\text{-}7)(v^3\text{-}44v^2+663v\text{-}3412)$
306	6	13	abc ade bdf ghi gjk glm	$\odot$	$\odot$	$\odot$					$\frac{1}{12}\Delta(v\text{-}7)(v^3\text{-}44v^2+651v\text{-}3216)$	$\frac{1}{12}\Delta(v\text{-}13)(v\text{-}9)(v^2\text{-}29v+222)$
307	6	13	abc ade bdf ghi gjk hlm	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$		$\frac{1}{2}\Delta(v\text{-}7)(v\text{-}16)(v^2\text{-}30v+249)$	$\frac{1}{2}\Delta(v^4\text{-}53v^3+1069v^2\text{-}9699v+32850)$
308	6	13	abc ade afg ahi ajk alm								$\frac{1}{1920}\Delta(v\text{-}11)(v\text{-}5)(v\text{-}7)(v\text{-}9)$	$\frac{1}{1920}\Delta(v\text{-}11)(v\text{-}5)(v\text{-}7)(v\text{-}9)$
309	6	13	abc ade afg ahi ajk blm								$\frac{1}{32}\Delta(v\text{-}7)(v\text{-}11)(v\text{-}13)(v\text{-}9)$	$\frac{1}{32}\Delta(v\text{-}7)(v\text{-}11)(v\text{-}13)(v\text{-}9)$
310	6	13	abc ade afg ahi bjk blm	$\odot$							$\frac{1}{16}\Delta(v\text{-}13)(v\text{-}9)(v\text{-}11)^2$	$\frac{1}{16}\Delta(v\text{-}7)(v\text{-}11)(v^2\text{-}26v+177)$
311	6	13	abc ade afg ahi bjk clm	$\odot$	$\odot$						$\frac{1}{16}\Delta(v\text{-}11)(v\text{-}13)(v\text{-}15)(v\text{-}7)$	$\frac{1}{16}\Delta(v\text{-}11)(v\text{-}9)(v^2\text{-}26v+181)$
312	6	13	abc ade afg ahi bjk dlm	$\odot$	$\odot$						$\frac{3}{8}\Delta(v\text{-}7)(v\text{-}13)(v^2\text{-}26v+181)$	$\frac{3}{8}\Delta(v\text{-}13)(v\text{-}9)(v^2\text{-}24v+151)$
313	6	13	abc ade afg ahi bjk jlm	$\odot$							$\frac{1}{4}\Delta(v\text{-}11)(v\text{-}13)(v\text{-}15)(v\text{-}7)$	$\frac{1}{4}\Delta(v\text{-}11)(v\text{-}9)(v\text{-}13)^2$
314	6	13	abc ade afg bhi dhj klm	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			$\Delta(v^4\text{-}50v^3+949v^2\text{-}8124v+26556)$	$\Delta(v\text{-}7)(v^3\text{-}42v^2+611v\text{-}3078)$
315	6	13	abc ade afg bhi bjk clm	$\odot$	$\odot$	$\odot$	$\odot$				$\frac{3}{32}\Delta(v\text{-}7)(v\text{-}11)(v^2\text{-}30v+229)$	$\frac{3}{32}\Delta(v\text{-}9)(v^3\text{-}39v^2+527v\text{-}2473)$
316	6	13	abc ade afg bhi bjk dlm	$\odot$	$\odot$	$\odot$	$\odot$				$\frac{3}{4}\Delta(v\text{-}7)(v^3\text{-}41v^2+575v\text{-}2759)$	$\frac{3}{4}\Delta(v\text{-}15)(v\text{-}9)(v^2\text{-}24v+159)$

Table 16. The six-line configurations VIII

317	6	13	abc ade afg bhi cjk dlm	⊙	⊙	⊙	⊙	⊙			$\frac{3}{4}\Delta(v\text{-}7)(v^3\text{-}43v^2+631v\text{-}3149)$	$\frac{3}{4}\Delta(v^4\text{-}50v^3+956v^2\text{-}8262v+26875)$
318	6	13	abc ade afg bhi cjk hlm	⊙	⊙	⊙	⊙	⊙			$\frac{3}{4}\Delta(v\text{-}15)(v\text{-}7)(v^2\text{-}28v+203)$	$\frac{3}{4}\Delta(v^4\text{-}50v^3+952v^2\text{-}8126v+25943)$
