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LARGE SETS OF t-DESIGNS FROM GROUPS

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Dedicated to Alex Rosa on the occasion of his 70th birthday

(*Communicated by Peter Horák*)

ABSTRACT. This paper addresses questions related to the existence and construction of *large sets* of t-(v,k,λ) designs. It contains material from my talk in the Combinatorial Designs Conference in honor of Alex Rosa's 70th birthday, which took place in beautiful Bratislava, in July, 2007. Naturally, only a small number of "highlight" topics could be included, and for the most part these involve the use of *symmetry*, that is, it is assumed that the particular designs or large sets of designs, are invariant under a prescribed group of automorphisms. I present almost no proofs, but give references so that the reader can find a much wider repertory of theorems and constructions in the literature. For completeness, I include the statement of a few recursive constructions. The latter are extremely important on their own right, and deserve extensive attention elsewhere. I hope the reader becomes interested in the intriguing open problems posed at the end of the paper and succeeds in solving some of them.

1. Introduction

Resolutions of t-designs were studied as early as 1847 by Reverend T. P. Kirkman [30], [31] who proposed the famous 15 schoolgirls problem ([18]). Kirkman's problem is equivalent to finding a *resolvable* 2-$(15,3,1)$ design with $r = 7$, and $b = 35$.

In the early 1980's combinatorial design practitioners were mostly immersed in questions of existence or non-existence of t-designs. Until 1983, simple t-designs with $t > 5$ had not been found, and many researchers believed that (simple) 6-designs would not exist. The construction of the first simple 6-designs in [47] surprised a few, but was quickly overshadowed by the work of Teirlinck who

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showed in [55] that simple t-designs exist for all t. Teirlinck's work made clear that the "high road" to the discovery of t-designs was via the construction of *large sets* of t-designs. Of course, t-(v, k, λ) designs with large t and small λ are rare. To this day no t-$(v, k, 1)$ designs have been found for $t \geq 6$.

Over the past two decades, large sets of t-designs appeared in recursive constructions. Unfortunately, for a given t, Teirlinck's constructions result in t-designs with extremely large values for the parameters v and λ. Subsequently, G. B. Khosrovshahi and S. Ajoodani-Namini [1], [2], [3], [28], [29], greatly contributed to the repertory of recursive methods. Of strong impact has been the work of Reinhard Laue and his group of researchers at Bayreuth [9], [10], [11], [41], [42], [43], particularly in the direct construction of t-designs and large sets. The names of researchers like W. Alltop, P. Cameron, C. Colbourn, Y. M. Chee, M. Dehon, R. H. F. Denniston, J. Dinitz, M. J. Grannell, T. S. Griggs, H. Hanani, A. Hartman, E. S. Kramer, D. Kreher, R. Laue, C. Lindner, R. Mathon, D. M. Mesner, R. Mullin, A. Rosa, S. P. Radziszowski, D. Stinson, Tran van Trung, R. Wilson, M. J. Sharry, A. Street, A. Wassermann, are intimately connected with t-designs, resolutions and large sets of t-designs. Other researchers, including this author, have contributed to the area with a number of results that appear in the bibliography. By means of these techniques many more parameter sets now yield constructible designs, with the value of t going up, the value of λ going down, and v taking infinitely many admissible values.

Recursive methods require a basic collection of large sets from which to start. Then infinite series of parameter sets are settled by recursion. In a number of articles, [2], [3], [28], [56], [61], the authors present methods for constructing "small" large sets which, in combination with already known small cases and the known recursive methods, handle many admissible parameter sets. These construction methods for starter large sets usually rely on assuming an appropriate group of automorphisms for the putative designs and large sets.

We do not address here any of the interesting questions that relate to obtaining large sets (or overlarge sets) of t-designs, by means of imprimitive group actions along the lines of [44].

2. Preliminaries

In this paper, V denotes a finite set of *points* with $|V| = v$. The parameters t and k are positive integers such that $0 < t < k \leq v$, and the collection of all k-subsets of V is denoted by $\binom{V}{k}$. For integers $s < r$, the symbol $[s, r]$ denotes the set of integers $\{s, s+1, \ldots, r\}$. The *symmetric group* on V will be denoted by $\mathscr{S}_V$. We denote the Galois field of order q by $\mathbb{F}_q$.

A *simple* t-(v,k,λ) *design*, $(V,\mathscr{B})$, is the point set V together with a collection $\mathscr{B}$ of k-element subsets of V, called *blocks*, such that every t-element subset of V is contained in precisely λ blocks.

If $(V,\mathscr{B})$ is a t-(v,k,λ) design, and $x \in V$, the *derived* design with respect to x is $\bigl(V-\{x\},\mathscr{D}\bigr)$, where $D \in \mathscr{D}$ if and only if $D = B - \{x\}$, for $x \in B \in \mathscr{B}$. Such a derived design is a $(t-1)$-$(v-1,k-1,\lambda)$ design.

If $(V,\mathscr{B})$ is a t-(v,k,λ) design, and $x \in V$, the *residual* design with respect to x is the design $\bigl(V-\{x\},\mathscr{R}\bigr)$, where $K \in \mathscr{R}$ if and only if $x \notin K \in \mathscr{B}$. A residual design is a $(t-1)$-$(v-1,k,\lambda')$ design, where $\lambda' = \lambda(v-k)/(k-t-1)$.

