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A SURVEY OF SKOLEM-TYPE SEQUENCES AND ROSA'S USE OF THEM

Nevena Francetić — Eric Mendelsohn

Dedicated to Alex Rosa on the occasion of his seventieth birthday
May he live in full strength (as Moses did) until one hundred twenty

(*Communicated by Peter Horák*)

ABSTRACT. Let D be a set of positive integers. A Skolem-type sequence is a sequence of $i \in D$ such that every $i \in D$ appears exactly twice in the sequence at positions a_i and b_i, and $|b_i - a_i| = i$. These sequences might contain empty positions, which are filled with null elements. Thoralf A. Skolem defined and studied Skolem sequences in order to generate solutions to Heffter's difference problems. Later, Skolem sequences were generalized in many ways to suit constructions of different combinatorial designs. Alexander Rosa made the use of these generalizations into a fine art. Here we give a survey of Skolem-type sequences and their applications.

1. Introduction

Searching for a solution to Heffter's difference problems, Skolem [97] studied the existence of a partition of the set $\{1, 2, \dots, 2n\}$ into n ordered pairs $\big\{(a_i, b_i) : \ i = 1, \dots, n, \ b_i - a_i = i\big\}$. Two years later, Langford [45] posed a question of ordering integers $i \in \{1, 2, \dots, n\}$ in a sequence such that each i appears twice in the sequence and there are exactly i other elements in between two copies of i. This started the study of Skolem and Langford sequences and their many generalizations. These sequences are applied in several areas such as triple systems ([25]), starters ([71], [72], [73]), balanced ternary

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designs ([13], [14], [63]), factorization of complete graphs ([21], [72]), and labelings of graphs ([4], [67], [56], [57]). Nowakowski established in his master's thesis [66] the connection between generalized Skolem and Langford sequences and the golden mean, Wythoff pairs and partitions of sets of numbers. Moreover, Langford sequences are applied in the construction of binary sequences with controllable complexity ([34]). As solutions to Heffter's difference problems, Skolem-type sequences are used in the generation of missile guidance codes resistant to random interference ([29]). Also, Mendelsohn and Shalaby used these sequences for labeling graphs to enhance testing of reliability of a communication network ([56]). Skolem-type sequences are applied in design of statistical models, such as a balanced sampling plan excluding contiguous units, and a balanced sampling plan avoiding the selection of adjacent units (cf. [100], [101], [102], [18], [24]). More recently, the enumeration of Langford sequences is used as a benchmark in testing new parallel processing algorithms ([35], [39]).

Although Rosa made a fine art of applying Skolem-type sequences and their generalizations to combinatorial designs, here we want to give a comprehensive survey on the Skolem-type sequences. Thus we include definitions of some sequences and applications which Rosa did not work on. Also, we give a large and representative bibliography of Skolem related sequences. (It would be hubris to say complete; further Rosa would find an omission within a few seconds if we claimed it to be complete).

2. Definitions and existence

There are many generalizations of Skolem sequences. Here we give their definitions and the existence conditions for each kind of sequence. Later, we shall consider the most frequently used techniques applied in proofs of the existence theorems.

Definition 2.1. ([8]) A *Skolem-type sequence* is a sequence $(s_1, s_2, \ldots, s_m)$ of positive integers $i \in D$ such that for each $i \in D$ there is exactly one $j \in \{1, 2, \ldots, m-i\}$ such that $s_j = s_{j+i} = i$.

This sequence is also called a generalized Skolem sequence in [49]. The integers $i \in D$ are often called *differences*, and positions s_j and s_{j+i} from the definition are sometimes called *elements*. Positions in the sequence not occupied by integers $i \in D$ contain null elements, denoted by 0, $*$, or ϵ (Rosa informally used X for these at the blackboard). The null elements in a sequence are also called *holes*. Historically, if the second last position in the sequence contains a null element, it is called a *hook*. However, some authors call all positions with null elements hooks.

Definition 2.2. A Skolem-type sequence is *k-extended* if it contains exactly one hole which is in position k. If a sequence has a hole in the penultimate position, it is called a *hooked* sequence.

Now, we consider some special cases of Skolem-type sequences.

Definition 2.3. ([97]) A *Skolem sequence* of order n is a partition of the set $\{1,2,3,\ldots,2n\}$ into a collection of disjoint ordered pairs $\{(a_i,b_i):\ i=1,\ldots,n\}$ such that $a_i<b_i$ and $b_i-a_i=i$.

Equivalently, a *Skolem sequence* of order n is a Skolem-type sequence with $m=2n$ and $D=\{1,2,\ldots,n\}$.

In the literature, Skolem sequences are also called pure Skolem sequences. Before the name Skolem sequences was widely established, these partitions were called 1, +1 systems by Skolem, (A,k)-systems by Rosa, and Nickerson sequences by Nowakowski. The first definition of Skolem sequences was originally given by Skolem, and it is very convenient for applications in the method of differences. Note that the most of sequences related to Skolem sequences can be considered as partitions of appropriate sets. The second definition was studied by Nickerson [64], and Nowakowski [66] proved that these two definitions are equivalent.

Definition 2.4. ([6]) A *k-extended Skolem sequence* of order n is a k-extended Skolem-type sequence with $m=2n+1$ and $D=\{1,2,\ldots,n\}$. A *hooked k-extended Skolem sequence* of order n is a Skolem-type sequence with $m=2n+2$, $D=\{1,2,\ldots,n\}$, and $s_k=s_{2n+1}=0$.

Definition 2.5. A *hooked Skolem sequence* of order n is a $(2n)$-extended Skolem sequence of order n.

For example, if $n=4$, the partition $\{(2,3),(6,8),(4,7),(1,5)\}$, or equivalently, the sequence $(4,1,1,3,4,2,3,2)$, is a (pure) Skolem sequence of order 4. On the other hand, the partition $\{(5,6),(1,3),(8,11),(9,13),(2,7),(4,10)\}$, or in equivalent notation the sequence $(2,5,2,6,1,1,5,3,4,6,3,0,4)$, is a hooked Skolem sequence of order 6.

In order to prove the existence of cyclic Steiner triple systems, Rosa [79] introduced a *split Skolem sequence* of order n which is an $(n+1)$-extended Skolem sequence of order n. Also, a *hooked split Skolem sequence* of order n is a hooked $(n+1)$-extended Skolem sequence of order n. In more recent literature, these sequences are also called *Rosa* and *hooked Rosa sequences* of order n.

Before giving more definitions, we present the existence theorems of the above defined sequences. The necessary and sufficient condition for the existence of a pure Skolem sequence was given by Skolem [97]. The existence problem for a hooked Skolem sequence was solved by O'Keefe [68].

Theorem 2.6. ([97], [68]) *A Skolem sequence of order* n *exists if and only if* $n \equiv 0, 1 \pmod 4$. *A hooked Skolem sequence of order* n *exists if and only if* $n \equiv 2, 3 \pmod 4$.

Theorem 2.7. ([79], [23]) *A Rosa sequence of order* n *exists if and only if* $n \equiv 0, 3 \pmod 4$.

A hooked Rosa sequence of order n *exists if and only if* $n \equiv 1, 2 \pmod 4$ *and* $n \neq 1$.

Much later the existence question of k-extended Skolem sequences was completely solved by Baker [6]. Linek and Jiang [50] determined the existence conditions of hooked k-extended Skolem sequences. Notice that adding a hook to a k-extended Skolem sequence changes the parity of k.

Theorem 2.8. ([6]) *A* k*-extended Skolem sequence of order* n *exists if and only if* $n \equiv 0, 1 \pmod 4$ *when* k *is odd, and* $n \equiv 2, 3 \pmod 4$ *when* k *is even.*

Theorem 2.9. ([50]) *A hooked* k*-extended Skolem sequence of order* n *exists if and only if* $(k, n) \neq (2, 1)$ *and if* k *is even, then* $n \equiv 0, 1 \pmod 4$, *or if* k *is odd, then* $n \equiv 2, 3 \pmod 4$.

Definition 2.10. An *excess Skolem sequence* of order n and surplus k ([91]) is a Skolem-type sequence with $m = 2n + 2$ and $D = \{1, 2, \ldots, n\}$ which in addition contains exactly two indices $u, v \in \{1, \ldots, 2n + 2\}$, $u < v$, $v \neq u + k$ and $s_u = s_{u+k} = s_v = s_{v+k} = k$.

The necessary conditions for the existence of these sequences are derived in [91] and their sufficiency is proved in [6]. In particular, if k and n are such that the conditions of the existence of an excess Skolem sequence hold, then there exists a k-extended Skolem sequence of order n, and we can add a surplus copy of k's as the first element of the sequence and in the place of the hook in the extended Skolem sequence.

Shalaby studied the existence of [hooked] near Skolem sequences. These sequences first appeared in some constructions related to 1-coverings of pairs by triples ([13]). Later they were widely used to prove existence of other Skolem related sequences as well as in other combinatorial designs.

Definition 2.11. A k*-near Skolem-type sequence* of order n is a Skolem-type sequence $(s_1, \ldots, s_m)$ with $D = \{1, 2, \ldots, n\} \backslash \{k\}$.

Definition 2.12. ([91]) If $k \leq n$, a *near Skolem sequence* of order n and defect k is a k-near Skolem-type sequence of order n with $m = 2n - 2$. This sequence is also called the k-near Skolem sequence of order n.

Definition 2.13. ([93]) When $k \leq n$, a k*-near Rosa sequence* of order n is a k-near Skolem-type sequence of order n with $m = 2n - 1$ and $s_n = 0$.

Sequences $(1,1,6,3,7,5,3,2,6,2,5,7)$ and $(5,7,1,1,6,5,3,0,7,3,6,2,0,2)$ are examples of a 4-near Skolem sequence of order 7 and a hooked 4-near Rosa sequence of order 7, respectively.

Shalaby proved the existence theorems for these sequences. Notice that again for a fixed order of a near Skolem sequence, adding a hook to it changes the parity of the defect. However, this is not the case with near Rosa sequences.

Theorem 2.14. ([91]) *A k-near Skolem sequence of order n exists if and only if $n \equiv 0,1 \pmod 4$ when k is odd, and $n \equiv 2,3 \pmod 4$ when k is even.*

A hooked k-near Skolem sequence of order n exists if and only if $n \equiv 0,1 \pmod 4$ when k is even, and $n \equiv 2,3 \pmod 4$ when k is odd.

Theorem 2.15. ([93]) *There exists a k-near Rosa sequence of order n if and only if either $n \equiv 0,3 \pmod 4$ and k is even, or $n \equiv 1,2 \pmod 4$ and k is odd, with the exceptions when $(n,k) = (3,2),(4,2)$.*

A hooked k-near Rosa sequence of order n exists if and only if either $n \equiv 0,3 \pmod 4$ and k is even, or $n \equiv 1,2 \pmod 4$ and k is odd, with the exceptions when $(n,k) = (2,1),(3,2)$.

Very little is known about the existence of extended near-Skolem sequences. Only the existence of the $(2n-3)$-extended and hooked $(2n-2)$-extended k-near Skolem sequences of order n has been established (cf. [78]).

In 1958 Langford posed the question about arranging two copies of integers $i \in \{1,\dots,n\}$ in a sequence such that there are exactly i integers between two copies of i (cf. [45]). For example, such a sequence of order 3 is $(3,1,2,1,3,2)$. Later this problem was generalized to define Langford sequences.

Definition 2.16. ([96]) A *Langford sequence* of defect d and length l is a Skolem-type sequence with $m = 2l$ and $D = \{d, d+1, \dots, d+l-1\}$.

Such a Langford sequence is also called a *perfect* Langford sequence.

A *hooked Langford sequence* of defect d and length l is a Skolem-type sequence with $m = 2l+1$, $D = \{d, d+1, \dots, d+l-1\}$ and $s_{2l} = 0$.

The largest difference in a Langford sequence, $d+l-1$, is sometimes called the *order* of the Langford sequence (cf. [6]). The k-extended and hooked k-extended Langford sequences were studied by Linek and Jiang [49], [50]. Similar to the definitions above, a *k-extended Langford sequence* of defect d and length l is a k-extended Skolem-type sequence with $m = 2l+1$ and $D = \{d, d+1, \dots, d+l-1\}$.

The problem which Langford originally stated was solved by Priday [74]. He showed that for every length l either a Langford or a hooked Langford sequence of defect $d = 2$ exists and he conjectured the necessary conditions for their existence. The generalization of the problem for any defect d was partially solved by Davies [26] and Bermond et al. [12], and it was completed by Simpson [96].

Theorem 2.17. ([96]) *A (perfect) Langford sequence of defect d and length l exists if and only if $l \geq 2d-1$, and $l \equiv 0,1 \pmod 4$ for d odd, or $l \equiv 0,3 \pmod 4$ for d even.*

A hooked Langford sequence of defect d and length l exists if and only if $l(l+1-2d)+2 \geq 0$, and $l \equiv 2,3 \pmod 4$ for d odd, and $l \equiv 1,2 \pmod 4$ for d even.

Proving the sufficiency of the existence conditions of k-extended Langford sequences of defect d and length l is a more difficult problem. Linek and Jiang gave necessary conditions for the existence of these sequences in [49].

Theorem 2.18. ([49]) *Given a k-extended Langford sequence of defect d and length l, the following conditions hold:*

- *if $l \equiv 0 \pmod 4$, then $k \equiv 1 \pmod 2$,*
- *if $l \equiv 1 \pmod 4$, then $k \equiv d \pmod 2$,*
- *if $l \equiv 2 \pmod 4$, then $k \equiv 0 \pmod 2$,*
- *if $l \equiv 3 \pmod 4$, then $k \equiv d+1 \pmod 2$,*

and $l \geq 2d-3$, $\frac{l}{2}(2d-1-l)+1 \leq k \leq \frac{l}{2}(l-2d+5)+1$. There is no 5-extended Langford sequence with defect 3 and length 4.

By considering an exhaustive list of cases and constructing required sequences for each of them, Linek and Jiang [49] proved that these conditions are also sufficient when the defect d is 2 or 3, except for $d=3$ and $(l,k) = (3,2),(3,6),(4,1),(4,5),(4,9)$ when no such sequence exists. In the same paper, they also settled the existence problem of hooked k-extended Langford sequences of defect 2.

For fixed length l, Linek and Mor [53] proved the sufficiency of the necessary conditions for the existence of extended Langford sequences only for small values of d, i.e. $d \leq \frac{l+4}{8}$, and large values of d, that is $d = \frac{l-1}{2}, \frac{l}{2}, \frac{l+1}{2}$. Moreover, they constructed the k-extended Langford sequences of defect 4 and length $l \in [10,27]$ which satisfy the necessary conditions. They also determined the existence of hooked k-extended Langford sequences of defect d and length l when $d = \frac{l-1}{2}, \frac{l}{2}, \frac{l+1}{2}$ (cf. [53]).