319	6	13	abc ade afg bhi djk flm	⊙	⊙	⊙	⊙	⊙			$\frac{1}{2}\Delta(v\text{-}7)(v^3\text{-}43v^2+639v\text{-}3277)$	$\frac{1}{2}\Delta(v^4\text{-}50v^3+952v^2\text{-}8182v+26447)$
320	6	13	abc ade afg bhi djk hlm	⊙	⊙	⊙	⊙	⊙			$3\Delta(v\text{-}7)(v^3\text{-}43v^2+635v\text{-}3217)$	$3\Delta(v^4\text{-}50v^3+948v^2\text{-}8070v+25755)$
321	6	13	abc ade afg bhi hjk hlm	⊙				⊙			$\frac{3}{16}\Delta(v\text{-}7)(v\text{-}11)(v^2\text{-}30v+229)$	$\frac{3}{16}\Delta(v\text{-}11)^2(v\text{-}13)^2$
322	6	13	abc ade afg bhi hjk ilm	⊙	⊙	⊙	⊙				$\frac{3}{8}\Delta(v\text{-}15)(v\text{-}7)(v^2\text{-}28v+203)$	$\frac{3}{8}\Delta(v\text{-}9)(v^3\text{-}41v^2+579v\text{-}2819)$
323	6	13	abc ade afg bhi hjk jlm	⊙	⊙	⊙	⊙				$\frac{3}{2}\Delta(v\text{-}7)(v^3\text{-}43v^2+631v\text{-}3149)$	$\frac{3}{2}\Delta(v\text{-}15)(v\text{-}9)(v^2\text{-}26v+185)$
324	6	13	abc ade bfg dfh cij klm	⊙	⊙	⊙	⊙	⊙	⊙	⊙	$\Delta(v^4\text{-}52v^3+1029v^2\text{-}9174v+30552)$	$\Delta(v\text{-}7)(v^3\text{-}44v^2+673v\text{-}3558)$
325	6	13	abc ade bfg dfh ijk ilm	⊙	⊙	⊙	⊙	⊙	⊙	⊙	$\frac{3}{8}\Delta(v^4\text{-}52v^3+1027v^2\text{-}9168v)+3917$	$\frac{3}{8}\Delta(v\text{-}7)(v^3\text{-}44v^2+675v\text{-}3584)$
326	6	13	abc ade bfg chi djk elm	⊙	⊙	⊙	⊙	⊙	⊙	⊙	$\frac{3}{16}\Delta(v\text{-}7)(v^3\text{-}45v^2+691v\text{-}3647)$	$\frac{3}{16}\Delta(v^4\text{-}52v^3+1038v^2\text{-}9300v+31001)$
327	6	13	abc ade bfg chi djk flm	⊙	⊙	⊙	⊙	⊙	⊙	⊙	$\frac{3}{2}\Delta(v\text{-}7)(v^3\text{-}45v^2+695v\text{-}3699)$	$\frac{3}{2}\Delta(v^4\text{-}52v^3+1034v^2\text{-}9236v+30741)$
328	6	13	abc ade bfg chi djk jlm	⊙	⊙	⊙	⊙	⊙	⊙		$\frac{3}{2}\Delta(v\text{-}7)(v^3\text{-}45v^2+699v\text{-}3751)$	$\frac{3}{2}\Delta(v^4\text{-}52v^3+1034v^2\text{-}9260v+30997)$
329	6	13	abc ade bfg dhi fhj klm	⊙	⊙	⊙	⊙	⊙	⊙		$\frac{2}{5}\Delta(v\text{-}16)(v\text{-}7)(v^2\text{-}29v+240)$	$\frac{2}{5}\Delta(v^2\text{-}23v+147)(v^2\text{-}29v+220)$
330	6	13	abc ade bfg dhi fjk hlm	⊙	⊙	⊙	⊙	⊙	⊙	⊙	$3\Delta(v\text{-}7)(v^3\text{-}45v^2+703v\text{-}3811)$	$3\Delta(v^4\text{-}52v^3+1030v^2\text{-}9180v+30577)$
331	6	14	abc ade bdf agh ijk lmn	⊙	⊙	⊙	⊙	⊙			$\frac{1}{12}\Delta(v\text{-}15)(v\text{-}7)(v^3\text{-}44v^2+679v\text{-}3732)$	$\frac{1}{12}\Delta(v\text{-}9)(v^4\text{-}57v^3+1251v^2\text{-}12619v+50016)$
332	6	14	abc ade bdf cgh ijk lmn	⊙	⊙	⊙	⊙	⊙	⊙		$\frac{1}{12}\Delta(v^5\text{-}68v^4+1882v^3\text{-}26556v^2+191493v\text{-}567216)$	$\frac{1}{12}\Delta(v\text{-}7)(v^4\text{-}59v^3+1339v^2\text{-}13917v+56124)$