It is well known that for each $s \in [0,t]$, every t-(v,k,λ) design is also an s-(v,k,λ_s) design, where $\lambda_s = \lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$. Thus, a set of necessary *divisibility conditions* for the existence of a t-(v,k,λ) design is that $\lambda\binom{v-s}{t-s} \equiv 0 \bmod \binom{k-s}{t-s}$, for $0 \le s < t$.

Two designs $\mathscr{D}_1 = (V_1,\mathscr{B}_1)$ and $\mathscr{D}_2 = (V_2,\mathscr{B}_2)$ are said to be *block-disjoint*, or simply *disjoint*, if $\mathscr{B}_1 \cap \mathscr{B}_2 = \emptyset$.

Suppose that $\mathscr{D} = (V,\mathscr{B})$ is a τ-(v,k,λ) design, and $\mathscr{R} = \bigl\{(V,\mathscr{B}_i)\bigr\}_{i=1}^{n}$ a collection of block-disjoint t-(v,k,λ') designs such that $\mathscr{B} = \bigcup_{i=1}^{n} \mathscr{B}_i$ we then say that $\mathscr{R}$ is a *t-resolution* of $\mathscr{D}$.

Let $\mathscr{R}_1 = \bigl\{(V,\Gamma_i)\bigr\}_{i=1}^{r}$ and $\mathscr{R}_2 = \bigl\{(V,\Delta_j)\bigr\}_{j=1}^{s}$ be t_1- and t_2-resolutions of $\mathscr{D} = (V,\mathscr{B})$ respectively. We say that $\mathscr{R}_1$ and $\mathscr{R}_2$ are *orthogonal* if and only if $|\Gamma_i \cap \Delta_j| \le 1$ for all $(i,j) \in [1,r]\times[1,s]$.

By a *large set* $\mathrm{LS}[N](t,k,v)$ we mean a collection $\mathscr{L} = \bigl\{(V,\mathscr{B}_i)\bigr\}_{i=1}^{N}$ of t-(v,k,λ) designs where $\{\mathscr{B}_i\}_{i=1}^{N}$ is a partition of $\binom{V}{k}$. Thus, a large set $\mathrm{LS}[N](t,k,v)$ is a t-resolution of the complete design $\binom{V}{k}$.

The number of blocks in a t-(v,k,λ) design is $b = \lambda_0 = \lambda\binom{v}{t}/\binom{k}{t}$. Thus, a necessary condition for a large set $\mathrm{LS}[N](t,k,v)$ to exist is that $Nb = \binom{v}{k}$. This is equivalent to $\lambda N = \binom{v-t}{k-t}$. Thus, N must divide $\binom{v-t}{k-t}$.

A group action $G|V$ is called *transitive* if V consists of a single G-orbit, it is said to be *t-homogeneous* if the induced action of G on $\binom{V}{t}$ is transitive. For brevity, by a *k-orbit* we mean an orbit of G in its induced action on $\binom{V}{k}$.

Let $\mathbb{B} = \{\mathscr{B}_i\}_{i=1}^{N}$ be the collection of designs in a large set $\mathscr{L}$. A group G is said to be an automorphism group of $\mathscr{L}$ if $\mathbb{B}^g = \mathbb{B}$ for all $g \in G$, that is, if $\mathscr{B}_i^g \in \mathbb{B}$ for all $\mathscr{B}_i \in \mathbb{B}$ and $g \in G$. Equivalently, we say that a large set with this property is G-invariant. If the stronger condition holds, that $\mathscr{B}_i^g = \mathscr{B}_i$ for all $\mathscr{B}_i \in \mathbb{B}$ and $g \in G$, we say that the large set $\mathbb{B}$ is $[G]$-invariant.

In 1976, E. S. Kramer and D. M. Mesner [35] stated a theorem which provides necessary and sufficient conditions for the existence of a G-invariant t-(v,k,λ) design. Beginning with a given group action $G|V$, the theorem allows for the construction of all such G-invariant t-designs. In 1999, the authors of [19] describe a slight generalization which provides means for constructing $[G]$-invariant *large sets* of t-(v,k,λ) designs. In particular, the authors of [19] turn their attention to t-homogeneous, G-semiregular large sets of t-designs.

3. Numerology

Some intriguing early questions on the possible existence of large sets of t-designs are mentioned below for their entertainment value, but also because some of these problems are still open.

A projective plane of order 2, also known as a *Fano* plane, is a 2-(7,3,1) design, $(V,\mathscr{B})$, where V is a set of 7 points, and $\mathscr{B}$ a set of 7 lines. Each line is incident with 3 points, and any two points of V lie on exactly one line. Since there are $\binom{7}{3} = 35$ 3-subsets of V it appears numerically possible to pack $\binom{V}{3}$ with $35/7 = 5$ mutually block-disjoint projective planes. Unfortunately this is not possible, and this fact was already known to Arthur Cayley, who allegedly had a very short proof of the fact.