More generalizations of Skolem sequences were needed for the construction of some combinatorial designs. Billington used sequences, which were later called near λ-fold and extended near λ-fold Skolem sequences to generate balanced ternary designs (cf. [63]). Moreover, the 2-fold Skolem sequences are used in the construction of 2-fold triple systems ([75]). There are also the usual variations: extended and near λ-fold sequences.

Definition 2.19. A *λ-fold Skolem-type sequence* $(s_1, s_2, \ldots, s_m)$ is a sequence of integers $i \in D$ such that for every $i \in D$ there are exactly λ disjoint pairs $(j, j+i)$ such that $s_j = s_{j+i} = i$.

Definition 2.20. ([9]) A *λ-fold Skolem sequence* of order n is a λ-fold Skolem-type sequence with $m = 2\lambda n$ and $D = \{1, \ldots, n\}$.

A *k-extended λ-fold Skolem sequence* of order n is a λ-fold Skolem-type sequence with $m = 2\lambda n + 1$, $D = \{1, \ldots, n\}$, and $s_k = 0$.

For example, $(3, 1, 1, 3, 3, 1, 1, 3, 2, 2, 2, 2)$ is a 2-fold Skolem sequence of order 3, and $(3, 1, 1, 3, 3, 1, 1, 3, 0, 2, 2, 2, 2)$ is a 9-extended 2-fold Skolem sequence of order 3. Some care is needed for definition of near λ-fold Skolem sequences, since the defect appears only in $\lambda - 1$ pairs in the sequence.

Definition 2.21. ([77]) An *m-near λ-fold Skolem sequence* $(s_1, s_2, \ldots, s_{2\lambda n-2})$ of order n and defect m is a sequence of integers $i \in \{1, \ldots, n\}$ such that if $i \neq m$, there are exactly λ disjoint pairs $(j, j + i)$ satisfying $s_j = s_{j+i} = i$, and there are exactly $\lambda - 1$ disjoint pairs $(j, j + m)$ satisfying $s_j = s_{j+m} = m$.

A *λ-fold Rosa sequence* of order n is a λ-fold Skolem-type sequence with $m = 2\lambda n + 2$, $D = \{1, 2, \ldots, n\}$, and exactly two null elements (cf. [9]). More specifically, a 2-fold Rosa sequence of order n has null elements in the positions $n + 1$ and $3n + 2$ (cf. [75]).

For instance, $(3, 1, 1, 3, 3, 1, 1, 3, 3, 0, 2, 3, 2, 2, 2, 2, 2, 2)$ is a 10-extended 1-near-3-fold Skolem sequence of order 3.

Note that a 1-fold Skolem-type sequence is a Skolem-type sequences. Hence, in the following, we assume that $\lambda \geq 2$. The existence of all of the variations to the λ-fold Skolem sequences is completely settled. We want to emphasize that the existence conditions for m-near and k-extended λ-fold Skolem sequences of order n are identical for the pairs (n, m) and (n, k). Here are the precise theorems.

Theorem 2.22. ([9]) *A λ-fold Skolem sequence of order n exists if and only if either $n \equiv 0, 1 \pmod 4$, or if $n \equiv 2, 3 \pmod 4$, then $\lambda \equiv 1 \pmod 2$.*

Theorem 2.23. ([9], [77]) *A k-extended, or a k-near λ-fold Skolem sequence of order n exists if and only if either $n \equiv 0, 1 \pmod 4$ and k is odd, or $n \equiv 2, 3 \pmod 4$, and k and λ are of the opposite parity.*

Theorem 2.24. ([77], [54]) *A k-extended m-near-λ-fold Skolem sequence of order n and defect m exists if and only if either*

(i) *$n \equiv 0, 1 \pmod 4$, and $k \equiv m \pmod 2$, or*

(ii) *$n \equiv 2, 3 \pmod 4$, λ odd, and $k \equiv m + 1 \pmod 2$, or*

(iii) *$n \equiv 2, 3 \pmod 4$, λ even, and $k \equiv m \pmod 2$.*

A λ-fold Rosa sequence of order n with null elements at positions k and m exists if and only if $n \equiv 0, 1 \pmod 4$ and $k \equiv m + 1 \pmod 2$, or conditions (ii) *or* (iii) *are satisfied.*

Corollary 2.25. ([75]) *A 2-fold Rosa sequence of order n exists if and only if $n \geq 2$.*

The concept of indecomposable triple systems led to the definition and study of indecomposable Skolem and Rosa sequences.

Definition 2.26. ([75]) A *k-indecomposable λ-fold [hooked] Skolem sequence* of order n is a λ-fold [hooked] Skolem sequence of order n such that for every i, j, $1 \leq i < j \leq 2\lambda n$ $[1 \leq i < j \leq 2\lambda n + 1]$ the subsequence starting at position i and ending at position j, inclusively, is not a k-fold [hooked] Skolem sequence of order m, $1 < m \leq n$.

An *indecomposable λ-fold [hooked] Skolem sequence* of order n is a k-indecomposable λ-fold [hooked] Skolem sequence of order n for every $k \leq \lambda$.

The sequence $(1, 1, 2, 2, 2, 2, 1, 1)$ is an indecomposable 2-fold Skolem sequence of order 2.

Definition 2.27. ([75]) A two-fold Rosa sequence of order n is *totally indecomposable* if it does not contain as a proper subsequence a Rosa sequence of order m, $m \leq n$.

For example, $(1, 1, 2, 0, 2, 2, 3, 2, 3, 3, 0, 3, 1, 1)$ is a totally indecomposable Rosa sequence of order 3.

Theorem 2.28. ([75]) *There exists an indecomposable λ-fold Skolem sequence of order n for all $n > 2$ but when λ is even, $n \equiv 2, 3 \pmod 4$.*

There exists an indecomposable hooked λ-fold Skolem sequence of order n if $n \equiv 2, 3 \pmod 4$, $n > 2$ and λ odd.

Theorem 2.29. ([75]) *A totally indecomposable two-fold Rosa sequence of order n exists for every $n \geq 2$.*

The idea of looped Langford sequences goes back to Priday [74], and it found its application in construction of extended near-Skolem sequences.

Definition 2.30. ([94]) A *looped Langford set* is a pair of sequences (L_n, K_n) of defect d and order n, $n \geq d$, where L_n is a $(2n-1)$-extended Langford sequence of defect d and length $n - d + 1$, and K_n is a $(2n)$-extended hooked Langford sequence of defect d and length $n - d + 1$.

For example,

$$\big((2, 4, 2, 7, 5, 4, 8, 6, 3, 5, 7, 3, 0, 6, 8), (2, 4, 2, 7, 5, 4, 6, 8, 3, 5, 7, 3, 6, 0, 0, 8)\big)$$

is a looped Langford set of defect 2 and order 8.

Gillespie and Utz [32] defined generalized Langford sequences and proposed a question about their existence. Later, the problem was extended to

Skolem ([27]), extended Skolem and near Skolem sequences ([92]). There are many necessary conditions for the existence of these sequences (cf. [47], [83], [27], [84], [92]), but the methods used to construct Skolem-type sequences seem to be inadequate for the construction of generalized Skolem-type sequences.

Definition 2.31. A *generalized Skolem-type sequence* of multiplicity k is a sequence $S = (s_1, s_2, \ldots, s_m)$ of integers $i \in D$ such that for each $i \in D$ there are exactly k positions in the sequence S, j_1, $j_2 = j_1 + i$, $\ldots$, $j_k = j_1 + (k-1)i$ and $s_{j_1} = s_{j_2} = \cdots = s_{j_k} = i$.

Dillon [27] defined a (k, n)-*generalized Skolem sequence* of order n and multiplicity k as a generalized Skolem-type sequence of multiplicity k with $m = kn$ and $D = \{1, 2, \ldots, n\}$. Similarly, *generalized* (k, n)-*Langford sequences* ([27]) and *generalized near Skolem sequences* ([92]) are defined.

A *generalized extended Skolem sequence* has the properties of a generalized Skolem sequence with the minimum number of positions occupied with null elements. The definition of a *generalized extended near Skolem sequences* is similar.

For example, $(4, 9, 16, 17, 4, 18, 19, 2, 4, 2, 9, 2, 4, 2, 13, 14, 5, 15, 16, 9, 17, 5, 8, 18, 12, 19, 5, 13, 9, 14, 8, 5, 15, 6, 16, 7, 12, 17, 8, 6, 13, 18, 7, 14, 19, 6, 8, 15, 12, 7, 16, 6, 3, 13, 17, 3, 7, 14, 3, 18, 12, 3, 15, 19)$ is a generalized Langford sequence of defect 2, length 18 and multiplicity 4 (cf. [84]).

Theorem 2.32. ([92]) *Let $k = p^e t$, where p is the smallest prime factor of k, and e and t are positive integers. Then, a generalized (k, n)-Skolem sequence exists only if $n \equiv 0, 1, \ldots, p-1 \pmod{p^{e+1}}$.*

For more information on generalized Skolem-type sequences, see [27], [92], [83], [84], [85], [86], [87], [88], [89], [90].

Further generalization of Langford sequences to infinite integer series and their relation to Beatty sequences can be found in [30].

2.1. Techniques used in the proofs of the existence theorems

In general, the existence problem of Skolem-type sequences is very difficult. In particular, Nordh [65] determined that the question if there exists a Skolem-type sequence of difference set D and length $m = 2|D|$ is **NP**-complete.

The necessary conditions in the existence theorems of Skolem related sequences mostly follow from testing the *parity* and the *density* conditions, or a variation of these conditions. Let S be a Skolem-type sequence with a set of differences $D = \{d_1, d_2, \ldots, d_n\}$, and a set of occupied positions $P = \{p_1, p_2, \ldots, p_{2n}\}$, $p_1 < p_2 < \cdots < p_{2n}$. Let $a_{d_i}, b_{d_i} \in P$ be the first and second position of d_i in the sequence S, respectively. If d_i is odd then $a_{d_i} \equiv b_{d_i} + 1 \pmod 2$, and if d_i is even, then $a_{d_i} \equiv b_{d_i} \pmod 2$. Hence, $a_{d_i} + b_{d_i} \equiv d_i \pmod 2$. Summing over i, we get the *parity condition*:

$$\sum_{i=1}^{2n} p_i \equiv \sum_{i=1}^{n} d_i \pmod 2.$$

Also, $d_i = b_{d_i} - a_{d_i}$, $\sum_{i=1}^{n} a_{d_i} \geq \sum_{i=1}^{n} p_i$ and $\sum_{i=1}^{n} b_{d_i} \leq \sum_{i=n+1}^{2n} p_i$. Hence, we get the *density condition*:

$$\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} b_{d_i} - \sum_{i=1}^{n} a_{d_i} < \Big(\sum_{i=n+1}^{2n} p_i\Big) - \Big(\sum_{i=1}^{n} p_i\Big).$$

As stated above, the parity and the density conditions were first used in the proof of the existence of k-extended Langford sequences (cf. [1], [49]). There is a variation of the parity and the density condition used when sets D and P are known. For example, in the proof of Theorem 2.17, assume we are given a Langford sequence L of length n and defect d. Let a_i and b_i be the first and the second position of the integer i in the sequence L, respectively. Then we have

$$\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{2n} i = \frac{2n(2n+1)}{2},$$
$$\sum_{i=1}^{n} b_i - \sum_{i=1}^{n} a_i = \sum_{i=d}^{d+n-1} i = \frac{1}{2}n(n+2d-1).$$

If we solve this system for $\sum_{i=1}^{n} a_i$, we get $\sum_{i=1}^{n} a_i = \frac{1}{4}n(3n-2d+3)$. On the other hand, $\sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$. The difference D of the exact and the minimum value of $\sum_{i=1}^{n} a_i$ has to be a non-negative integer, hence

$$D = \frac{1}{4}n(3n-2d+3) - \frac{1}{2}n(n+1) = \frac{1}{4}n(n-2d+1) \geq 0.$$

Since D is an integer, it follows that 4 divides either n or $n-2d+1$. If d is even, this implies that $n \equiv 0,3 \pmod 4$, and if d is odd, we have to have $n \equiv 0,1 \pmod 4$.

The sufficiency of the existence conditions is generally proved by *directly constructing* the required sequences. New sequences can also be formed by *concatenating* the existing sequences. For example, if we concatenate a Skolem sequence of order n and a [hooked] Langford sequence of defect $n+2$ and length l, we get a [hooked] $(n+1)$-near Skolem sequence of order $n+l+1$ (cf. [91]).

On the other hand, the *reverse* of a hooked Skolem sequence is a 2-extended Skolem sequence. We can concatenate the reverse of a sequence with another sequence. Also, we can *hook* a hooked sequence, HS, and the reverse of another hooked sequence, RHS. Position the first element of RHS at the place of a hook in the HS, and put the last element of the HS at the place of the hook in the RHS. For example,

$$L = (12, 10, 8, 6, 4, 11, 9, 7, 4, 6, 8, 10, 12, 5, 7, 9, 11, 0, 5)$$

is a hooked Langford sequence of defect 4 and length 9. *Hooking* $(1, 1, 2, 0, 2)$ with L, we get

$$(1, 1, 2, 5, 2, 11, 9, 7, 5, 12, 10, 8, 6, 4, 7, 9, 11, 4, 6, 8, 10, 12),$$

which is a 3-near Skolem sequence of order 12 (cf. [91]).

Moreover, Linek and Jiang [50] present several direct constructions of hooked extended Skolem sequences by taking a *sum* (*union*) of two Skolem-type sequences. Let null elements in the sequence be represented by 0. If S_1 and S_2 are two Skolem-type sequences of equal length, such that at a given position i not both S_1 and S_2 contain an integer, then the sum is easily seen as an element-wise sum of these sequences. Obviously, if S_1 and S_2 are not of the same length, then we can extend the shorter sequence by adding null elements to its end. For example, given $S_1 = (1, 1, 0, 0, 0, 3, 0, 5, 3, 0, 0, 0, 5)$ and $S_2 = (0, 0, 2, 6, 2, 0, 4, 0, 0, 6, 4)$, then $S_1 + S_2 = (1, 1, 2, 6, 2, 3, 4, 5, 3, 6, 4, 0, 5)$.

On the other hand, given a Skolem-type sequence with $i \in D$, we can *double* it by replacing each null element with two null elements and each integer i with the integer $2i$ and a null element. Usually, a double of an existing sequence is summed with a partial sequence built explicitly for that purpose (cf. [6]). For example, Skolem-type sequences of odd consecutive integers were useful as auxiliary sequences in several constructions. In particular, the existence of a Skolem-type sequence $(s_1, \ldots, s_{4n+3})$ of integers $i \in \{2j + 1 : \ j = 0, \ldots, 2n\}$ such that either $s_3 = 0$ or $s_5 = 0$ was proved by *recursive* constructions. These sequences are denoted by 3-ext $\mathscr{O}_{4n+1}$ and 5-ext $\mathscr{O}_{4n+1}$, respectively (cf. [6]). For example, $(7, 9, 0, 5, 1, 1, 3, 7, 5, 3, 9)$ is a 3-ext $\mathscr{O}_9$. Later, these sequences were named *extended odd Langford sequences* ([51]).