333	6	14	abc ade bdf ghi gjk lmn	⊙	⊙	⊙	⊙	⊙	⊙		$\frac{1}{12}\Delta(v\text{-}15)(v\text{-}7)(v^3\text{-}46v^2+741v\text{-}4272)$	$\frac{1}{12}\Delta(v\text{-}9)(v^4\text{-}59v^3+1345v^2\text{-}14157v+58950)$
334	6	14	abc ade afg ahi ajk lmn								$\frac{1}{960}\Delta(v\text{-}11)(v\text{-}13)(v\text{-}15)(v\text{-}7)(v\text{-}9)$	$\frac{1}{960}\Delta(v\text{-}11)(v\text{-}13)(v\text{-}15)(v\text{-}7)(v\text{-}9)$
335	6	14	abc ade afg ahi bjk lmn	⊙	⊙						$\frac{1}{24}\Delta(v\text{-}9)(v\text{-}13)(v\text{-}15)(v^2\text{-}24v+155)$	$\frac{1}{24}\Delta(v\text{-}13)(v\text{-}15)(v\text{-}7)(v^2\text{-}26v+189)$
336	6	14	abc ade afg ahi jkl jmn	⊙							$\frac{1}{128}\Delta(v\text{-}11)(v\text{-}15)(v\text{-}9)(v\text{-}13)^2$	$\frac{1}{128}\Delta(v\text{-}7)(v\text{-}11)(v\text{-}13)(v^2\text{-}30v+237)$
337	6	14	abc ade afg bhi bjk lmn	⊙	⊙	⊙	⊙				$\frac{1}{32}\Delta(v\text{-}9)(v^4\text{-}54v^3+1112v^2\text{-}10410v+37767)$	$\frac{1}{32}\Delta(v\text{-}15)(v\text{-}7)(v^3\text{-}41v^2+591v\text{-}2999)$
338	6	14	abc ade afg bhi cjk lmn	⊙	⊙	⊙	⊙	⊙			$\frac{1}{16}\Delta(v\text{-}15)(v\text{-}7)(v^3\text{-}43v^2+647v\text{-}3389)$	$\frac{1}{16}\Delta(v\text{-}9)(v^2\text{-}24v+183)(v^2\text{-}32v+259)$
339	6	14	abc ade afg bhi djk lmn	⊙	⊙	⊙	⊙	⊙			$\frac{1}{4}\Delta(v\text{-}7)(v^4\text{-}58v^3+1300v^2\text{-}13390v+53523)$	$\frac{1}{4}\Delta(v\text{-}9)(v^4\text{-}56v^3+1206v^2\text{-}11936v+46497)$

TABLE 17. The six-line configurations IX

340	6	14	abc ade afg bhi hjk lmn	⊙	⊙	⊙	⊙	⊙			$\frac{1}{4}\Delta)(v\text{-}15)(v\text{-}7)(v^3\text{-}43v^2+647v\text{-}3437)$	$\frac{1}{4}\Delta(v^5\text{-}65v^4+1706v^3\text{-}22618v^2+151749v\text{-}413349)$
341	6	14	abc ade afg bhi jkl jmn	⊙	⊙	⊙	⊙	⊙			$\frac{3}{16}\Delta(v^5\text{-}65v^4+1702v^3\text{-}22470v^2+149529v\text{-}399161)$	$\frac{3}{16}\Delta(v\text{-}7)(v^4\text{-}58v^3+1296v^2\text{-}13238v+52127)$
342	6	14	abc ade afg hij hkl hmn	⊙				⊙			$\frac{1}{192}\Delta(v\text{-}11)^2(v\text{-}13)^2(v\text{-}15)$	$\frac{1}{192}\Delta(v\text{-}7)(v\text{-}11)(v\text{-}15)(v^2\text{-}30v+245)$
343	6	14	abc ade afg hij hkl imn	⊙	⊙	⊙	⊙				$\frac{1}{16}\Delta(v\text{-}9)(v^4\text{-}56v^3+1194v^2\text{-}11560v+43157)$	$\frac{1}{16}\Delta(v\text{-}15)(v\text{-}7)(v^3\text{-}43v^2+647v\text{-}3389)$
344	6	14	abc ade bfg dfh ijk lmn	⊙	⊙	⊙	⊙	⊙	⊙	⊙	$\frac{1}{24}\Delta(v^5\text{-}67v^4+1823v^3\text{-}25229v^2+177912v\text{-}513576)$	$\frac{1}{24}\Delta(v\text{-}7)(v^4\text{-}59v^3+1351v^2\text{-}14325v+59688)$