$$A = \begin{matrix}
7&3&3&3&3&3&3&3&1&1&1&1&1&1&1&0&0&0&0&0&0&0&1&1&1&1&1&1&1&0\\
3&7&1&1&1&1&1&1&3&0&0&0&3&0&3&1&1&1&1&1&1&1&3&0&3&3&0&0&0&1\\
3&1&7&1&1&1&1&1&3&3&0&0&0&3&0&1&1&1&1&1&1&1&0&3&0&3&3&0&0&1\\
3&1&1&7&1&1&1&1&0&3&3&0&0&0&3&1&1&1&1&1&1&1&0&0&3&0&3&3&0&1\\
3&1&1&1&7&1&1&1&3&0&3&3&0&0&0&1&1&1&1&1&1&1&0&0&0&3&0&3&3&1\\
3&1&1&1&1&7&1&1&0&3&0&3&3&0&0&1&1&1&1&1&1&1&3&0&0&0&3&0&3&1\\
3&1&1&1&1&1&7&1&0&0&3&0&3&3&0&1&1&1&1&1&1&1&3&3&0&0&0&3&0&1\\
3&1&1&1&1&1&1&7&0&0&0&3&0&3&3&1&1&1&1&1&1&1&0&3&3&0&0&0&3&1\\
1&3&3&0&3&0&0&0&7&1&1&1&1&1&1&3&0&3&0&0&0&3&1&1&1&1&1&1&1&3\\
1&0&3&3&0&3&0&0&1&7&1&1&1&1&1&3&3&0&3&0&0&0&1&1&1&1&1&1&1&3\\
1&0&0&3&3&0&3&0&1&1&7&1&1&1&1&0&3&3&0&3&0&0&1&1&1&1&1&1&1&3\\
1&0&0&0&3&3&0&3&1&1&1&7&1&1&1&0&0&3&3&0&3&0&1&1&1&1&1&1&1&3\\
1&3&0&0&0&3&3&0&1&1&1&1&7&1&1&0&0&0&3&3&0&3&1&1&1&1&1&1&1&3\\
1&0&3&0&0&0&3&3&1&1&1&1&1&7&1&3&0&0&0&3&3&0&1&1&1&1&1&1&1&3\\
1&3&0&3&0&0&0&3&1&1&1&1&1&1&7&0&3&0&0&0&3&3&1&1&1&1&1&1&1&3\\
0&1&1&1&1&1&1&1&3&3&0&0&0&3&0&7&1&1&1&1&1&1&3&0&3&0&0&3&3&1\\
0&1&1&1&1&1&1&1&0&3&3&0&0&0&3&1&7&1&1&1&1&1&3&3&0&3&0&0&3&1\\
0&1&1&1&1&1&1&1&3&0&3&3&0&0&0&1&1&7&1&1&1&1&3&3&3&0&3&0&0&1\\
0&1&1&1&1&1&1&1&0&3&0&3&3&0&0&1&1&1&7&1&1&1&0&3&3&3&0&3&0&1\\
0&1&1&1&1&1&1&1&0&0&3&0&3&3&0&1&1&1&1&7&1&1&0&0&3&3&3&0&3&1\\
0&1&1&1&1&1&1&1&0&0&0&3&0&3&3&1&1&1&1&1&7&1&3&0&0&3&3&3&0&1\\
0&1&1&1&1&1&1&1&3&0&0&0&3&0&3&1&1&1&1&1&1&7&0&3&0&0&3&3&3&1\\
1&3&0&0&0&3&3&0&1&1&1&1&1&1&1&3&3&3&0&0&3&0&7&1&1&1&1&1&1&0\\
1&0&3&0&0&0&3&3&1&1&1&1&1&1&1&0&3&3&3&0&0&3&1&7&1&1&1&1&1&0\\
1&3&0&3&0&0&0&3&1&1&1&1&1&1&1&3&0&3&3&3&0&0&1&1&7&1&1&1&1&0\\
1&3&3&0&3&0&0&0&1&1&1&1&1&1&1&0&3&0&3&3&3&0&1&1&1&7&1&1&1&0\\
1&0&3&3&0&3&0&0&1&1&1&1&1&1&1&0&0&3&0&3&3&3&1&1&1&1&7&1&1&0\\
1&0&0&3&3&0&3&0&1&1&1&1&1&1&1&3&0&0&3&0&3&3&1&1&1&1&1&7&1&0\\
1&0&0&0&3&3&0&3&1&1&1&1&1&1&1&3&3&0&0&3&0&3&1&1&1&1&1&1&7&0\\
0&1&1&1&1&1&1&1&3&3&3&3&3&3&3&1&1&1&1&1&1&1&0&0&0&0&0&0&0&7
\end{matrix}$$

It is known that, up to isomorphism, there is a unique plane of order 2. Thus the symmetric group $\mathscr{S}_7$ acts transitively on all Fano planes on $V = [1,7]$. Since the automorphism group of a Fano plane is $PSL_3(2)$, of order 168, there are in all $7!/168 = 30$ distinct Fano planes on V. An easy computation yields the

30×30 matrix A above, whose rows and columns are indexed by the 30 planes, and where $A(i,j)$ is the number of blocks (lines) the ith plane has in common with the jth plane.

Thus, planes i and j are disjoint if $A(i,j) = 0$. A quick inspection of matrix A shows that there are pairs of disjoint planes but that there are no cliques of disjoint planes of size 3 or greater.

Observe next that PG(2,3), the projective plane of order 3, is a 2-(13,4,1) design, with 13 lines, each incident with 4 points. Again, PG(2,3) is unique up to isomorphism. Since $\binom{13}{4}/13 = 55$, it is conceivable that a large set of 55 planes of order 3 will exist. In fact, such a large set of projective planes does exist ([17]).

PG(2,4), the projective plane of order 4, has 21 lines, each of size 5, and $\binom{21}{5}/21 = 969$. Again, it is numerically possible that there exists a large set of 969 planes of order 4. The existence, or otherwise, of such a large set, is still an unsettled problem.