Now we look into a technique called *pivoting* of an element, which is very useful in constructions of extended sequences. We will illustrate it on an example from a construction in [53]. The sequence $(6, 7, 3, 4, 5, \underline{3}, 6, \underline{4}, 7, \underline{5})$ is a Langford sequence of defect 3 and length 5. We can pivot the second copy of integer 3 to the beginning of the sequence to get $(3, 6, 7, 3, 4, 5, 0, 6, \underline{4}, 7, \underline{5})$, a 7-extended Langford sequence of defect 3 and length 5. Similarly, we can pivot integers 4 and 5 in the original sequence.

L i n e k [52] was interested in constructing Langford sequences which start with a prescribed difference. Hence, he developed a technique called *twinning sequence S at distance k*. Let S be a Langford sequence of defect d and length l which starts with difference j. Then, consider the sequence $S' = S0^kS$ and let i be a difference in S. Then substitute the first and the fourth occurrence of i in S' with the appropriate difference. Also, substitute the second and the third occurrence of i in S' with the appropriate difference. Hence, the resulting sequence is a Skolem-type sequence with differences $\{2l+k-i:\ i \in \{d, d+1, \ldots, d+l-1\}\} \cup \{2l+k+i:\ i \in \{d, d+1, \ldots, d+l-1\}\}$, and it starts with the integer $l+k+j$. For example, twinning the sequence $(4, 2, 3, 2, 4, 3)$ at distance 2, we get $(12, 10, 11, 6, 4, 5, 0, 0, 4, 6, 5, 10, 12, 11)$.

HILL CLIMBING ALGORITHM

Randomly choose an unused integer $i \in \{1, 2, \ldots, n\}$.
If no such i exists, the Skolem sequence S is found, output a_i, b_i EXIT.
If there exists two available positions a_i and b_i in S such that
    $|b_i - a_i| = i$ put i at positions a_i and b_i in S, and mark i as used.
Else let a_k be the first available position in S.
    If $a_k + i$ is an admissible position in the sequence S containing
    an integer j then
        Remove j and its pair from the sequence S,
        mark j as unused, put i at a_k and $a_k + i$ and mark it used.
        $m := m + 1$
        If $m > M$ then mark all i unused and $m := 0,\ n := n + 1$
    Else let b_k be the last available position in S. Then,
        Remove integer j at position $b_k - i$ and its pair from
        the sequence S, mark integer j as unused, put the integer
        i at positions b_k and $b_k - i$ in S and mark i as used.
        $m := m + 1$
        If $m > M$ then mark all i unused and $m := 0,\ n := n + 1$
    If $n > N$ then output 'FAILED', EXIT.

The hooked k-extended Skolem sequences and their existence were studied by L i n e k and J i a n g [50]. They obtained the required sequences of small orders by a computer program based on the *hill-climbing* algorithm ([95]). This is an efficient probabilistic algorithm for obtaining a Skolem sequence of large order. To construct a Skolem sequence S of order n, the following steps are repeated in the *hill-climbing algorithm* according to two parameters: N the number of times to repeat the algorithm before declaring it a failure, and M the number of consecutive times the inner loop does not decrease the number of unused integers before it reports a failure.

3. Triple systems

For the first application of Skolem sequences, we give some constructions of triple systems.

A *triple system* ([23]) is a balanced incomplete block design with block size three. In other words, a triple system is a pair (V, B), where V is set of elements, $|V| = v$, and B is a multiset of 3-subsets (called also triples or blocks) of a V. Moreover, each unordered pair of elements of V appears in exactly λ triples in B. We denote such a triple system by $TS(v, \lambda)$. If $\lambda = 1$, the triple system is called a *Steiner triple system*, and it is denoted by $STS(v)$.

A multigraph λK_v is a graph on v vertices with every two vertices joined by λ edges. The necessary conditions for the existence of triple systems $TS(v, \lambda)$ are obtained if we consider the triple system $TS(v, \lambda)$ as an edge-partition of λK_v into triangles.

Summarized as in [25], a triple system $TS(v, \lambda)$ exists only if

$$\begin{array}{lll} v \equiv 1,3 \pmod 6 & \text{if} & \lambda \equiv 1,5 \pmod 6 \\ v \equiv 0,1 \pmod 3 & \text{if} & \lambda \equiv 2,4 \pmod 6 \\ v \equiv 1 \pmod 2 & \text{if} & \lambda \equiv 3 \pmod 6 \\ v \neq 2 & \text{if} & \lambda \equiv 0 \pmod 6. \end{array}$$

Let (V, B) and (W, D) be two triple systems and $\varphi\colon V \to W$ a mapping. Then φ is an *isomorphism* of triple systems (V, B) and (W, D) if $D = \big\{\{\varphi(a), \varphi(b), \varphi(c)\} : \{a, b, c\} \in B\big\}$. If φ is an isomorphism of (V, B) onto itself, then φ is called an *automorphism*.

3.1. Heffter's difference problems

Heffter [38] gives a direct construction of $STS(6n + 1)$ and $STS(6n + 3)$ under the assumption that there exists a partition of a specific set into specific triples. Hence he poses two problems:

Heffter's first difference problem (denoted $HDP_1(n)$):
can a set $\{1, 2, \ldots, 3n\}$ be partitioned into n ordered triples (a_i, b_i, c_i), $i = 1, \ldots, n$, such that either $a_i + b_i - c_i \equiv 0 \pmod{6n + 1}$ or $a_i + b_i + c_i \equiv 0 \pmod{6n + 1}$. Such triple is called a *difference triple*.

Heffter's second difference problem (denoted $HDP_2(n)$):
can a set $\{1, 2, \ldots, 3n+1\} \setminus \{2n+1\}$ be partitioned into difference triples, that is into ordered triples (a_i, b_i, c_i), $i = 1, \ldots, n$, such that either $a_i + b_i - c_i \equiv 0 \pmod{6n+3}$ or $a_i + b_i + c_i \equiv 0 \pmod{6n+3}$.

This method of partitioning a specific set into difference triples has later been applied in many combinatorial designs. We will mention some of them as applications of Skolem-type sequences in this survey.

Heffter's difference problems were first solved by Peltesohn [69] in 1939, which predates Skolem. Now, we see how to generate a solution to these problems given a [hooked] Skolem or [hooked] Rosa sequence, and hence construct a cyclic Steiner triple system. We present these results as in [25].

Consider a Skolem sequence of order n as a partition of the set $\{1, 2, \ldots, 2n\}$ into ordered pairs $\big\{(a_i, b_i) : \ b_i - a_i = i, \ i = 1, \ldots, n\big\}$. Skolem [97] proved that the set of triples $\big\{(i, n + a_i, n + b_i) : \ i = 1, \ldots, n\big\}$ is a solution to $HDP_1(n)$, when $n \equiv 0, 1 \pmod 4$, that is when a Skolem sequence of order n exists. Similarly, given a hooked Skolem sequence of order n, $\big\{(a_i, b_i) : b_i - a_i = i, \ i = 1, \ldots, n\big\}$, if $b_j = 2n + 1$ then the set of difference triples $\big\{(i, n + a_i, n + b_i) : \ i \neq j, \ \ i = 1, \ldots, n\big\} \cup \big\{\big(j, n + a_j, 6n + 1 - (b_j + n)\big)\big\}$ is a solution to $HDP_1(n)$, when $n \equiv 2, 3 \pmod 4$. Later, Rosa introduced split and hooked split Skolem sequences in [79], which in the same way yield a solution to $HDP_2(n)$, for $n \geq 1$ (cf. [98]).

To explain how to construct a cyclic Steiner triple system given a solution to one of Heffter's difference problems, we introduce the concept of *base blocks* or *starter blocks*. Let (V, B) be a triple system which admits an automorphism φ. Then the free group generated by φ acts on B and partitions it into equivalence classes called orbits. A representative of an orbit is a base block, and the set of base blocks completely determines B under φ.

Given a solution $\big\{(a_i, b_i, c_i) : \ i = 1, \ldots, n\big\}$ to $HDP_1(n)$, the triples $\big\{\{0, a_i, a_i + b_i\} : i = 1, \ldots, n\big\}$ are base blocks of a cyclic $STS(6n+1)$ on $V = \mathbb{Z}_{6n+1}$ under the automorphism $\varphi = (0\ 1\ \ldots\ 6n)$. That is, $B = \big\{\{j, a_i + j, a_i + b_i + j\} : i = 1, \ldots, n, \ j \in \mathbb{Z}_{6n+1}\big\}$.

Similarly, a solution $\big\{(a_i, b_i, c_i) : \ i = 1, \ldots, n\big\}$ to $HDP_2(n)$ implies the existence of a cyclic $STS(6n+3)$ on $V = \mathbb{Z}_{6n+3}$ under the automorphism $(0\ 1\ \ldots\ 6n+2)$ generated from base blocks $\big\{\{0, a_i, a_i + b_i\} : \ i = 1, \ldots, n\big\} \cup \{0, 2n+1, 4n+2\}$. The difference triple $\{0, 2n+1, 4n+2\}$ is called the *short block* (in [23]) or the *short orbit* (in [25]).

Theorem 3.1. ([25]) *A cyclic Steiner triple system of order* v *exists if and only if* $v \equiv 1, 3 \pmod 6$ *and* $v \neq 9$.

For example, $S = (3,1,1,3,2,0,2) = \{(2,3),(5,7),(1,4)\}$ is a hooked Skolem sequence of order 3. It gives the following solution to $HDP_1(3)$: $\{(1,5,6), (2,8,10),(3,4,7)\}$, which generates the base blocks $\{0,1,6\}$, $\{0,2,9\}$, $\{0,3,7\}$ to a cyclic $STS(19)$.

3.2. Partial Heffter's difference problems

Mendelsohn and Rosa [58] call a solution to $HDP_1(n)$ a *Heffter system* $HS(n)$. A $HS(n)$ is *pure* if all of its difference triples $\{a,b,c\}$ are such that $a+b=c$. On the other hand, it is *almost pure* if all but one difference triple is of this kind. A *partial Heffter system* $PHS(m)$ is a set of m difference triples $\{(a_i,b_i,c_i) : \ i = 1,\dots,m, \ a_i + b_i = c_i\}$. Mendelsohn and Rosa [58] considered a question of completing a $PHS(m)$ to a $HS(n)$ for some integer n.

Theorem 3.2. ([58]) *A pure* $HS(n)$ *can be completed to a* $HS(N)$ *for all* $N \geq 7n+1$.

Sketch and/or highlights of the proof. The statement of the theorem follows from the existence of a Langford sequence of defect $d = 3n+1$ and length $l \geq 6n+1$, $\{(a_i,b_i) : \ i = 1,\dots,l\}$. Then the required difference triples are $\{(i+3n, a_i+3n, b_i+3n) : \ i = 1,\dots,l\}$. □

Lemma 3.3. *Let* $\mathscr{H}$ *be a* PHS *which contains only one difference triple* $(1,n,n+1)$, $n \geq 2$. *Then* $\mathscr{H}$ *can be completed to a* $HS(N)$ *for all* $N \geq 7n-6$.

Sketch and/or highlights of the proof. The required difference triples are generated from two [hooked] Langford sequence, one of defect 2 and length $n-2$ and another of defect $n+2$ and length $l \geq 2n+3$. □

3.3. A recursive construction of $TS(2n+7,\lambda)$

Given a $STS(n)$, Rosa presented a recursive construction of $STS(2n+7)$ in [48], [80], [82]. Let (V,B) be a $STS(n)$, and let $U = \{u_1,\dots,u_n\}$ and $W = \{w_1,\dots,w_7\}$ be mutually disjoint and disjoint from V. Then, our goal is to edge-partition K_{2n+7} on $V \cup U \cup W$ into triangles. Let (W,D) be a $STS(7)$. Moreover, let $E = \{\{u_i,u_{i+1},u_{i+3}\} : \ i \in \mathbb{Z}_n\}$. For $m = \frac{n-1}{2}$, let $S = \{(a_i,b_i) : i = 1,\dots,m, \ b_i - a_i = i\}$ be a [hooked] Skolem sequence of order m. Now, let $Y = U \backslash \{u_i : \ i \in \{a_j,b_j\}, \ j = 4,\dots,m\}$. Hence $|Y| = 7$ and we can relabel vertices of $Y = \{u_{j_i} : \ i = 1,\dots,7\}$. Let $F = \{\{v_k, u_{j_i+k-1}, w_i : \ k = 1,\dots,n, \ i = 1,\dots,7\}$ and $G = \{\{v_k, u_{a_i+k-1}, u_{b_i+k-1}\} : \ k = 1,\dots,n, \ i = 4,\dots,m\}$. Then the required edge-partition is $B \cup D \cup E \cup F \cup G$.

An *automorphism-free* Steiner triple system ([48]) is a Steiner triple system which admits only the trivial automorphism. Lindner and Rosa [48] have constructed an automorphism-free STS of order 15, 19, 25, 27, and 33. They have also proved that if we apply the previous construction to an automorphism-free $STS(n)$, the resulting $STS(2n+7)$ is also automorphism-free.

Moreover, the above construction is easily generalized to a recursive construction of a $TS(2n+7,\lambda)$ given a $TS(n,\lambda)$ (cf. [25, p. 43]). The main difference is that when λ is even, the generalized construction uses the Petersen's theorem on the existence of a 2-factorization of a regular multigraph of even degree instead of a [hooked] Skolem sequence.

3.4. k-rotational triple systems

A triple system $TS(v,\lambda)$ is *k-rotational* ([25]) if it admits an automorphism which has k cycles of equal length and one fixed point. An (f,k)-rotational $STS(v)$ ([40]) is a Steiner triple system which admits an automorphism which has f fixed points and k cycles of length $\frac{v-f}{k}$. Hence, a k-rotational $STS(v)$ is a $(1,k)$-rotational $STS(v)$.

Study of rotational triple systems started with Rosa's work [70] on 1-rotational Steiner triple systems, which he constructed using Skolem sequences. Cho [16] generalized this idea to triple systems $TS(v,\lambda)$. Cho noticed that k-multiple of a 1-rotational $TS(v,\lambda)$ is a 1-rotational $TS(v,k\lambda)$. Hence, it suffices to show the existence of a 1-rotational $TS(v,\lambda)$ for $\lambda = 2,3,6$, which Cho constructed using Skolem and hooked Skolem sequences in a similar way to Rosa's. We present the Rosa's result and the main theorem on the existence of a 1-rotational $TS(v,\lambda)$. For more details on these constructions see [25, Section 7.3].

Lemma 3.4. ([70]) *A 1-rotational $STS(v)$ exists if and only if $v \equiv 3,9 \pmod{24}$.*

Sketch and/or highlights of the proof. The necessary conditions are proved by examining the orbits of triples under the desired automorphism.