345	6	14	abc ade bfg chi djk lmn	⊙	⊙	⊙	⊙	⊙	⊙	⊙	$\frac{1}{4}\Delta(v\text{-}7)(v\text{-}15)(v^3\text{-}45v^2+715v\text{-}4055)$	$\frac{1}{4}\Delta(v^5\text{-}67v^4+1826v^3\text{-}25334v^2+178605v\text{-}505959)$
346	6	14	abc ade bfg chi jkl jmn	⊙	⊙	⊙	⊙	⊙	⊙		$\frac{1}{16}\Delta(v\text{-}15)(v\text{-}7)(v^3\text{-}45v^2+719v\text{-}4083)$	$\frac{1}{16}\Delta(v^5\text{-}67v^4+1822v^3\text{-}25210v^2+178257v\text{-}517203)$
347	6	14	abc ade bfg dhi fjk lmn	⊙	⊙	⊙	⊙	⊙	⊙	⊙	$\frac{1}{2}\Delta(v\text{-}7)(v^4\text{-}60v^3+1394v^2\text{-}14940v+62349)$	$\frac{1}{2}\Delta(v^5\text{-}67v^4+1822v^3\text{-}25162v^2+176409v\text{-}497235)$
348	6	14	abc ade bfg dhi jkl jmn	⊙	⊙	⊙	⊙	⊙	⊙	⊙	$\frac{3}{8}\Delta(v\text{-}15)(v\text{-}17)(v\text{-}7)(v^2\text{-}28v+243)$	$\frac{3}{8}\Delta(v^5\text{-}67v^4+1818v^3\text{-}25014v^2+175077v\text{-}500583)$
349	6	14	abc ade bfg hij hkl imn	⊙	⊙	⊙	⊙	⊙	⊙	⊙	$\frac{3}{16}\Delta(v^5\text{-}67v^4+1818v^3\text{-}25030v^2+174773v\text{-}490727)$	$\frac{3}{16}\Delta(v\text{-}7)(v^4\text{-}60v^3+1398v^2\text{-}15036v+63073)$
350	6	15	abc ade bdf ghi jkl mno	⊙	⊙	⊙	⊙	⊙	⊙		$\frac{1}{324}\Delta(v\text{-}7)(v^5\text{-}77v^4+2425v^3\text{-}39195v^2+326718v\text{-}1128816)$	$\frac{1}{324}\Delta(v^6\text{-}84v^5+2982v^4\text{-}57340v^3+630909v^2\text{-}3774672v)+29739$
351	6	15	abc ade afg ahi jkl mno	⊙	⊙						$\frac{1}{1152}\Delta(v\text{-}13)(v\text{-}15)(v\text{-}17)(v\text{-}7)(v^2\text{-}25v+186)$	$\frac{1}{1152}\Delta(v\text{-}13)(v\text{-}9)(v^4\text{-}55v^3+1145v^2\text{-}10709v+38202)$
352	6	15	abc ade afg bhi jkl mno	⊙	⊙	⊙	⊙	⊙			$\frac{1}{48}\Delta(v\text{-}9)(v^5\text{-}72v^4+2112v^3\text{-}31678v^2+244575v\text{-}786330)$	$\frac{1}{48}\Delta(v\text{-}15)(v\text{-}7)(v^4\text{-}59v^3+1357v^2\text{-}14541v+61914)$
353	6	15	abc ade afg hij hkl mno	⊙	⊙	⊙	⊙	⊙			$\frac{1}{96}\Delta(v^6\text{-}81v^5+2752v^4\text{-}50214v^3+519301v^2\text{-}2890185v+6779898)$	$\frac{1}{96}\Delta(v\text{-}15)(v\text{-}7)(v^4\text{-}59v^3+1349v^2\text{-}14293v+59610)$

Table 18. The six-line configurations X

354	6	15	abc ade bfg chi jkl mno	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\frac{1}{144}\Delta(v\text{-}7)(v^5\text{-}76v^4+2372v^3\text{-}38114v^2+316203v\text{-}1086498)$	$\frac{1}{144}\Delta(v^6\text{-}83v^5+2912v^4\text{-}55382v^3+603601v^2\text{-}3584391v+9100350)$