The legendary 5-(24,8,1) design, discovered by E. Witt in 1938 has 759 blocks of size 8 (see [60]). Its successive derived designs with respect to one point at a time, are 4-(23,7,1), 3-(22,6,1), and 2-(21,5,1) designs with 253, 77 and 21 blocks respectively. The 2-(21,5,1) design is in fact isomorphic to the projective plane of order 4 discussed above. We now observe that $\binom{24}{8}/759 = \binom{23}{7}/253 = \binom{22}{6}/77 = \binom{21}{5}/21 = 969$. Thus it is conceivable for a large set of LS[969](5,8,24) to exist, and this would imply the existence of LS[969](4,7,23), LS[969](3,6,22) and LS[969](2,5,21).

The inversive plane $S(3,4,10)$, consists of 30 blocks, each of size 4, and $\binom{10}{4}/30 = 7$. It is conceivable that there exists a large set of 7, block disjoint $S(3,4,10)'s$. Unfortunately this large set does not exist ([33]).

We note that for any natural number n, n^2+n+1 divides $\binom{n^2+n+1}{n+1}$. A projective plane of order n, if such exists, is a $2-(n^2+n+1, n+1, 1)$ design, and can be viewed as having point set $V = [1, n^2+n+1]$. It is numerically feasible that a large set of projective planes of order n exists, partitioning $\binom{V}{n+1}$ into $N = \binom{n^2+n+1}{n+1}/(n^2+n+1)$ mutually disjoint planes. It was conjectured in 1978 by Magliveras [46] that such a large set will exist for all $n \geq 3$, provided that n is the order of a projective plane. Although Kramer and Magliveras have found sets of over 600 mutually disjoint 2-(21,5,1)'s (and smaller sets of disjoint 5-(24,8,1)'s) by probabilistic means, no one has yet found a large set with these parameters.

There is a unique up to isomorphism 2-(9,3,1) design, with 12 blocks. Now, $\binom{9}{3}/12 = 7$ and indeed there are large sets LS[7](2,3,9) partitioning the 84 triples of a 9-set into mutually disjoint 2-(9,3,1)'s. In fact much more can be said here. There exists a system of three mutually orthogonal large sets $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$, where

$\mathscr{L}_i = \{D_{i,j}\}_{j=1}^{7}$. Note that each $D_{i,j}$ above is a 2-(9,3,1) design containing 12 blocks. It is interesting to note here that a system $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$ gives rise to a $7 \times 7 \times 7$ cube so that the (i, j, k)th cell of the cube contains $D_{1,i} \cap D_{2,j} \cap D_{3,k}$ i.e. either a single 3-subset or empty. In fact, there are, up to isomorphism, two such systems $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$.

Such a cube, called a *Steiner tableau* by E. S. Kramer and D. M. Mesner [34], will clearly have the property that all $\binom{9}{3}$ triples occur in it, and the non-empty cells of any 7×7 subplane, parallel to the faces of the cube, will constitute a 2-(9,3,1) design. We display below the 7 horizontal planes of such a Steiner tableau.

A Steiner tableau by Kramer and Mesner.

			147			
	568					
	169				138	
489						
246		125	278		239	
			367			
		345	579			

	289		139	679		459
237	157					
	346					478
		168			124	
358				256		

					356	
			348			
				145	268	379
259		578				
					167	189
	123					
				247	469	

			158			
					789	
			235			
	267					
		569				368
457		248		349		146
		137		129		

	378				257	
128	359	467				
			249			
179						
134					458	
			689			
156		236				

			126	468		
	245	149				
				589		567
369				238	347	135
		279		178		

						234
				136		269
						127
			456			
	479			357		
					159	
678	148	389				258

4. Recursive constructions

In this section we state without proof a number of results that are central in the theory of large sets of t-designs. These are *recursive* constructions and require certain *seed* designs before the recursive steps can take over. When the seed design conditions are satisfied the recursive constructions will usually yield infinite families of large sets of designs.

Although the following easy observation does not really belong to recursive constructions, it is convenient to place it in this section.

Proposition 4.1. *If there exists a* $\mathrm{LS}[M](t, k, v)$ *and* $N|M$, *then there exists an* $\mathrm{LS}[N](t, k, v)$.

The following result is also implicit in the 1991 work of Khosrovshahi and Ajoodani-Namini [28].

Proposition 4.2. (Y. M. Chee and S. S. Magliveras [16]) *If an* $\mathrm{LS}[M](t,k,v)$ *and a* $\mathrm{LS}[N](t,k+1,v)$ *exist, then there exists a* $\mathrm{LS}[\gcd(M,N)](t, k+1, v+1)$.

We summarize some additional known recursive constructions of large sets of t-designs.

Theorem 4.1. (Teirlinck, 1989, [56]) *For every natural number* t *let* $\lambda(t) = \mathrm{lcm}\{\binom{t}{m} : m = 1,\dots,t\}$, $\lambda^*(t) = \mathrm{lcm}\{1,2,\dots,t+1\}$, *and* $\ell(t) = \prod_{i=1}^{t} \lambda(i)\lambda^*(i)$. *Then, for all* $N > 0$, *there is an* $\mathrm{LS}[N](t,t+1,t+N\ell(t))$.

Theorem 4.2. (Khosrovshahi, Ajoodani-Namini, 1991, [28]) *If there are* $\mathrm{LS}[N](t,t+1,v)$, *and* $\mathrm{LS}[N](t,t+1,w)$, *then there is also an* $\mathrm{LS}[N](t,t+1,v+w-t)$.