Let $V = \{\infty\} \cup \mathbb{Z}_{v-1}$. We want to construct (V,B), a 1-rotational Steiner triple system, which admits the automorphism $\varphi = (\infty)(0\ 1\ 2\ \dots\ v-2)$. When $v \equiv 3,9 \pmod{24}$, let $v = 6n+3$. Then $n \equiv 0,1 \pmod 4$, so there exists a Skolem sequence S of order n. Denote S as a set of ordered pairs $\{(a_i,b_i) : \ b_i - a_i = i, \ i = 1,\dots,n\}$. Then, $\{\infty, 0, \frac{v-1}{2}\}$ is a short block and $\{\{0,i,b_i+n\} : \ i = 1,\dots,n\}$ are base blocks of B under the automorphism φ. □

Theorem 3.5. ([16]) *If $\lambda > 1$, a 1-rotational $TS(v,\lambda)$ exists if and only if a $TS(v,\lambda)$ exists. If $\lambda = 1$, there exists a 1-rotational $STS(v)$ if and only if $v \equiv 3,9 \pmod{24}$.*

The problem of the existence of a 2-rotational Steiner triple system was solved by Phelps and Rosa [70]. Later Cho [17] gave the necessary conditions for the existence of 3- and 4-rotational Steiner triple systems and proved their sufficiency. Jiang [41] solved the existence problem of a k-rotational $STS(v)$ for $k = 5, 7, 11$, and Rosa [81], Doyen [28], and Teirlinck [99] did it for $k = (v-1)/2$. Finally, Colbourn and Jiang [22] completely settled the existence problem of k-rotational $STS(v)$. We illustrate an application of Skolem-type sequences by the Phelps and Rosa's construction of a 2-rotational $STS(v)$, when $v \equiv 1 \pmod{24}$. More details can be found in [25], too.

Theorem 3.6. ([70]) *There exists a 2-rotational* $STS(v)$ *if and only if* $v \equiv 1, 3, 7, 9, 15, 19 \pmod{24}$.

Sketch and/or highlights of the proof. When $v \equiv 1 \pmod{24}$, let $v = 24n+1$, $V = (\mathbb{Z}_{12n} \times \{1,2\}) \cup \{\infty\}$ and let $(\infty)(0_1\ 1_1\ 2_1\ \ldots\ (12n-1)_1)$ $(0_2\ 1_2\ 2_2\ \ldots\ (12n-1)_2)$ be the required automorphism. Consider the following base blocks of B, such that (V, B) is a 2-rotational $STS(v)$ under φ: $\{\infty, 0_1, (6n)_1\}$, $\{\infty, 0_2, (6n)_2\}$, $\{0_1, (4n)_1, (8n)_1\}$, $\big\{\{0_1, i_1, (b_i-1)_2\} :\ i = 1, \ldots, 6n-1,\ i \neq 4n\}$ and $\{0_1, (a_{4n}-1)_2, (b_{4n}-1)_2\big\}$, where $\big\{(a_i, b_i) :\ i = 1, \ldots, 6n-1\big\}$ is a [hooked] Skolem sequence of order $6n-1$. If n is odd, take $\{0_2, 1_2, 2_1\}$ for a base triple, or otherwise if n is even, take $\{0_2, 2_2, 3_1\}$. Finally, if $n = 2$, take also base triples $\{0_2, 1_2, 10_2\}$, $\{0_2, 5_2, 11_2\}$, and $\{0_2, 3_2, 7_2\}$. Otherwise if $n \neq 2$, take base triples $\big\{\{0_2, (c_i+2n)_2, (d_i+2n)_2\} :\ i \in D(n)\big\}$, where when n is odd, $\big\{(c_i, d_i) :\ i \in D(n)\big\}$ is a $(2n)$-extended Langford sequence of defect 2 and length $2n-1$, and set of differences $D(n) = \{2, 3, \ldots, 2n\}$. When n is even, $\big\{(c_i, d_i) :\ i \in D(n)\big\}$ is a $(2n)$-extended 2-near Skolem sequence of order $2n$ with set of differences $D(n) = \{1, 3, 4, \ldots, 2n\}$. The existence of the latter Skolem-type sequence is proved by a direct construction. □

Theorem 3.7. ([22]) *For positive integers v and k such that* $1 \le k \le (v-1)/2$, *a k-rotational* $STS(v)$ *exists if and only if* $v \equiv 1, 3 \pmod 6$, $v \equiv 1 \pmod k$, *and if* $k = 1$, $v \equiv 3 \pmod 6$, *and if* $(v-1)/k$ *is even, then* $v \not\equiv 7, 13, 15, 21 \pmod{24}$.

Note that, Jiang [40] completely solved the existence problem of an $(f, 2)$-rotational Steiner triple system of order v with $f > 1$.

Finally, we present a related concept of bicyclic Steiner triple systems. A $STS(v)$ is *bicyclic* if it admits an automorphism which consists of two disjoint cycles of length v_1 and v_2 such that $v = v_1 + v_2$. Obviously, a 1-rotational $STS(v)$ is bicyclic since it admits an automorphism which has one point fixed and a $(v-1)$-cycle. The necessary conditions for the existence of a bicyclic $STS(v)$ are given by Colbourn and Rosa [25]. The sufficiency of these conditions is proved by exhibiting direct constructions of these systems using Skolem-type sequences (cf. [50]), similarly as in the above proofs. In particular, L i n e k and J i a n g applied most of the already defined Skolem related sequences. They also used a Skolem-type sequence $(s_1, \ldots, s_{24n+15})$ of integers $i \in \{1, \ldots, 12n+9\} \setminus \{8n+5, 12n+8\}$ with $s_{6n+4} = 0$, and a Skolem-type sequence $(s_1, \ldots, s_{24n+21})$ of integers $i \in \{1, \ldots, 12n+12\} \setminus \{8n+7, 12n+11\}$ with $s_{6n+5} = 0$, for $n \geq 0$. For a proof of the following theorem see [50].

Theorem 3.8. ([50]) *There exists a bicyclic Steiner triple system $STS(v)$ admitting an automorphism which consists of two disjoint cycles of length v_1 and v_2, where $v_1 + v_2 = v$ and $1 < v_1 < v_2$ if and only if $v \equiv 1, 3 \pmod 6$, $v_1 \equiv 1, 3 \pmod 6$, $v_1 \neq 9$, $v_1 | v_2$ and*

(i) *if $v_1 \equiv 1 \pmod 6$, then $v_2 \equiv 2v_1 \pmod{6v_1}$,*

(ii) *if $v_1 \equiv 3 \pmod 6$, then $v_2 \equiv 0 \pmod{2v_1}$.*

3.5. Disjoint cyclic Steiner and Mendelsohn triple systems of order v

Two Steiner triple systems, (V, B_1) and (V, B_2), are *disjoint* ([5]) if $B_1 \cap B_2 = \emptyset$. If (V, B_1) and (V, B_2) are cyclic, then they are disjoint if they have no orbits in common. Since every cyclic $STS(6n+3)$ contains a short base block $\{0, 2n+1, 4n+2\}$, they are never disjoint. Denote by $n_c(v)$ the maximum number of pairwise disjoint cyclic $STS(v)$'s on the set of vertices $\{0, 1, \ldots, v-1\}$ which admit the automorphism $(0\ 1\ \ldots\ v-1)$.

Since a [hooked] Skolem sequence of order n directly implies a solution to Heffter's first difference problem and generates base blocks of a cyclic $STS(6n+1)$, the research of disjoint cyclic $STS(6n+1)$ led to the definition and study of disjoint [hooked] Skolem sequences ([5]). Two [hooked] Skolem sequences of order n, $\{(a_i, b_i) : i = 1, \ldots, n\}$ and $\{(c_i, d_i) : i = 1, \ldots, n\}$ are *disjoint* ([5]) if $\{a_i, b_i\} \neq \{c_i, d_i\}$ for all $i = 1, \ldots, n$. A Skolem sequence S is *reverse disjoint* if S and the reverse of S are disjoint.

Given a [hooked] Skolem sequence of order n, $\{(a_i, b_i) : i = 1, \ldots, n\}$, it generates two disjoint sets of base blocks of a cyclic $STS(6n+1)$:

(i) $\{\{0, i, b_i + n\} : i = 1, \ldots, n\}$ and

(ii) $\{\{0, a_i + n, b_i + n\} : i = 1, \ldots, n\}$.

Moreover, two disjoint [hooked] Skolem sequences of order n generate disjoint sets of base blocks of cyclic $STS(6n+1)$. Therefore, two disjoint [hooked] Skolem sequences of order n imply the existence of four mutually disjoint cyclic Steiner triple systems of order $6n+1$.

Theorem 3.9. ([5]) *There exist at least four mutually disjoint Skolem sequences of order n, when $n \equiv 0, 1 \pmod 4$ and $n \geq 4$. Hence, if $v \equiv 1, 7 \pmod{24}$, $v \geq 25$, then $n_c(v) \geq 8$.*

Theorem 3.10. ([5]) *There are at least three mutually disjoint hooked Skolem sequences of order n, for $n \equiv 2, 3 \pmod 4$, $n \geq 6$. Therefore, if $v \equiv 13, 19 \pmod{24}$, $v \geq 37$, then $n_c(v) \geq 6$.*

A simple consequence of these theorems is a lower bound on the number of disjoint cyclic Mendelsohn triple systems. A cyclic 3-set $\langle a, b, c\rangle$ contains ordered pairs (a, b), (b, c) and (c, a). A *Mendelsohn triple system* ([59]) of order v, $MTS(v)$, is a pair (V, B) where $|V| = v$ and B is a set of cyclic 3-subsets of V such that each ordered pair of elements in V is contained in exactly one member of B. The definitions of cyclic and disjoint $MTS(v)$ are similar to the definitions of cyclic and disjoint $STS(v)$, respectively.

If (V, B) is a $STS(v)$, then (V, B') is a $MTS(v)$, where $B' = \big\{\langle a, b, c\rangle, \langle a, c, b\rangle : \{a, b, c\} \in B\big\}$. If we denote by $m_c(v)$ the maximum number of mutually disjoint cyclic $MTS(v)$, then $m_c(v) \geq 8$ when $v \equiv 1, 7 \pmod{24}$ and $v \geq 25$, and $m_c(v) \geq 6$ when $v \equiv 13, 19 \pmod{24}$ and $v \geq 37$.

3.6. Simple and indecomposable triple systems

A triple system $TS(v, \lambda)$ is called *simple* ([76]) if each triple in its set of triples appears exactly once. On the other hand, if (V, B) is a $TS(v, \lambda)$, it is *indecomposable* if B cannot be written as disjoint union of sets B_1 and B_2 such that (V, B_1) is a $TS(v, \lambda_1)$ and (V, B_2) is a $TS(v, \lambda_2)$ and $\lambda_1 + \lambda_2 = \lambda$.

Similarly to the way one constructs cyclic $STS(v)$ using Skolem and Rosa sequences in Section 3.1, one can construct cyclic $TS(v, 2)$ from two-fold Skolem and Rosa sequences. Here we present four constructions which under certain conditions give indecomposable $TS(v, 2)$.

<u>Construction 1.</u> ([75], [76]) Let $S_1 = \big\{(a_i, b_i), (c_i, d_i) : \ i = 1, \ldots, n, \ b_i - a_i = d_i - c_i = i\big\}$ be a two-fold Skolem sequence of order n. Then the triples $\big\{\{i, a_i + n, b_i + n\}, \{6n + 1 - i, c_i + n, d_i + n\} : \ i = 1, \ldots, n\big\}$ partition the set $\{1, \ldots, 6n\}$ into difference triples. Therefore, $\big\{\{0, i, b_i + n\}, \{0, i, d_i + n\} : i = 1, \ldots, n\big\}$ are base blocks of a $TS(6n + 1, 2)$ on $V = \mathbb{Z}_{6n+1}$ under the automorphism $(0\ 1\ \ldots\ 6n)$. Moreover, if S_1 is such that there is a pair $(x_i, y_i) \in \big\{(a_i, b_i), (c_i, d_i) : \ i = 1, \ldots, n, \ b_i - a_i = d_i - c_i = i\big\}$ and $x_i + y_i = 4n + 1$, then the resulting $TS(6n + 1, 2)$ is indecomposable.

<u>Construction 2.</u> ([75], [76]) Let $S_2 = \big\{(a_i, b_i), (c_i, d_i) : \ i = 1, \ldots, n, \ b_i - a_i = d_i - c_i = i\big\}$ be a two-fold Rosa sequence of order n. Then, base blocks of a cyclic $TS(6n + 3, 2)$ on $V = \mathbb{Z}_{6n+3}$ under the automorphism $(0\ 1\ \ldots\ 6n + 2)$ are $\big\{\{0, i, b_i + n\}, \{0, i, d_i + i\} : \ i = 1, \ldots, n\big\} \cup \{0, 2n + 1, 4n + 2\} \cup \{0, 2n + 1, 4n + 2\}$. If S_2 is such that there is a pair $(x_i, y_i) \in \big\{(a_i, b_i), (c_i, d_i) : \ i = 1, \ldots, n, \ b_i - a_i = d_i - c_i = i\big\}$ satisfying $x_i + y_i = 4n + 3$, then the resulting $TS(6n + 3, 2)$ is indecomposable.

Construction 3. ([76]) If $S_3 = \{(a_i, b_i) : i = 1, \ldots, n\}$ is a Skolem sequence of order n, then, $\{0, i, b_i + n\}$ are base blocks of a cyclic two-fold triple system $TS(3n+1, 2)$. Moreover, if we consider S_3 as a sequence $S_3 = (s_1, \ldots, s_{2n})$, and if $s_{2n-1} = s_{2n} = 1$, then the resulting $TS(3n+1, 2)$ is indecomposable.

Construction 4. ([76]) If $S_4 = \{(a_i, b_i) : i = 1, \ldots, n\}$ is a Rosa sequence of order n, then $\{\{0, i, b_i + n + 1\} : i = 1, \ldots, n\}$ are base blocks of a cyclic $TS(3n+3, 2)$. Again, if we consider S_4 as a sequence $S_4 = (s_1, \ldots, s_{2n+1})$, and if $s_{2n} = s_{2n+1} = 1$, then the obtained $TS(3n+3, 2)$ is indecomposable.

Theorem 3.11. ([76]) *The $TS(v, 2)$'s in* constructions 1. *and* 3. *are simple.*

Theorem 3.12. ([76]) *An indecomposable simple two-fold triple system $TS(v, 2)$ exists if and only if $v \equiv 0, 1, 3, 4, 7, 9 \pmod{12}$ and either $v = 4$ or $v \geq 12$.*

3.7. Partial triple systems

The following definitions about partial triple systems are from [25].

A *partial triple system $PTS(v, \lambda)$* consists of a set V of v elements and a set B of 3-subsets of V such that every unordered pair of elements of V appears in at most λ triples in B. A *leave* of a $PTS(v, \lambda)$ (V, B) is a multigraph on the vertex set V such that an edge $\{x, y\}$ appears $\lambda - i$ times, where i is the number of triples in B which contain the pair $\{x, y\}$.