355	6	15	abc ade bfg dhi jkl mno	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\frac{1}{24}\Delta(v\text{-}7)(v^5\text{-}76v^4+2372v^3\text{-}38162v^2+317979v\text{-}1102338)$	$\frac{1}{24}\Delta(v^6\text{-}83v^5+2908v^4\text{-}55122v^3+597213v^2\text{-}3514395v+8811846)$
356	6	15	abc ade bfg hij hkl mno	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\frac{1}{16}\Delta(v^6\text{-}83v^5+2904v^4\text{-}54886v^3+591833v^2\text{-}3453735v+8473326)$	$\frac{1}{16}\Delta(v\text{-}15)(v\text{-}7)(v^4\text{-}61v^3+1457v^2\text{-}16323v+73438)$
357	6	15	abc ade fgh fij klm kno	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\frac{1}{128}\Delta(v^6\text{-}83v^5+2900v^4\text{-}54634v^3+586053v^2\text{-}3402243v+8381734)$	$\frac{1}{128}\Delta(v\text{-}7)(v^5\text{-}76v^4+2368v^3\text{-}38010v^2+315663v\text{-}1086538)$
358	6	16	abc ade afg hij klm nop	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$			$\frac{1}{2592}\Delta(v\text{-}15)(v\text{-}7)(v^5\text{-}76v^4+2372v^3\text{-}38258v^2+321675v\text{-}1139346)$	$\frac{1}{2592}\Delta(v\text{-}9)(v^6\text{-}89v^5+3348v^4\text{-}68290v^3+799097v^2\text{-}5113125v+14104242)$
359	6	16	abc ade bfg hij klm nop	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\frac{1}{432}\Delta(v\text{-}19)(v^6\text{-}81v^5+2796v^4\text{-}52594v^3+569025v^2\text{-}3363165v+8526546)$	$\frac{1}{432}\Delta(v\text{-}7)(v^6\text{-}93v^5+3684v^4\text{-}79858v^3+1003329v^2\text{-}6959457v+20910474)$
360	6	16	abc ade fgh fij klm nop	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\frac{1}{384}\Delta(v\text{-}15)(v\text{-}7)(v^5\text{-}78v^4+2510v^3\text{-}41972v^2+368513v\text{-}1378158)$	$\frac{1}{384}\Delta(v^7\text{-}100v^6+4331v^5\text{-}105398v^4+1558023v^3\text{-}14006088v^2+70947981v\text{-}155698974)$
361	6	17	abc ade fgh ijk lmn opq	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\frac{1}{10368}\Delta(v\text{-}7)(v^7\text{-}111v^6+5376v^5\text{-}147718v^4+2496249v^3\text{-}26058123v^2+156349926v\text{-}417905568)$	$\frac{1}{10368}\Delta)(v^8\text{-}118v^7+6153v^6\text{-}185350v^5+3531139v^4\text{-}43610058v^3+341396307v^2\text{-}1551728826v+3145570848)$
362	6	18	abc def ghi jkl mno pqr	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\frac{1}{1399680}\Delta(v^9\text{-}137v^8+8419v^7\text{-}304873v^6+7176053v^5\text{-}113957779v^4+1222096713v^3\text{-}8544363291v^2+35399189214v\text{-}66407856480)$	$\frac{1}{1399680}\Delta(v\text{-}7)(v^8\text{-}130v^7+7509v^6\text{-}252310v^5+5409883v^4\text{-}76069158v^3+687999087v^2\text{-}3677611842v+8940261600)$