Theorem 4.3. (Qiu-Rong Wu, 1991, [61]) *If there exist large sets* $\mathrm{LS}[N](t,k,v)$, $\mathrm{LS}[N](t,k,w)$, $\mathrm{LS}[N](k-2,k-1,v-1)$, $\mathrm{LS}[N](k-2,k-1,w-1)$, *then there exists a large set* $\mathrm{LS}[N](t,k,v+w-k+1)$.

Corollary 4.1. *If there exist large sets* $\mathrm{LS}[N](t,k,v)$, *and* $\mathrm{LS}[N](k-2,k-1, v-1)$, *then there exist large sets* $\mathrm{LS}[N](t,k,v+m(v-k+1))$ *for all* $m \geq 0$.

An interesting construction by Ajoodani-Namini produces new large sets of $(t+1)$-designs, from a large set of t-designs.

Theorem 4.4. (Ajoodani-Namini, 1996, [2]) *Let* p *be a prime. If an* $\mathrm{LS}[p](t,k,v)$ *exists, then large sets* $\mathrm{LS}[p](t+1,pk+i,pv+j)$ *also exist for* $0 \leq j < i < p$.

5. When things are small

We seek an $\mathrm{LS}[N](t,k,v)$ which is invariant under a suitable group $G \leq \mathscr{S}_V$. Let V, t, v, k, λ and G be as above.

Let $\mathscr{D} = \{\Delta_1,\dots,\Delta_r\}$ be the collection of all orbits of G acting on the family of all $t-(v,k,\lambda)$ designs with point set V, and $\mathscr{F} = \{\Gamma_1,\dots,\Gamma_s\}$ the collection of all G-orbits on $\binom{V}{k}$. We define an incidence matrix $\mathbf{M} = (m_{ij})$ by:

$$m_{ij} = |D \cap \Gamma_j| \cdot |\Delta_i| / |\Gamma_j|$$

where D is any fixed design in orbit Δ_i. Thus, m_{ij} is the number of blocks of design D belonging to the G-orbit Γ_j of k-sets, modified by the normalizing factor $|\Delta_i|/|\Gamma_j|$. The following theorem first appeared in [38].

Theorem 5.1. *There exists a G-invariant large set of t-(v,k,λ) designs if and only if there exists $\mathbf{u} \in \{0,1\}^r$ such that $\mathbf{u} \cdot \mathbf{M} = \mathbf{j}$, where $\mathbf{j}$ is the s-dimensional column vector of all 1's.*

We remark that any rows of $\mathbf{M}$ that contain entries greater than one can be deleted, since the corresponding entry of $\mathbf{u}$ must be zero in any solution of $\mathbf{u} \cdot \mathbf{M} = \mathbf{j}$.

Let $\mathscr{L}$ be a G-invariant LS$[N](t,k,v)$, and $\pi \in \mathscr{S}_V$. It is clear that $\mathscr{L}^\pi$ will be a G^π-invariant LS$[N](t,k,v)$. Thus, if $\pi \in N(G)$ (the normalizer of G in $\mathscr{S}_V$), then $\mathscr{L}^\pi$ is G-invariant. This observation can be used in determining the isomorphism types of G-invariant LS$[N](t,k,v)$.

Proposition 5.1. *There are exactly 5 non-isomorphic $\mathbb{Z}_7$-invariant* LS$[42](2,5,11)$ *and* LS$[42](3,6,12)$ *large sets with component designs* 2-$(11,5,2)$*'s and* 3-$(12,6,2)$*'s respectively.*

Proof. There is a unique, up to isomorphism, 2-$(5,11,2)$ design whose full automorphism group is $PSL_2(11)$. Hence, there are a total of $[S_{11} : PSL_2(11)] = 11!/660 = 60480$ distinct 2-$(11,5,2)$'s on V. It is easy to obtain the 8640= 60480/7 $\mathbb{Z}_7$-orbits of these designs, the $\binom{11}{5}/7 = 66$ $\mathbb{Z}_7$-orbits on 5-sets and compute the 8640×66 matrix $\mathbf{M}$. An APL program "SYNTH" I wrote (which determines all solutions to 0-1 knapsacks of the type we see in Theorem 5.1), and Brendan McKay's graph isomorphism program did the rest. □

6. Coherence

Certain incidence matrices have come to be known as the *Kramer-Mesner* (KM) matrices, from the 1976 observation of E. S. Kramer and D. M. Mesner that allows one to construct t-designs invariant under a prescribed group.

Similar ideas were considered and used earlier by P. Dembowski [21], [22], D. R. Hughes [27] in 1957, and E. T. Parker [51] in 1957, under the topic *tactical decompositions*. Donald G. Higman's work on *Coherent Configurations* [25], [26] (1967-68) also predates the work of Kramer and Mesner. In 1973, R. M. Wilson [59] used similar matrices in his famous paper "*The necessary conditions for t-designs are sufficient for something*". Indeed, the KM matrices are the Wilson matrices, appropriately fused by the action $G|V$.

Moreover, very similar coherence techniques were used by S. S. Magliveras in his Ph.D. dissertation [45] for studying subgroup structures of groups (1970), and by M. H. Klin [32] in his Ph.D. dissertation (1974). Historically, it seems that the idea and use of **tactical objects** goes back to E. H. Moore in the late 1890's [50].