An *incomplete triple system $ITS(v, w; \lambda)$* of order v, index λ, with hole of size w is a $PTS(v, \lambda)$ (V, B) such that there exists $W \subseteq V$, $|W| = w$, and for every $x, y \in W$, no triple in B contains the pair $\{x, y\}$, but for every $x \in V \backslash W$ and $y \in V$, there are exactly λ triples in B containing the pair $\{x, y\}$. If there exists a triple system $TS(w, \lambda)$ (W, D), then it is called a *subsystem* of a triple system $(V, B \cup D)$.

A *maximal partial triple system $MPT(v, \lambda)$* is a $PTS(v, \lambda)$ (V, B) such that there is no $PTS(v, \lambda)$ (V, B') and $B \subset B'$. If the leave of a $PTS(v, \lambda)$ contains no triangles, then the $PTS(v, \lambda)$ is maximal.

Baker [6] gives a construction of a cyclic partial triple system $PTS(6n+3, 1)$ generated from a k-extended Skolem sequence $S = \{(a_i, b_i) : i = 1, \ldots, n\}$ of order n. Let $V = \mathbb{Z}_{6n+3}$, and consider the following sets of base triples:

$$\begin{aligned} \mathscr{B}_1 &= \{(0, i, b_i + n) : i = 1, \ldots, n\} \\ \mathscr{B}_2 &= \{(0, a_i + n, b_i + n) : i = 1, \ldots, n\}. \end{aligned}$$

Developing either $\mathscr{B}_1$ or $\mathscr{B}_2$ modulo $6n + 3$, we generate a $PTS(6n + 3, 1)$. Moreover, the resulting partial triple systems are disjoint, that is, they have no triples in common. Also, they are maximal partial triple systems, unless $k = n + 1$ and $n \equiv 0, 3 \pmod 4$.

The necessary conditions for the existence of an $ITS(v, w; \lambda)$ are easily obtained, and [hooked] Skolem or [hooked] Rosa sequences are applied in direct constructions of these systems.

Lemma 3.13. ([25]) *An $ITS(v,w;\lambda)$, when $\lambda > 0$, exists only if $v \geq 2w+1$ or $v = w$, and*

$$\lambda\left(\binom{v}{2} - \binom{w}{2}\right) \equiv 0 \pmod 3,$$
$$\lambda(v-1) \equiv \lambda(v-w) \equiv 0 \pmod 2.$$

If there exists an $ITS(v,w;\lambda)$ $(V,\mathscr{B})$ with a hole W, then let $X = V\backslash W$, and consider a multigraph G on the vertex set X with the edge set $\bigcup_{w\in W} E_w$, where $E_w = \big\{\{x,y\} : \{x,y,w\} \in \mathscr{B}\big\}$. Notice that $\{E_w : w \in W\}$ is a λ-factorization of G. Moreover, $\left(X, \bigcup_{w\in W} E_w \cup \mathscr{B}'\right)$, where $\mathscr{B}' = \{B : B \in \mathscr{B}, B \cap W = \emptyset\}$, is a λK_{v-w}. Such a partition of the edge set of λK_{v-w} is called $(v-w,w;\lambda)$*-decomposition* (cf. [25]).

Obviously, an $ITS(v,w;\lambda)$ exists if and only if there exists a $(v-w,w;\lambda)$-decomposition. Proofs of the following lemmas are all based on application of a [hooked] Skolem or Rosa sequence to choose a set of triples $\mathscr{B}'$. We shall illustrate this on a sample case. Then, a λ-factorization of the remaining edges is obtained from either the Petersen's theorem ([25, Theorem 1.22, p. 21]) about 2-factorization, or the Stern and Lenz's, or the Hartman's theorem about 1-factorization of cyclic graphs of even order ([25, Theorem 1.16, p. 19]).

Lemma 3.14. ([25]) *A $(v,w;\lambda)$-decomposition exists for all $w = t-1,\ldots$ $\ldots, 6n+t-1$ such that $w \equiv t-1 \pmod{6/\gcd(6,\lambda)}$, when $v = 6n+t$, $t \geq 1$, $n \equiv 0,1 \pmod 4$ and $t\lambda$ is even.*

S k e t c h a n d/o r h i g h l i g h t s o f t h e p r o o f. Let $\big\{(a_i,b_i) : i = 1,\ldots$ $\ldots, n\big\}$ be a Skolem sequence. Now, take $\mathscr{O}_i = \big\{\{j, a_i+j+n, b_i+j+n\} : j = 0,\ldots,v-1\big\}$ and define $\mathscr{O}_k = \mathscr{O}_i$ when $k \equiv i \pmod n$. Then, $\mathscr{B}' = \bigcup_{i=1}^{l} \mathscr{O}_i$ where $l = \lambda n - (\lambda(w-(t-1)))/6$. □

Lemma 3.15. ([25]) *A $(v,w;\lambda)$-decomposition exists for all $w = t-1,\ldots$ $\ldots, 6n+t-1$ such that $w \equiv t-1 \pmod{6/\gcd(6,\lambda)}$, when $v = 6n+t$, $t \geq 3$, $n \equiv 2,3 \pmod 4$ and $t\lambda$ is even.*

Lemma 3.16. ([25]) *Let $n \geq 0$, $v = 6n+t$ and $t \equiv 0 \pmod 3$. Let $\delta \in \{0,1,2\}$, such that $\delta\lambda$ is even. Then, a $(v,w;\lambda)$-decomposition exists for all $w = t-1-\delta,\ldots,v-1-\delta$ such that $w \equiv t-1-\delta \pmod{6/\gcd(6,\lambda)}$, when $t\lambda$ is even.*

Theorem 3.17 (Doyen-Wilson). ([25]) *An $ITS(v,w;1)$ with a subsystem of order w exists if $w \equiv 1,3 \pmod 6$ or $w = 0$, $v \equiv 1,3 \pmod 6$, and $v \geq 2w+1$.*

On the other hand, Lemmas 3.14 and 3.15 have as a corollary a theorem about the existence of incomplete triple systems, whose hole is not necessarily a subsystem.

Theorem 3.18. ([25]) *An $ITS(v, w; 1)$ exists if $w \equiv v \equiv 5 \pmod 6$ and $v \geq 2w + 1$.*

Now, we consider leaves of partial triple systems when $\lambda = 1$, called 1-leaves. Since leave of a $PTS(v)$ is obtained by removing triangles from a complete graph K_v, the necessary conditions for a graph to be a leave of a $PTS(v)$ follow.

A graph is said to be *quadratic* if every vertex has degree zero or two. Colbourn and Rosa [19] applied method of differences with Langford sequences to establish that every quadratic graph which satisfies necessary conditions is a 1-leave.

Theorem 3.19. *Let G be a graph with e edges, and whose every vertex has degree zero or two. If $e \equiv 0 \pmod 3$ when $v \equiv 1, 3 \pmod 6$, or $e \equiv 1 \pmod 3$ when $v \equiv 5 \pmod 6$, then G is a 1-leave, except if $v = 7$ and $G = 2\{C_3\} \cup K_1$, or $v = 9$ and $G = C_4 \cup C_5$.*

The *neighborhood* graph $N(z)$ of a $TS(v, \lambda)$ (V, B) is a graph on vertex set $V \setminus \{z\}$ and edge set $\big\{\{x, y\} : \{x, y, z\} \in B\big\}$. Colbourn and Rosa [20] used the same approach with method of differences and Langford sequences to prove the following theorem.

Theorem 3.20. *Every 2-regular multigraph G on $v \equiv 0, 1 \pmod 3$ vertices is a neighborhood graph of an element in a $TS(v, 2)$, except when $G = C_2 \cup C_3$ or $G = C_3 \cup C_3$.*

4. λ-coverings and balanced ternary designs

The fact that a Skolem sequence of order n generates a partition of the set $\{1, 2, \ldots, 3n\}$ into difference triples is very useful for constructions of a 1-covering of pairs by triples, when repeated elements are allowed in the triples (cf. [13], [5]). A similar property of many Skolem-type sequences, especially 1-near Skolem sequences, is also very useful. Billington defines near Skolem sequences in terms of "pairings" (cf. [63], [13]), but uses 1-near Skolem sequences in her constructions, which are equivalent to Langford sequences of defect 2.

A t-(v, k, λ) *covering* ([23]) is a pair (V, B), where V is a set of v elements and B is a set of k-subsets of V, called blocks, such that every t-subset of elements in V occurs in at least λ blocks in B, when $v \geq k \geq t$. Repeated elements are allowed in blocks of B as well as in t-subsets of V. The *covering number* $C_\lambda(v, k, t)$ is the least number of blocks in a t-(v, k, λ) covering.

Theorem 4.1. ([13]) $C_1(v,3,2) = \lceil \frac{v(v+3)}{6} \rceil$.

Sketch and/or highlights of the proof. The inequality $C_1(v,3,2) \geq \lceil \frac{v(v+3)}{6} \rceil$ follows from observation that there must exist a block in B of the form $\{a,a,b\}$ for all $a \in V$. Then calculate the minimum number of 3-subsets of V.

The opposite inequality is proved by providing a construction of a 2-$(v,3,1)$ covering with exactly $\lceil \frac{v(v+3)}{6} \rceil$ blocks. We illustrate the application of Skolem-type sequences in the case when $v \equiv 1,7 \pmod{24}$. Let $v = 6n+1$. There exists a 1-near Skolem sequence of order n, $\{(a_i,b_i) : i = 2,\dots,n\}$. Take $\{0,0,3n-1\} \cup \{\{0,i,b_i+n\} : i = 2,\dots,n\}$ for base blocks developed modulo $6n-1 = v-2$. The remaining triples are: $\{\{0+2j,1+2j,6n-1\} : j = 0,\dots,3n-2\} \cup \{\{1+2j,2+2j,6n\} : j = 0,\dots,3n-2\} \cup \{\{6n-2,0,6n-1\},\{6n-1,6n-1,6n\},\{6n,6n,0\}\}$. □

Two t-(v,k,λ) coverings of a set are *disjoint* ([5]) if they have no blocks in common. Let $\max_1(v)$ be the maximum number of mutually disjoint 2-$(v,3,1)$ coverings. Similarly as in the construction of a disjoint $STS(v)$, the existence of disjoint [hooked] Skolem sequences and disjoint near Skolem sequences together with a modification of the above construction of 2-$(v,3,1)$ coverings yield four disjoint 2-$(v,3,1)$ coverings.

Theorem 4.2. ([5]) *A reverse disjoint k-near Skolem sequence of order n exists if and only if $n \equiv 0,1 \pmod 4$ and k is odd, or $n \equiv 2,3 \pmod 4$ and k is even.*

Theorem 4.3. ([5]) *If $v \not\equiv 4 \pmod 6$ and $v > 3$, then $\max_1(v) \geq 4$.*

A *balanced ternary design* $BTD(V,B;\rho_1,\rho_2,R;K,\Lambda)$ ([23]) is a design on V elements arranged in B sets all of size K, called blocks, such that each element appears exactly once in ρ_1 blocks and exactly twice in ρ_2 blocks, and each unordered pair of distinct elements appears Λ times in B. Hence, each element appears $\rho_1 + 2\rho_2 = R$ times in B, and if m_{ib} equals number of times an element i appears in a block b, then $\sum_{b=1}^{B} m_{ib}m_{jb} = \Lambda$ for all $i \neq j$.

Billington [13], [14] completely determined all $CBTD(\rho_2,3,2;V)$ i.e. cyclic $BTD(V,B;\rho_1,\rho_2,R;3,2)$. The main idea for the construction of a $CBTD(\rho_2,K,\Lambda;V)$ resembles the method of the construction of a cyclic Steiner triple systems. These constructions apply a variety of Skolem-type sequences, as well as sequences which satisfy all properties of 2-fold Skolem sequences with two hooks and in addition, one position is shared by two elements in a sequence.

As an illustration, we present a construction of a $CBTD(1,3,2;6n)$ (cf. [13]). An n-near 2-fold Skolem sequence, $S(n)$, with two null elements at the positions $n+1$ and $3n+1$ exists if and only if n is even. The $S(2) = \{(1,2),(4,5),(6,8)\}$.

To generate a $CBTD(1,3,2;12)$ from $S(2)$, take base blocks $\big\{\{0,0,4\},\{0,1,3\},\{0,1,6\},\{0,2,5\}\big\}$ and develop them modulo 12.

Otherwise, for $n \geq 4$, n even, $S(n) = \big\{(2+i, n+1-i) : i = 1,\dots,n/2-1\big\} \cup \big\{(n+1+i, 2n+3-i) : i = 1,\dots,n/2\big\} \cup \big\{(2n+2+i, 3n+1-i) : i = 1,\dots,n/2-2\big\} \cup \big\{(3n+1+i, 4n+1-i) : i = 1,\dots,n/2-1\big\} \cup \big\{(1,2), (3n/2+2, 5n/2+1), (5n/2+2, 7n/2+1)\big\}$. Then, sequence $S(n)$ gives difference triples, which give the following base blocks $\big\{\{0,0,2n\},\{0,1,n+1\}, \{0,n-1,(7n-2)/2\},\{0,n-1,(5n-1)/2\}\big\} \cup \big\{\{0,n-1-2i,2n-i\} : i = 1,\dots,n/2-1\big\} \cup \big\{\{0,n+2-2i,4n-i\} : i = 1,\dots,n/2\big\} \cup \big\{\{0,n-1-2i, 3n-1-i\} : i = 1,\dots,n/2-2\big\} \cup \big\{\{0,n-2i,2n-i\} : i = 1,\dots,n/2-1\big\}$ of a $CBTD(1,3,2;6n)$. Notice that the construction of $CBTD(1,3,2;6n)$ depends on the construction of $S(n)$.

5. Graph decompositions

Here we shall look into the relationship between Skolem-type sequences, starters, and factorizations of complete graphs. Also, we present the concept of cyclic m-cycle systems, and how Skolem-type sequences relate to them.

An *r-factor* ([23]) of a multigraph G is an r-regular spanning submultigraph of G, i.e. a submultigraph whose vertex set coincides with vertex set of G and whose vertices all have degree r. An *r-factorization* of a multigraph G is a set $\{F_1, F_2, \dots, F_k\}$, where F_i's are edge-disjoint r-factors of G and the edge set of G is partitioned by the edge sets of F_i's.

Given an additive group G of odd order $|G| = 2n+1$, a *starter* ([23]) is a set of unordered pairs $\big\{\{a_i,b_i\} : i = 1,\dots,n\big\}$ such that $\bigcup_{i=1}^{n} \{a_i,b_i\} = G \setminus \{0\}$ and $\big\{\pm(a_i - b_i) : i = 1,\dots,n\big\} = G \setminus \{0\}$. On the other hand, given an additive group G of even order $|G| = 2n$, an *even starter* ([72]) is a set of unordered pairs $\big\{\{a_i,b_i\} : i = 1,\dots,n-1\big\}$ such that $\bigcup_{i=1}^{n-1} \{a_i,b_i\} = G \setminus \{0,m\}$ and $\big\{\pm(a_i - b_i) : i = 1,\dots,n-1\big\} = G \setminus \{0,k\}$, where $m \neq 0$, $k \neq 0$.

A Skolem and an extended Skolem sequence of order n can be seen as a starter in $\mathbb{Z}_{2n+1}$ and an even starter in $\mathbb{Z}_{2n+2}$, respectively. For example, $(4,5,1,1,4,3,5,2,3,2)$ is a Skolem sequence of order 5, or equivalently $\big\{\{3,4\}, \{8,10\},\{6,9\},\{1,5\},\{2,7\}\big\}$ is a starter in $\mathbb{Z}_{11}$. Also, $(2,0,2,1,1)$ is a 2-extended Skolem sequence of order 2, or equivalently $\big\{\{4,5\},\{1,3\}\big\}$ is an even starter in $\mathbb{Z}_6$.

However, not all starters are Skolem sequences. For example, the *patterned* starter $\big\{\{g,-g\} : g \in G \setminus \{0\}\big\}$ is *not* a Skolem sequence.

There is a 1-factorization of a complete graph on $2n$ vertices, K_{2n}, known since the mid 19th century called $GK(K_{2n})$, generated from starters (when asked what the GK stood for, Rosa replied "**G**od **K**nows" — *EM*). Given a starter S in Z_{2n+1}, let $F = S \cup \{\infty, 0\}$. Denote by $F + i = \big\{\{a+i, b+i\} : \{a,b\} \in F\big\}$. Then $\big\{F + i : i \in \mathbb{Z}_{2n+1}\big\}$ is a 1-factorization of K_{2n+2} on vertex set $V = \{\infty\} \cup \mathbb{Z}_{2n+1}$ (cf. [10]). Similarly, given an even starter ES in $\mathbb{Z}_{2n}$, where m is the omitted element in the definition of an even starter above, let $F_1 = ES \cup \big\{\{\infty_1, m\}, \{\infty_2, 0\}\big\}$ and $F_2 = \{\infty_1, \infty_2\} \cup \big\{\{i, i+n\} : i = 0, \dots, n-1\big\}$. Then $\big\{F_1 + i : i \in \mathbb{Z}_{2n}\big\} \cup \{F_2\}$ is a 1-factorization of K_{2n+2} on the vertex set $\mathbb{Z}_{2n} \cup \{\infty_1, \infty_2\}$ (cf. [10]).

The relationship between starters, Skolem sequences and 1-factorizations of K_{2n} was studied in [71], [72], [73]. Particular interest was given to perfect 1-factorizations. A 1-factorization $\{F_1, \dots, F_k\}$ of a graph G is *perfect* ([23]) if the union of the edge sets of F_i and F_j is a Hamiltonian cycle for ever $i, j \in \{1, \dots, k\}$, $i \neq j$. Also, it was of interest to count the number of orthogonal perfect 1-factorizations, because of their connection to perfect room squares. The 1-factorizations $\mathscr{F}_1$ and $\mathscr{F}_2$ of a multigraph are *orthogonal* ([23]) if every 1-factor in $\mathscr{F}_1$ has at most one edge in common with each 1-factor in $\mathscr{F}_2$.

Pike and Shalaby [72] calculated that there is a greater probability to generate a perfect 1-factorization from Skolem sequences constructed by the hill-climbing algorithm, than from starters not induced by Skolem sequences. In [73], they improved known results on the number of non-isomorphic perfect 1-factorizations of K_{2n} for $9 \leq n \leq 20$, $n \neq 19$, by considering starters generated from Skolem sequences. Also, they improved the known values of the largest number of mutually orthogonal perfect 1-factorizations of K_n for $n \in \{18, 22, 26, 28, 32, 34, 36\}$.

Colbourn and Rosa [21] were interested in constructing a 2-factorization of K_{2n} with a given number of repeated edges, where a repeated edge is a cycle of length 2. In some constructions of the desired 2-factorizations, they applied Skolem and Langford sequences.

Finally, Skolem-type sequences are applied in the decomposition of graphs into cycles. Given a graph G on vertex set V, let $\mathscr{C}$ be a collection of cycles of length m which partition the set of edges of G. Then $(V, \mathscr{C})$ is called an *m-cycle system* of G (cf. [31]). The m-cycle system $(V, \mathscr{C})$ is *resolvable* if $\mathscr{C}$ can be partitioned into 2-factors of G.

A bijective mapping φ of vertex set V onto itself is an automorphism of $(V, \mathscr{C})$ if the image of every cycle $C \in \mathscr{C}$ under φ is again in $\mathscr{C}$. Similar to triple systems, a $(V, \mathscr{C})$ is 1-rotational if it admits an automorphism which is a $(|V|-1)$-cycle. We illustrate how Skolem-type sequences are used in the proof of the following theorem. We denote the fixed vertex by ∞, and assume that $V = \{\infty\} \cup \mathbb{Z}_{|V|-1}$.

We shall construct $\mathscr{C}$ so that $\varphi = (\infty)(0\ 1\ 2\ \ldots\ |V|-2)$ is an automorphism of $(V, \mathscr{C})$.

Theorem 5.1. ([31]) *There exists a* 1*-rotationally resolvable* 4*-cycle system of* $2K_v$ *if and only if* $v \equiv 0 \pmod 4$.

Sketch and/or highlights of the proof. The necessary conditions for existence are obtained from study of orbits of $\mathscr{C}$ under φ.

On the other hand, the sufficiency is proved by two constructions. Let $v = 4n+4$. If $n \equiv 0, 3 \pmod 4$, then there exists a Rosa sequence of order n, $\{(a_i, b_i) :\ i = 1, \ldots, n\}$. Hence, the 4-cycles $\{(2a_i - 1, 2b_i - 1, 2a_i, 2b_i) :\ i = 1, \ldots, n\} \cup \{(\infty, 0, 2n+1, 2n+2)\}$ are base cycles and base resolution class under φ of a 1-rotationally resolvable 4-cycle system of $2K_{4n+4}$.

Similarly, if $n \equiv 1, 2 \pmod 4$, then there exists $\{(c_i, d_i) :\ i = 1, \ldots, n\}$, an n-extended Skolem sequence of order n. If also $(c_n, d_n) = (n+1, 2n+1)$, then $\{(2c_i - 1, 2d_i - 1, 2c_i, 2d_i) :\ i = 1, \ldots, n-1\} \cup \{(0, 2n, 4n+1, 2n-1), (\infty, 2n+1, 2n+2, 4n+2)\}$ are base cycles (and a base resolution class) under φ of a 1-rotationally resolvable 4-cycle system of $2K_{4n+4}$. □

Notice that a $STS(v)$ is actually a 3-cycle system on K_v. Bryant, Gavlas and Ling [15] extended the idea of building cyclic $STS(v)$ from difference triples to construct cyclic m-cycle systems of K_{2mt+1}, when $m \geq 3$ and $t \geq 1$, and of $K_{2mt+2} - F$ when $mt \equiv 0, 3 \pmod 4$, $m \geq 3$, $t \geq 1$, and F is a 1-factor of K_{2mt+2}. They defined Skolem-type difference m-tuples, and [hooked] Skolem-type m-cycle difference sets which correspond to base blocks of a cyclic m-cycle system of a graph. They used the fact that a Skolem sequence of order n partitions the set $\{1, \ldots, 2n\}$ and the edges corresponding to the ordered pairs of the Skolem sequence have lengths 1 to n, to construct the desired sets of difference m-tuples.

6. Graph labeling

The relationship between Skolem sequences and graph labelings was studied by several authors. It is interesting that Skolem sequences were applied in many different ways to graph labeling. We present two, the graceful labeling of trees and cycles, and the Skolem labeling.

6.1. Graceful labeling of graphs

Let $G = (V, E)$ be a graph on a vertex set V with an edge set E. The injective mapping $\varphi \colon V \to \{0, 1, \ldots, |E|\}$ is graceful labeling of G if $\Phi \colon E \to \{1, \ldots, |E|\}$, where $\Phi\big(\{x,y\}\big) = |\varphi(x) - \varphi(y)|$ is a bijection. If graph G admits a graceful labeling then G is graceful. The weight of an edge $\{x,y\}$ is $\Phi(\{x,y\})$.

Lee and Shee [46] defined a variation of graceful labeling called Skolem graceful labeling. A graph $G = (V,E)$ is *Skolem graceful*, if there exists an injective mapping $\varphi\colon V \to \{1,2,\dots,|V|\}$ such that the induced mapping $\Phi\colon E \to \{1,2,\dots,|E|\}$ is a bijection. This labeling is also called *node graceful* ([36]). Given a Skolem sequence $\big\{(a_i,b_i) : \ i = 1,\dots,n\big\}$ of order n, taking edges $\{a_i,b_i\}$ on the vertex set $\{1,2,\dots,2n\}$, we get n copies of a path of length one, denoted by nP_2. Obviously, the graph nP_2 is Skolem graceful if and only if $n \equiv 0,1 \pmod 4$. Also, nP_2 can be considered as a 1-factor of a graph on the same vertex set, which is the starting step of the constructions of graceful trees in [61] and [62].

Abrham [4] showed the correspondence between graceful simple 2-regular graphs containing at most one cycle of odd length and a particular kind of Skolem sequences. Let $S = (s_0,s_1,\dots,s_{2n+1})$ be a Skolem sequence of order $n+1$, such that if $s_j = s_{j+i} = i$, then either $j+i \le n$ or $j > n$ for $i \ne n+1$, and if n is odd, also for $i \ne \frac{n+1}{2}$. When n is odd, if $i = \frac{n+1}{2}$ and $s_j = s_{j+i} = \frac{n+1}{2}$, then $j \le n$ and $j+i > n$. Now, we can consider S also as a set of pairs $\big\{(a_i,b_i) : \ i = 1,\dots,n+1\big\}$. To label a 2-regular graph G on n vertices, we pick a cycle in G and a direction in which to go around the cycle. If n is odd, we first label the edge of weight $l = \frac{n+1}{2}$, otherwise we can label the first edge by any weight l smaller than $n+1$. Label vertices of the chosen edge by a_l, b_l modulo $n+1$ following the chosen direction. Now, we pick a label for the vertex adjacent to the last labeled vertex b_l. Let $k \equiv b_l + n + 1 \pmod{2n+2}$, and let $k \in \{a_j,b_j\}$, where $(a_j,b_j) \in S$, and without loss of generality, assume $k = a_j$. Then label the next vertex by b_j congruent modulo $n+1$. Continue the process until all vertices are labeled.

For example, consider the sequence $(2,3,2,5,3,4,1,1,5,4)$ and a square $ABCD$. If we first label the edge AB and assign the weight 4 to it, then we get the following labeling: $A = 0$, $B = 4$, $C = 1$, and $D = 2$.

A variation of graceful labeling is ordered graceful labeling. A graceful labeling of a tree T is *ordered graceful* if when the edges of the tree T are directed from the vertex with greater label to the vertex with smaller label, every vertex has either indegree or outdegree equal to 0. Nowakowski and Whitehead [67] applied Skolem sequences of order n to construct 2^n distinct ordered graceful labelings of a 2-star of order $6n+1$ with central vertex labeled 1. They modified this idea to cover all remaining cases, and proved that there are 2^n distinct ordered graceful labelings of a 2-star of order $6n+1$ or $6n+3$ with central vertex labeled 1. Here we present the idea of the construction which uses Skolem sequences.

Let G be a 2-star of order $6n+1$, and $S = \{(a_i, b_i) : i = 1, \ldots, n\}$ be a Skolem sequence of order n (hence, $n \equiv 0, 1 \pmod 4$). Then, we can label any three 2-paths from the center by $(1, 6n+2-i, a_i+n+1)$, $(1, 5n+2-a_i, 5n+2-b_i)$ and $(1, b_i+n+1, i+1)$, or by $(1, 6n+2-i, 5n+2-b_i)$, $(1, 5n+2-a_i, i+1)$ and $(1, b_i+n+1, a_i+n+1)$ for an $i \in \{1, \ldots, n\}$. It follows that there are 2^n distinct ordered graceful labelings of G.

6.2. Skolem labeled graphs

The problem of developing an efficient schedule for testing a communication network reliability such that every node, link and distance are tested led to the definition of Skolem labeled graphs, given by Mendelsohn and Shalaby [56]. In the following, let $G = (V, E)$ be an undirected graph and $d_G(u, v)$ be equal to the smallest length of a path connecting vertices u and v.

Definition 6.1. ([56]) A *(strongly) d-Skolem labeled graph* is a triple (G, φ, d) such that

(i) $\varphi\colon V \to \{d, d+1, \ldots, d+n-1\}$,
(ii) for every $i \in \{d, d+1, \ldots, d+n-1\}$ there are exactly two vertices $a_i, b_i \in V$ such that $\varphi(a_i) = \varphi(b_i) = i$ and $d_G(a_i, b_i) = i$, and
(iii) if $G' = (V, E')$ where $E' \subset E$, then (G', φ, d) violates (ii).

If a triple (G, φ, d) satisfies only properties (i) and (ii), then it is called a *weakly d-Skolem labeled graph*. Also, a *hooked d-Skolem labeled graph* is a triple (G, φ, d) such that $\varphi\colon V \to \{0\} \cup \{d, d+1, \ldots, d+n-1\}$ and it satisfies the properties (ii) and (iii) from the above definition. A hooked Skolem labeling of a graph G with the smallest possible number of hooks is called a *minimum hooked Skolem labeling*. By convention, if the value of d in a Skolem labeling is not stated, it is understood that $d = 1$.

Obviously, given a Skolem sequence $S = (s_1, s_2, \ldots, s_{2n})$ of order n, the path of length $2n-1$, s_1 ___ s_2 ___ $\ldots$ ___ s_{2n} is Skolem labeled. Therefore, the existence of Skolem and hooked Skolem sequences is equivalent to Skolem and hooked Skolem labelings of paths. Similarly, a (hooked) Langford sequence of defect d and length l yields a d-Skolem labeled path of length $2l-1$.

Mendelsohn and Shalaby [56] completely settled the existence of minimum hooked Skolem labeling of paths and cycles. Note that at most two hooks are needed for any path or a cycle to be hooked Skolem labeled. Hence, a Skolem labeled graph (V, E) containing an induced path or a cycle of length n with least number of vertices has $|V| \leq n+2$. In proofs of these theorems, Mendelsohn and Shalaby label paths and cycles directly.

Mendelsohn and Shalaby also showed in [56] that any tree on v vertices can be embedded as an induced subgraph of a Skolem labeled tree on $O(2v)$

vertices. Moreover, every graph with v vertices can be embedded as an induced subgraph of a Skolem labeled graph on $O(v^3)$ vertices.

A two-dimensional Skolem array of order n is a generalization of Skolem-type sequences, which yields a Skolem labeling of a ladder graph $K_2 \Box P_n$, the Cartesian product of K_2 and a path of length $n-1$, P_n (cf. [7]). We present several definitions related to Skolem arrays.