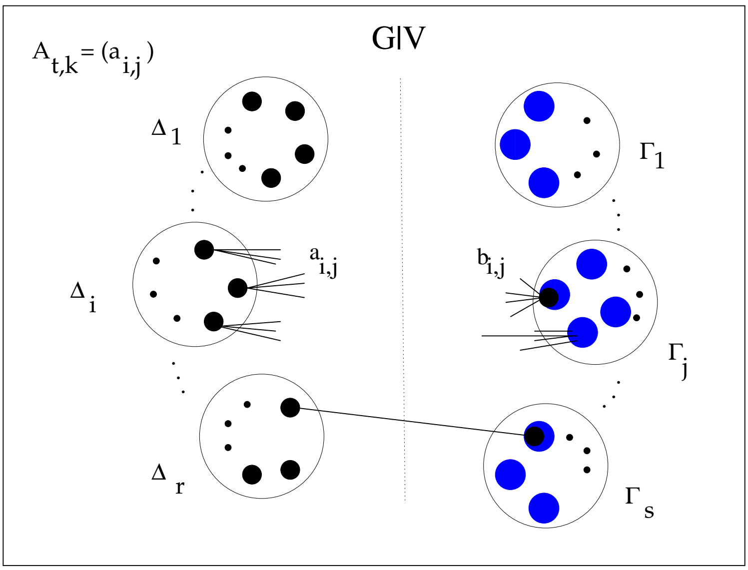
Recall that a t-(v,k,λ) design $(V,\mathscr{B})$ is *G-invariant* if $B^g \in \mathscr{B}$ for all $B \in \mathscr{B}$ and $g \in G$. Thus, if a t-(v,k,λ) design $(V,\mathscr{B})$ is *G-invariant*, then $\mathscr{B}$ is the union of G-orbits on $\binom{V}{k}$. The KM matrices present precise conditions under which G-orbits of $\binom{V}{k}$ can be selected to form a t-(v,k,λ) design, as well as conditions for the existence of large sets $\mathrm{LS}[N](t,k,v)$.

Let $G|V$ be a group action, and $t < k \leq v/2$. Let $\Delta_1, \Delta_2, \ldots, \Delta_r$ and $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ be the orbits of G on $\binom{V}{t}$ and $\binom{V}{k}$ respectively. The *Kramer-Mesner matrix* is defined to be the $r \times s$ matrix $\mathbf{A_{t,k}} = (a_{i,j})$ with:

$$a_{i,j} = |\{K \in \Gamma_j : \ T \subset K\}|,$$

for any fixed $T \in \Delta_i$.

Kramer-Mesner matrices



The *dual Kramer-Mesner matrix* is defined to be the $r \times s$ matrix $\mathbf{B_{t,k}} = (b_{i,j})$ where:

$$b_{i,j} = |\{T \in \Delta_i : \ T \subset K\}|,$$

for any fixed $K \in \Gamma_j$.

The diagram above depicts the actions induced by $G|V$ on $\binom{V}{t}$ and $\binom{V}{k}$, respectively, with the additional feature that each t-subset of $T \subset V$ on the left, is joined by an edge to each k-subset $K \subset V$ on the right, whenever $T \subset K$.

In this complete bipartite $[\binom{v}{t}, \binom{v}{k}]$ graph, the valence of each t-subset is $\binom{v-t}{k-t}$, and the valence of each k-subset is $\binom{k}{t}$. The number of edges emanating from any fixed $T \in \Delta_i$ and terminating anywhere in Γ_j is $a_{i,j}$. On the other hand the number of edges emanating from any particular $K \in \Gamma_j$ and terminating in Δ_i is $b_{i,j}$.

More generally, given $G|V$ and $s \leq \frac{v}{2}$, let $\Delta_1^{(s)}, \ldots, \Delta_{\rho(s)}^{(s)}$ be the G-orbits on $\binom{V}{s}$, and let $\ell_i^{(s)} = |\Delta_i^{(s)}|$, $1 \leq i \leq \rho(s)$. Then, the following lemma holds (also see [36]):

Lemma 6.1. *Let $\mathbf{A_{t,k}}$, $\mathbf{B_{t,k}}$, and $\ell_i^{(s)}$ be as defined above, and let $\ell^{(s)} = \left(\ell_1^{(s)}, \ldots, \ell_{\rho(s)}^{(s)}\right)$.*

(i) *If $t \leq s \leq k$, then $\mathbf{A_{t,k}} = \binom{k-t}{k-s}\mathbf{A_{t,s}}\mathbf{A_{s,k}}$.*

(ii) *$\mathbf{A_{t,k}}$ has constant row sum $\binom{v-t}{k-t}$.*

(iii) *$\ell_i^{(t)}\mathbf{A_{t,k}}(i,j) = \ell_j^{(k)}\mathbf{B_{t,k}}(i,j)$.*

(iv) *$\binom{k}{t}\ell^{(k)} = \ell^{(t)}\mathbf{A_{t,k}}$.*

When $\omega = |G| + \left|\binom{V}{k}\right|$ is relatively small, $\mathbf{A_{t,k}}$ can be computed directly with relative ease. However, when ω is large, computation of the G-orbits of s-subsets is not straightforward. One can easily compute the number $\rho(s)$ of G-orbits on s-subsets, and can also compute a collection Q_s of s-subsets containing representatives from each G-orbit of s-subsets ([36]). Deciding how to trim down Q_s into a complete collection of G-orbit representatives on s-subsets is usually difficult, and in [36] the notion of *invariant functions* is used to solve the problem. We will not discuss these techniques here. However, we remark that for $t < k \leq v/2$, it is easier to first compute the matrices $\mathbf{B_{t,t+1}}$, and from these compute $\mathbf{A_{t,t+1}}$, using Lemma 6.1. Then, applying statement (i) of the lemma yields the matrices $\mathbf{A_{t,k}}$ for $k - t > 1$.