Definition 6.2. ([7]) A *Skolem array* SA of order n is a $2 \times n$ array of integers $i \in \{1, 2, \ldots, n\}$ such that each i occurs exactly twice in the array at positions (a_i, b_i) and (c_i, d_i) and $|c_i - a_i| + |d_i - b_i| = i$. The pair (i, i) is *split* in SA if integer i appears in both rows of SA. Otherwise, the pair (i, i) is *nonsplit.*

Definition 6.3. ([7]) A Skolem array SA of order n is *split* if every integer $i \in \{1, 2, \ldots, n\}$ occurs once in each row of SA. Moreover, if n is even, SA is *vertically split* if each integer $i \in \{1, 2, \ldots, n\}$ appears exactly once in the first $\frac{n}{2}$ columns of SA.

For example,

2	3	1	4
4	2	1	3

is a split Skolem array of order 4 and

4	2	3	2
3	1	1	4

is a vertically split Skolem array of order 4.

Theorem 6.4. ([7]) *There exists a Skolem array of order* n *if and only if* $n \equiv 0, 1 \pmod 4$.

A split Skolem array of order n *exists if and only if* $n \equiv 0, 1 \pmod 4$.

Theorem 6.5. ([7]) *There exist a Skolem labeled ladder graph* $K_2 \Box P_n$ *if and only if* $n \equiv 0, 1 \pmod 4$.

Further study of Skolem labeled graphs, in particular of trees and grids, $P_m \Box P_n$, is pursued in [33].

The configuration of communication networks with one central hub which directs information to its destination has the structure of a generalized k-windmill. A *k-windmill* ([8]) is a tree consisting of k paths of equal length, called vanes, rooted at the pivot, central vertex whose degree is at least three. A *generalized k-windmill* is a k-windmill whose vanes might have different lengths.

Mendelsohn and Shalaby [57] completely classified k-windmills with respect to Skolem labeling. They proved that if a tree $T = (V, E)$ has an even number of vertices, then the sum $\sum_{v \in V} d_T(u, v) \pmod 2$, called the *parity* *of the tree* T, is independent of the choice of a vertex $u \in V$. Considering the parity of a Skolem labeled tree and the relationship between the length of the vanes and the number of vanes in a Skolem labeled windmill, they derived the necessary conditions, called the parity condition and the degeneracy condition. Sufficiency of these conditions is proved by a direct construction of the required Skolem labelings. In the following, let n denote the length of the vane of a k-windmill. Note that if a k-windmill with v vertices has a Skolem labeling, then $(v-1)/k \in \mathbb{Z}$ and the longest path in the k-windmill is at least equal to $v/2$. Hence, only 3-windmills can be Skolem labeled.

Lemma 6.6 (Skolem parity condition). ([57]) *If* T *is a Skolem labeled tree on* $2n$ *vertices, then either* $n \equiv 0, 3 \pmod 4$ *and the parity of* T *is even, or* $n \equiv 1, 2 \pmod 4$ *and the parity of* T *is odd.*

Lemma 6.7 (Degeneracy condition). ([57]) *If* T *is a Skolem labeled* k*-windmill with vanes of length* n*, then* $k \leq 2n$*.*

Theorem 6.8. ([57]) *A* 3*-windmill admits a Skolem labeling if* $n \equiv 1, 7 \pmod 8$*, a hooked Skolem labeling with one hook if* $n \equiv 0 \pmod 2$*, except when* $n = 2$*, and a hooked Skolem labeling with two hooks when* $n \equiv 3, 5 \pmod 8$*, except when* $n = 3$*.*

Theorem 6.9. ([57]) *A* k*-windmill admits a minimum hooked Skolem labeling if and only if* $k \leq 2n$*.*

Baker and Manzer [8] studied Skolem labelings of generalized 3-windmills. They also give a new necessary condition, called the nondegeneracy condition, for the existence of a Skolem labeled generalized k-windmill.

Lemma 6.10 (Nondegeneracy condition). *A Skolem labeled generalized* k*-windmill with* $2n$ *vertices and vanes of length* $x_1, x_2, \ldots, x_k$ *satisfies the inequality:*

$$n(n+1) \leq \sum_{i=1}^{n} x_i(x_i+1).$$

Theorem 6.11. *A generalized* 3*-windmill* G *has a Skolem labeling if and only if* G *satisfies the* Skolem parity condition 6.6.

Proof. (Sketch) Sufficiency is proved by providing a labeling for various cases. All types of generalized Skolem sequences are used, as well as all methods applied in proofs of the existence of these sequences. The main idea is to label most of the vertices of two vanes using one Skolem-type sequence, and label the remaining vertices using another Skolem-type sequence. Also, Baker and Manzer defined and used a special kind of Langford sequence, called the symmetric Langford sequence SL_d^{2d-1} of defect d and length $2d-1$. A sequence SL_d^{2d-1} is constructed in the following way: $(i, 2i)$ for $i = d, d+1, \ldots, 2d-1$, and

$(1+i, 2d+2i+1)$ for $i = 0, 1, \ldots, d-2$. Note that integers i in position $(i, 2i)$, for $i = d, \ldots, 2d-1$, can be pivoted to generate a $(4d-2i-1)$-extended Langford sequence. □

Corollary 6.12. *A generalized 3-windmill which cannot be Skolem labeled, has a weak hooked Skolem labeling with at most three hooks.*

7. Enumeration

Since pure and hooked Skolem sequences directly generate a solution to Heffter's first difference problem, a lower bound on the number of distinct pure and hooked Skolem sequences implies a lower bound on the number of solutions to Heffter's first difference problem. Similarly, a lower bound on the number of distinct Rosa sequences provides a lower bound on the number of solutions to Heffter's second difference problem. On the other hand, the enumeration of Langford sequences of defect 2 and length n is used as a benchmark for testing new parallel processing algorithms (cf. [35], [39]).

Abrham [2] gave a construction using additive permutations, called Skolem permutations, to determine a lower bound on the number of Skolem sequences. An ordered set $X = (x_1, x_2, \ldots, x_n)$, $x_1 < x_2 < \cdots < x_n$, is called a *base set of an additive permutation* if there exists a permutation $Y = (y_1, y_2, \ldots, y_n)$ of the elements in X such that $X + Y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ is again a permutation of X.

A relation between Skolem permutations and extended Skolem sequences was established in [1]. If $S = (s_1, s_2, \ldots, s_{2n+1})$ is an extended Skolem sequence of order n, then the permutation $Y = (y_1, y_2, \ldots, y_{2n+1})$, where if $s_j = s_{j+i} = i$ then $y_j = i$ and $y_{j+i} = -i$ for $i \in \{1, 2, \ldots, n\}$, is an additive permutation of the base set $X = (-n, -n+1, \ldots, 0, \ldots, n-1, n)$. The permutation Y is called a Skolem permutation of X. Moreover, it is easy to see that the sequence $Y' = (y'_1, y'_2, \ldots, y'_{2n+1})$ where $y'_j = |y_j|$ for $j = 1, \ldots, 2n+1$, is the original sequence S. Hence, we can count the number of Skolem permutations of X in order to obtain a lower bound on the number of extended Skolem sequences.

To construct Skolem permutations, Abrham applied Langford sequences. Similarly as a Skolem sequence, Langford sequence of defect d and length n, $L = \{(a_i, b_i) : i = d, \ldots, d+n-1\}$ such that $\bigcup_{i=1}^{n} \{a_i, b_i\} = \{d, d+1, \ldots \ldots, d+2n-1\}$, partitions the set $\{d, d+1, \ldots, d+3n-1\}$ into n difference triples $\{i, a_i + n, b_i + n\}$, where $i = d, \ldots, d+n-1$. On the other hand, each difference triple $\{a, b, a+b\}$ generates the following two 3-cycles in a Skolem permutation of the set $X = \{-(d+3n-1), \ldots, -d, d, \ldots, d+3n-1\}$: $(a, b, -a-b)(-b, -a, a+b)$ or $(-a-b, b, a)(a+b, -a, -b)$. Hence, there are at least 2^n such permutations.

Theorem 7.1. ([1]) *Let σ_n be the number of distinct Skolem sequences of order n, and let ϵ_n denote the number of extended Skolem sequences of order n, with unspecified placement of a null element. Then, if $n \equiv 0, 1 \pmod 4$, $\sigma_n \geq 2^{\lfloor n/3 \rfloor}$, and for any $n \geq 1$, $\epsilon_n \geq 2^{\lfloor n/3 \rfloor}$.*

The extended Skolem sequences constructed in the proof of Theorem 7.1 when $n \equiv 0, 3 \pmod 4$ are Rosa sequences of order n.

Corollary 7.2. *There are at least $2^{\lfloor n/3 \rfloor}$ distinct Rosa sequences of order n for $n \equiv 0, 3 \pmod 4$.*

Theorem 7.3. ([11]) *Let h_n and $\overline{h_n}$ denote the number of distinct hooked Skolem and hooked Rosa sequences of order n, respectively. Then, $h_n \geq 2^{\lfloor n/3 \rfloor}$ when $n \equiv 2, 3 \pmod 4$ and $n \geq 46$. Moreover, $\overline{h_n} \geq 2^{\lfloor n/3 \rfloor}$ when $n \equiv 1, 2 \pmod 4$ and $n \geq 117$.*

Abrham [4] also established a connection between a graceful labeling of cycles C_{n-1} and Skolem sequences of order n (see Section 7.1). Hence, the lower bound on the number of distinct graceful labelings of cycles of order $n-1$ slightly improves the lower bound on the number of the distinct Skolem sequences.

Theorem 7.4. ([3]) *There are at least $0.15077 \times 3^{9n/40}$ distinct Skolem sequences of order n, for $n \equiv 0 \pmod 4$ and $n \geq 53$. On the other hand, there are at least $3^{(n-3)/4}$ distinct Skolem sequences of order n when $n \equiv 1 \pmod 4$ and $n \geq 53$.*

Baker, Kergin, and Bonato calculated a lower bound on number of distinct Skolem arrays of order n, which directly follows from the recursive construction of vertically split Skolem arrays (cf. [7]).

Theorem 7.5. *Let $vs(n)$ and $s(n)$ be the numbers of distinct vertically split Skolem arrays and Skolem arrays of order n, respectively. Then,*

$$\begin{aligned} vs(n) &\geq 2^{\frac{3(n-4)}{4}} \cdot 12, \\ s(n) &\geq 2^{\frac{3n}{4}-2} \cdot 31 \qquad \textit{if} \quad n \equiv 0 \pmod 4, \\ s(n) &\geq 2^{\frac{3(n-1)}{4}-2} \cdot 3 \qquad \textit{if} \quad n \equiv 1 \pmod 4. \end{aligned}$$

The exact numbers of distinct Skolem, hooked Skolem and extended Skolem sequences of small orders are given in [23]. Also, there are given the exact numbers of distinct Langford, hooked Langford and extended Langford sequences of defect 2 and small lengths. The web-site [60] provides some information about computation of the number of distinct Langford sequences of defect 2.

8. Open problems and conjectures

Reid and Shalaby examine in [77] some more variations of Skolem-type sequences and propose three conjectures. They conjecture that the necessary conditions for the existence of k-extended m-near Skolem sequence, (m_1, m_2)-near Skolem sequences and (p, q)-extended m-near Skolem sequences are sufficient.

As we have mentioned, it is conjectured that necessary conditions for the existence of generalized Skolem-type sequences are also sufficient (cf. [47], [92]). Computer search for these sequences of small order supports this conjecture.

Mendelsohn and Rosa [58] conjectured that there exists a constant $c(n)$ such that every partial Heffter system $PHS(n)$ can be completed to a Heffter system $HS(N)$ for all $N \geq c(n)$.

In [72], Pike and Shalaby stated two open problems relating Skolem sequences and perfect 1-factorizations of complete graphs. They asked if there is a direct or recursive construction of Skolem sequences which would give perfect 1-factorizations of complete graphs. Also, they note that it is not known if there exists a Rosa sequence of any order which would yield a perfect 1-factorization of the corresponding complete graph.

Mendelsohn and Shalaby conjectured in [56] that the only biconnected Skolem labeled graphs are cycles of length n, when $n \equiv 0, 2 \pmod 8$.

Baker and Manzer [8] conjectured that a generalized k-windmill satisfying the Skolem parity condition 6.6 and nondegeneracy condition 6.10 admits a (strong) Skolem labeling. They also conjectured that a generalized 3-windmill can be strongly Skolem labeled if it satisfies the Skolem parity condition 6.6.

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REFERENCES

[1] ABRHAM, J.—KOTZIG, A.: *Skolem sequences and additive permutations*, Discrete Math. **37** (1981), 143–146.

[2] ABRHAM, J.: *Exponential lower bounds for the number of Skolem and extremal Langford sequences*, Ars Combin. **22** (1986), 187–198.

[3] ABRHAM, J.—KOTZIG, A.: *Estimates of the number of graceful valuations of cycles*, Congr. Numer. **77** (1990), 213–216.

[4] ABRHAM, J.: *Graceful 2-regular graphs and Skolem sequences*, Discrete Math. **93** (1991), 115–121.

[5] BAKER, C.—SHALABY, N.: *Disjoint Skolem sequences and related disjoint structures*, J. Combin. Des. **1** (1993), 329–345.

[6] BAKER, C. A.: *Extended Skolem sequences*, J. Combin. Des. **3** (1995), 363–379.

[7] BAKER, C.—KERGIN, P.—BONATO, A.: *Skolem arrays and Skolem labellings of ladder graphs*, Ars Combin. **63** (2002), 97–107.

[8] BAKER, C. A.—MANZER, J. D. A.: *Skolem-labeling of generalized three-vane windmills*, Australas. J. Combin. (To appear)

[9] BAKER, C. A.—SHALABY, N.—SHARAY, A.—NOWAKOWSKI, R. J.: *M-fold and extended m-fold Skolem sequences*, Util. Math. **45** (1994), 153–167.

[10] BALL, W. W. R.: *Mathematical Recreations and Problems*, MacMillan, London, 1892.

[11] BENNETT, G. K.—GRANNELL, M. J.—GRIGGS, T. S.: *Exponential lower bounds for the number of Skolem-type sequences*, Ars Combin. **73** (2004), 101–106.

[12] BERMOND, J. C.—BROUWER, A. E.—GERMA, A.: *Systèmes de triplets et différences associées*. In: Problèmes combinatoireset théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), 1978, pp. 35–38.

[13] BILLINGTON, E. J.: *On lambda coverings of pairs by triples, repeated elements allowed in triples*, Util. Math. **21** (1982), 187–203.

[14] BILLINGTON, E. J.: *Cyclic balanced ternary designs with block size three and index two* (Fourteenth Australasian conference on combinatorial mathematics and computing (Dunedin, 1986)), Ars Combin. **23B** (1987), 215–232.

[15] BRYANT, D.—GAVLAS, H.—LING, A. C. H.: *Skolem-type difference sets for cyclic systems*, Electron. J. Combin. **10** (2003), Research Paper 38, 12 pp. (electronic).

[16] CHO, C. J.: *Rotational triple systems*, Ars Combin. **13** (1982), 203–209.

[17] CHO, C. J.: *Rotational Steiner triple systems*, Discrete Math. **42** (1982), 153–159.

[18] COLBOURN, M. J.—COLBOURN, C. J.: *Cyclic block designs with block size* 3, European J. Combin. **2** (1981), 21–26.

[19] COLBOURN, C. J.—ROSA, A.: *Quadratic leaves of maximal partial triple systems*, Graphs Combin. **2** (1986), 317–337.

[20] COLBOURN, C. J.—ROSA, A.: *Element neighborhoods in twofold triple systems*, J. Geom. **30** (1987), 36–41.

[21] COLBOURN, C. J.—ROSA, A.: *Repeated edges in* 2*-factorizations*, J. Graph Theory **14**, (1990), 5–24.

[22] COLBOURN, C. J.—JIANG, Z.: *The spectrum for rotational Steiner triple systems*, J. Combin. Des. **4** (1996), 205–217.

[23] *CRC Handbook of Combinatorial Designs* (2nd ed.) (C. J. Colbourn, J. H. Dinitz, eds.), CRC Press, Boca Ration, FL, 2006.

[24] COLBOURN, C. J.—LING, A. C. H.: *A class of partial triple system with applications in survey sampling*, Comm. Statist. Theory Methods **27** (1998), 1009–1018.

[25] COLBOURN, C. J.—ROSA, A.: Triple Systems, Clarendon Press/Oxford University Press, New York, 1999.

[26] DAVIES, R. O.: *On Langford's problem (II)*, Gaz. Math. **43** (1959), 253–255.

[27] DILLON, J. F.: *The generalized Langford-Skolem problem.* In: Proceedings of the Fourth Southeastern Conference on Combinatorics, Graph Theory and Computing, Florida Atlantic Univ., Boca Raton, Fla., 1973, pp. 237–247.

[28] DOYEN, J.: *A note on reverse Steiner triple systems*, Discrete Math. **1** (1971-72), 315–319.

[29] ECKLER, A. R.: *Construction of missile guidance codes resistant to random interference*, Bell Syst. Tech. J. **39** (1960), 973–994.

[30] EGGLETON, R. B.—FRAENKEL, A. S.—SIMPSON, R. J.: *Beatty sequences and Langford sequences*, Discrete Math. **111** (1993), 165–178.

[31] FU, H. L.—MISHIMA, M.: *1-rotationally resolvable 4-cycle systems of* $2K_v$, J. Combin. Des. **10** (2002), 116–125.

[32] GILLESPIE, R. S.—UTZ, W. R.: *A generalized Langford problem*, Fibonacci Quart. **4** (1966), 184–186.

[33] GRAHAM, A. J.—PIKE, D. A.—SHALABY, N.: *Skolem labelled trees and* $P_s \square P_t$ *Cartesian products*, Australas. J. Combin. **38** (2007), 101–115.

[34] GROTH, E. J.: *Generation of binary sequences with controllable complexity*, IEEE Trans. Inform. Theory **IT-17** (1971), 288–296.

[35] HABBAS, Z.—KRAJECKI, M.—SINGER, D.: *Parallelizing combinatorial search in shared memory.* In: Proceedings of the Fourth European Workshop on Open MP, Rome, Italy, 2002.

[36] HARARY, F.—HSU, D. F: *Node-graceful graphs*, Comput. Math. Appl. **15** (1988), 291–298.

[37] HAYASAKA, T.—SAITO, S.: *Langford sequences: a progress report*, Gaz. Math. **63** (1979), 261–262.

[38] HEFFTER, L.: *Über Tripelsysteme*, Math. Ann. **49** (1897), 101–112.

[39] JAILLET, C.—KRAJECKI, M.: *Solving the Langford problem in parallel.* In: Proceedings of the Third International Symposium on Parallel and Distributed Computing (ISPDC 2004), 2004, pp. 83–90.

[40] JIANG, Z.: *$(F, 2)$-rotational Steiner triple systems.* In: Combinatorics Advances (Tehran, 1994), pp. 181–206.

[41] JIANG, Z.: *On rotational Steiner triple systems: five, seven and eleven*, Ars Combin. **47** (1997), 201–221.

[42] KÖHLER, E.: *Über das Oberwolfacher Problem.* In: Beiträge zur geometrischen Algebra (Proc. Sympos., Duisburg, 1976). Lehrbücher Monograph. Geb. Exakten Wissensch., Math. Reihe 21, Birkhauser, Basel-Boston, 1977, pp. 189–201.

[43] KÖHLER, E.: *Bemerkungen über Langfordsequenzen.* In: Numerische Methoden bei Optimierungsaufgaben, Band 3 (Tagung, Math. Forschungsinst., Oberwolfach, 1976). Internat. Ser. Numer. Math. **36** (1977), 137–145.

[44] KÖHLER, E.: *Über die Konstruktion optimaler Versuchspläne mit Hilfe von Skolem-Sequenzen.* In: Numerische Methoden bei Optimierungsaufgaben, Band 3 (Tagung, Math. Forschungsinst., Oberwolfach, 1976). Internat. Ser. Numer. Math. **36** (1977), 147–154.

[45] LANGFORD, C. D.: *Problem*, Gaz. Math. **43** (1958), 288.

[46] LEE, S. M.—QUACH, L. H.—WANG, S. S.: *Skolem-gracefulness of graphs which are union of paths and stars*, Congr. Numer. **61** (1988), 59–64.

[47] LEVINE, E.: *On the generalized Langford problem*, Fibonacci Quart. **6** (1968), 135–138.

[48] LINDNER, C. C.—ROSA, A.: *On the existence of automorphism free Steiner triple systems*, J. Algebra **34** (1975), 430–443.

[49] LINEK, V.—JIANG, Z.: *Extended Langford sequences with small defects*, J. Combin. Theory Ser. A **84** (1998), 38–54.

[50] LINEK, V.—JIANG, Z.: *Hooked k-extended Skolem sequences*, Discrete Math. **196** (1999), 229–238.

[51] LINEK, V.: *Note concerning an odd Langford sequence*, European J. Combin. **22** (2001), 79–83.

[52] LINEK, V.: *Extending Skolem sequences, how can you begin?*, Australas. J. Combin. **27** (2003), 129–137.

[53] LINEK, V.—MOR, S.: *On partitions of $\{1,\dots,2m+1\}\setminus\{k\}$ into differences $d,\dots,d+m-1$: Extended Langford sequences of large defect*, J. Combin. Des. **12** (2004), 421–442.

[54] LINEK, V.—SHALABY, N.: *The existence of (p,q)-extended Rosa sequences*, Discrete Math. **308** (2008), 1583–1602.

[55] LORIMER, P.: *A method of constructing Skolem and Langford sequences*, Southeast Asian Bull. Math. **6** (1982), 115–119.

[56] MENDELSOHN, E.—SHALABY, N.: *Skolem labelled graphs*, Discrete Math. **97** (1991), 301–317.

[57] MENDELSOHN, E.—SHALABY, N.: *On Skolem labelling of windmills*, Ars Combin. **53** (1999), 161–172.

[58] MENDELSOHN, E.—ROSA, A.: *Completing partial solutions to Heffter's difference problem*, Bull. Inst. Combin. Appl. (To appear).

[59] MENDELSOHN, N. S.: *A natural generalization of Steiner triple systems.* In: Computers in Number Theory (Proc. Sci. Res. Council Atlas Sympos. No. 2, Oxford, 1969), Academic Press, London, 1971, pp. 323–338.

[60] MILLER, J. E.: Langford's problem. http://www.lclark.edu/~miller/langford.html.

[61] MORGAN, D.: *Gracefully labeled trees from Skolem sequences*, Congr. Numer. **142** (2000), 41–48.

[62] MORGAN, D.—REES, R.: *Using Skolem and hooked-Skolem sequences to generate graceful trees*, J. Combin. Math. Combin. Comput. **44** (2003), 47–63.

[63] MORGAN, E.: *Balanced ternary designs with block size three.* In: Combinatorial Math. VII. Lecture Notes in Math. 829, Springer-Verlag, Berlin-New York, 1980, pp. 186–198.

[64] NICKERSON, R. S.: *Problem E1845; Solution*, Amer. Math. Monthly **73** (1966), 81; **74** (1967), 591–595.

[65] NORDH, G.: *$\mathscr{NP}$-completeness of generalized multi-Skolem sequences*, Discrete Appl. Math. **155** (2007), 2061–2068.

[66] NOWAKOWSKI, R. J.: *Generalizations of the Langford-Skolem Problem.* M.Sc. Thesis, University of Calgary, Canada, 1975.

[67] NOWAKOWSKI, R. J.—WHITEHEAD, C. A.: *Ordered graceful labellings of the 2-star*, Discrete Math. **233** (2001), 183–191.

[68] O'KEEFE, E. S.: *Verification of a conjecture of Th. Skolem*, Math. Scand. **9** (1961), 80–82.

[69] PELTESOHN, R: *Eine Lösung der beiden Heffterschen Differenzenprobleme*, Compos. Math. **6** (1939), 251–257.

[70] PHELPS, K. T.—ROSA, A.: *Steiner triple systems with rotational automorphisms*, Discrete Math. **33** (1981), 57–66.

[71] PIKE, D. A.—SHALABY, N.: *Starters, Skolem sequences and perfect 1-factorizations (Progress report)*. In: Proceedings of the Thirty-first Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 2000). Congr. Numer. **45** (2000), 147–155.

[72] PIKE, D. A.—SHALABY, N.: *The use of Skolem sequences to generate perfect one-factorizations*, Ars Combin. **59** (2001), 153–159.

[73] PIKE, D. A.—SHALABY, N.: *Non-isomorphic perfect one-factorizations from Skolem sequences and starters*, J. Combin. Math. Combin. Comput. **44** (2003), 23–32.

[74] PRIDAY, C. J.: *On Langford's problem I*, Gaz. Math. **43** (1959), 250–253.

[75] REES, R.—SHALABY, N.—SHARARY, A.: *Indecomposable Skolem and Rosa sequences*, Australas. J. Combin. **21** (2000), 3–12.

[76] REES, R.—SHALABY, N.: *Simple and indecomposable twofold cyclic triple systems from Skolem sequences*, J. Combin. Des. **8** (2000), 402–410.

[77] REID, C.—SHALABY, N.: *The existence of near-λ-fold and extended near-λ-fold Skolem sequences*, Congr. Numer. **177** (2005), 171–182.

[78] REID, C.—SHALABY, N.: *The existence of two new classes of near-Skolem type sequences*, Int. Math. Forum **1** (2006), 909–932.

[79] ROSA, A.: *A note on cyclic Steiner triple systems*, Mat.-Fyz. Časopis Slov. Akad. Vied **16** (1966), 285–290 (Slovak).

[80] ROSA, A.: *Steiner triple systems and their chromatic number*, Acta Math. Univ. Comenian. (N.S.) **24** (1970), 159–174.

[81] ROSA, A.: *On reverse Steiner triple systems*, Discrete Math. **2** (1972), 61–71.

[82] ROSA, A.: *Algebraic properties of designs and recursive constructions*. In: Proceedings of the Conference on Algebraic Aspects of Combinatorics (Univ. Toronto, Toronto, Ont.), Utilitas Mathematica Publ., Winnipeg 1975, pp. 183–200.

[83] ROSELLE, D. P.—THOMASSON, T. C. Jr.: *On generalized Langford sequences*, J. Combin. Theory Ser. A **11** (1971), 196–199.

[84] SAITO, S.—HAYASAKA, T.: *The Langford* $(4, n)$*-sequence: a trigonometric approach*, Discrete Math. **28** (1979), 81–88.

[85] SAITO, S.—HAYASAKA, T.: *Langford sequences: some connections with complete and additive permutations*, Discrete Math. **84** (1990), 63–70.

[86] SAITO, S.—HAYASAKA, T.: *Theory of the perfect* (s, n)*-sequence based on* $F(i, k)$, Res. Rep. Miyagi Nat. College Tech. **19** (1983), 73–80.

[87] SAITO, S.: *Harmonic analysis of the Langford sequence*, Res. Rep. Miyagi Nat. College Tech. **16** (1980), 99–103.

[88] SAITO, S.—HAYASAKA, T.: A perfect (s, n)-sequence based on $F(k, i)$, Res. Rep. Miyagi Nat. College Tech. **14** (1977), 101–108.

[89] SAITO, S.—HAYASAKA, T.: *The extension of the generalized Langford sequence*, Res. Rep. Miyagi Nat. College Tech. **13** (1976), 133–135.

[90] SAITO, S.—HAYASAKA, T.: *The generalized Langford sequence*, Res. Rep. Miyagi Nat. College Tech. **12** (1975), 93–98.

[91] SHALABY, N.: *The existence of near-Skolem and hooked near-Skolem sequences*, Discrete Math. **135** (1994), 303–319.

[92] SHALABY, N.—AL-GWAIZ, M. A.: *Generalized hooked, extended, and near-Skolem sequences*, J. Combin. Math. Combin. Comput. **26** (1998), 113–128.

[93] SHALABY, N.: *The existence of near-Rosa and hooked near-Rosa sequences*, Discrete Math. **261** (2003), 435–450.

[94] SHALABY, N.—STUCKLESS, T.: *The existence of looped Langford sequences*, Crux Mathematicorum **26** (2000), 86–92.

[95] SHARAF ELDIN, A.—SHALABY, N.—AL-THUKAIR, F.: *Construction of Skolem sequences*, Int. J. Comput. Math. **70** (1998), 333–345.

[96] SIMPSON, J. E.: *Langford sequence: perfect and hooked*, Discrete Math. **44** (1983), 97–104.

[97] SKOLEM, TH.: *On certain distribution of integers in pairs with given differences*, Math. Scand. **5** (1957), 57–68.

[98] SKOLEM, TH.: *Some remarks on the triple systems of Steiner*, Math. Scand. **6** (1958), 273–280.

[99] TEIRLINCK, L.: *The existence of reverse Steiner triple systems*, Discrete Math. **6** (1973), 299–300.

[100] WEI, R.: *Cyclic BSEC of block size 3*, Discrete Math. **250** (2002), 291–298.

[101] ZHANG, J.—CHANG, Y.: *The spectrum of cyclic BSEC with block size three*, Discrete Math. **305** (2005), 312–322.

[102] ZHANG, J.—CHANG, Y.: *The spectrum of cyclic* $\mathrm{BSA}(v,3,\lambda;\alpha)$ *with* $\alpha = 2,3$, J. Combin. Des. **13** (2005), 313–335.

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Department of Mathematics
University of Toronto
Toronto, ON M5S 3G3
CANADA

E-mail: nevena.francetic@utoronto.ca
mendelso@math.utoronto.ca