The point of the Kramer-Mesner matrices is the following theorem:

Theorem 6.1. (Kramer-Mesner, 1976, [35]) *There exists a G-invariant t-(v,k,λ) design $(V,\mathscr{B})$ if and only if there exists a vector $\mathbf{u} \in \{0,1\}^{s\times 1}$ satisfying the equation:*

$$\mathbf{A_{t,k}} \cdot \mathbf{u} = \lambda\,\mathbf{j} \tag{1}$$

where $\mathbf{j}$ is the r-dimensional vector of all ones.

C. A. Cusack and S. S. Magliveras gave a slight generalization of the above theorem for the existence of G-invariant large sets of t-designs.

Theorem 6.2. (Cusack and Magliveras, 1999, [19]) *A* [G]*-invariant* LS$[N](t,k,v)$ *exists if and only if there exists a matrix* $\mathbf{u} \in \{0,1\}^{s\times N}$*, with constant row sum* 1*, satisfying the matrix equation:*

$$\mathbf{A_{t,k}} \cdot \mathbf{u} = \lambda(\mathbf{j},\ldots,\mathbf{j}) = \lambda \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix}_{r\times N} . \tag{2}$$

Corollary 6.1. *Let* $\mathbf{U}$ *denote the* $N \times s$ *matrix whose rows are all vectors* $\mathbf{u^T}$ *satisfying equation* (1)*. Then, there exists a* ***large set*** *of* G*-invariant* t-(v,k,λ) *designs if and only if there exists a vector* $\mathbf{w} \in \{0,1\}^{1\times N}$ *such that* $\mathbf{w}\cdot\mathbf{U} = \mathbf{j}$.

Example 6.1. Let $G = \langle(1,2,3)(4,5,6)(7,8,9)\rangle$, $V = \{1,\ldots,9\}$ and t-$(v,k,\lambda) =$ 2-(9,3,1).

Representatives of the orbits of G on $\binom{V}{2}$ and $\binom{V}{3}$ are: 12, 14, 15, 16, 17, 18, 19, 45, 47, 48, 49, 78, and 123, 124, 125, 126, 127, 128, 129, 145, 146, 147, 148, 149, 156, 157, 158, 159, 167, 168, 169, 178, 179, 189, 456, 457, 458, 459, 478, 479, 489, 789, respectively, where, for brevity, we write (for example) 189 for the 3-set $\{1,8,9\}$. Computation of the matrix $A_{2,3}$ yields:

$$\mathbf{A_{2,3}} = \begin{pmatrix}
\mathbf{1}&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&1&1&\mathbf{1}&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&1&0&0&0&0&1&1&1&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&\mathbf{1}&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&0&0&\mathbf{1}&0&0&0&1&0&0&1&0&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&0&0&0&1&0&0&0&1&0&0&\mathbf{1}&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&1&0&0&0&0&1&0&0&0&\mathbf{1}&0&0&1&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&\mathbf{1}&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&1&0&0&0&1&0&0&0&0&1&1&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&\mathbf{1}&1&0&0&0&0&0&0&0&1&1&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&\mathbf{1}&0&0&0&0&0&1&0&1&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&1&1&1&\mathbf{1}
\end{pmatrix}$$

where the rows and columns are indexed according to the G-orbits on 2-sets and 3-sets in the order given by the orbit representatives above.

There are exactly 21 solutions which are presented as the rows of $\mathbf{U}$ representing the 21 2-(9,3,1) designs invariant under G.

$$\mathbf{U} = \begin{pmatrix}
\mathbf{1}&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&\mathbf{1}&0&\mathbf{1}&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&\mathbf{1}\\
1&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1\\
1&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&1\\
0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&\mathbf{1}&0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&0&0&0&0\\
0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&\mathbf{1}&0&0&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0\\
0&0&0&\mathbf{1}&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&\mathbf{1}&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&\mathbf{1}&0&0&0&\mathbf{1}&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0\\
0&0&0&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&\mathbf{1}&0&0&0&0&0&\mathbf{1}&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&\mathbf{1}&\mathbf{1}&0&0&0&0&0&0&0&0&\mathbf{1}&0&0&0&0&0&0&0&0&0&0&\mathbf{1}&0&0\\
0&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0
\end{pmatrix}$$

The 1's in a row of $\mathbf{U}$ select the G-orbits on 3-subsets which come together to form the particular 2-(9,3,1) design invariant under G. It is now clear that if two rows of $\mathbf{U}$ are orthogonal, they give rise to two disjoint designs. Moreover, it is equally clear that a vector $\mathbf{L} \in \{0,1\}^{1\times 21}$, such that $\mathbf{L}\cdot\mathbf{U} = \mathbf{J}$, $\mathbf{J}$ the all 1's vector, gives rise to a large set of 2-(9,3,1) designs, the designs corresponding to the 1's in $\mathbf{L}$. These large sets are of course $[G]$-invariant since each constituent design is fixed by G.

Collecting all possible solutions $\mathbf{L}$ to

$$\mathbf{L}\cdot\mathbf{U} = \mathbf{J}$$

as the rows of a new matrix $\mathbf{\Lambda}$ yields the 6 $[G]$-invariant large sets of 2-(9,3,1)'s:

$$\mathbf{\Lambda} = \begin{pmatrix}
\mathbf{1}&0&0&\mathbf{1}&0&0&0&0&\mathbf{1}&0&\mathbf{1}&0&0&\mathbf{1}&0&0&0&\mathbf{1}&\mathbf{1}&0&0\\
1&0&0&0&1&0&1&0&0&0&0&1&0&0&1&1&0&0&0&1&0\\
0&1&0&0&1&0&1&0&0&0&0&1&1&0&0&0&1&0&0&0&1\\
0&1&0&0&0&1&0&1&0&1&0&0&0&1&0&0&0&1&1&0&0\\
0&0&1&1&0&0&0&0&1&0&1&0&1&0&0&0&1&0&0&0&1\\
0&0&1&0&0&1&0&1&0&1&0&0&0&0&1&1&0&0&0&1&0
\end{pmatrix}$$

If we like playing this game of looking for *complete sets of orthogonal rows of* 0-1 *matrices* we may ask to find *large sets of large sets*, or **super large** sets. Indeed, there are exactly two solutions as presented by the rows of matrix $\mathbf{\Sigma}$ below. One may ask whether these complete sets of *orthogonal* large sets are orthogonal in the same sense as in our definition in the Preliminaries section. It is somewhat disappointing that our new kind of orthogonality is different from the one presented earlier, and does not yield Steiner tableaux. Large sets orthogonal in the new sense, are orthogonal because constituent designs of one large set are not constituents of the other and this is certainly an interesting notion to be further explored.

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 \end{pmatrix}.$$

Finally we note that the two super large sets are themselves orthogonal and complementary as is clear from matrix $\mathbf{\Sigma}$.

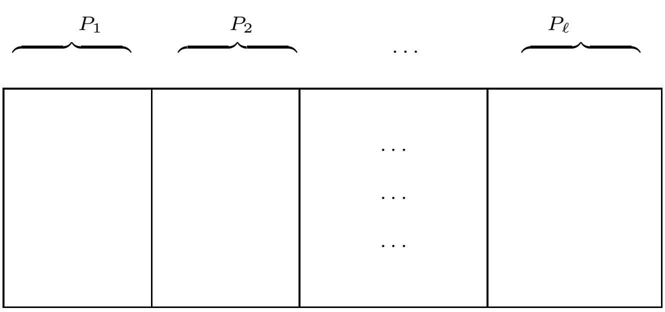
7. Semiregular large sets

Theorem 6.2 can be restated in a way which makes it transparent:

Remark 7.1. If $\mathbf{A} = \mathbf{A_{t,k}}$ is the KM matrix for a group action $G|V$, with $1 \leq t < k$, and if there exists a **partition** $\{P_1, P_2, \ldots, P_\ell\}$ of the columns of $\mathbf{A}$ such that the sum of column vectors within each part P_j is equal to $\lambda\mathbf{j}$, then there exists a $[G]$-invariant large set of t-(v, k, λ) designs.

If we were to reorder the columns of $\mathbf{A} = \mathbf{A_{t,k}}$, according to the blocks of the partition, then $\mathbf{A}$ takes the form:

$\mathbf{A_{t,k}} =$



Suppose now that $G|V$ is a t-homogeneous action. In this case, there is a single G-orbit on the t-subsets of V, and consequently $\mathbf{A_{t,k}}$ is a single-row matrix. The following proposition is a consequence of Remark 7.1.

$\mathbf{A_{t,k}}$:

	Γ_1	Γ_2	Γ_3		Γ_{r-1}	Γ_r
Δ_1	λ_1	λ_2	λ_3	$\cdots$	λ_{r-1}	λ_r

Proposition 7.1. *If $G|V$ is t-homogeneous, then any union of G-orbits of k-sets (corresponding to the columns of $\mathbf{A} = \mathbf{A_{t,k}}$) is a t-design. Moreover, if there is a partition of the columns of $\mathbf{A}$ which yields the same sum (λ) in each part, then, there is a large set of t-(v, k, λ) designs.*

7.1. The easy case

In particular, if $G|V$ is t-homogeneous, and all G-orbits of k-subsets have the **same length**, then, the collection of all G-orbits of k-subsets forms a large set of t-(v, k, λ) designs.

Now, suppose that $G|V$ is t-homogeneous, and that for some k, $1 \leq t < k < v$, no k-subset of V is fixed by any element of $G^* = G - \{1\}$, then each G-orbit of k-subsets is **regular**, forms a t-(v, k, λ) design with $b = |G|$, and the collection of all G-orbits of k-subsets of V forms a large set.

We call these "**G-semiregular** large sets" of t-designs.

7.2. Semiregular large sets from $\mathbf{PSL_2(q)}$

We now turn our attention to an infinite family of well understood groups which are 3-homogeneous. For $q \equiv 3 \pmod 4$, $G = PSL_2(q)$ is 3-homogeneous on the $q + 1$ points of the projective line $V = \mathbb{F}_q \cup \{\infty\}$.

Each element of G:

i) belongs to a cyclic subgroup of order $(q + 1)/2$ and fixes 0 points of V, or

ii) belongs to a cyclic subgroup of prime order p (where $q = p^a$), and fixes exactly 1 point of V, or

iii) belongs to a cyclic subgroup of order $(q - 1)/2$ and fixes exactly 2 points of V.

iv) Each element $x \in G$ is semiregular on the points not fixed by x, i.e. all cycles of x on $V - \operatorname{fix} x$ have the same length.

The table below exhibits a sample of prime powers $q \equiv 3 \pmod 4$, and for each such q, the values of the parameter k for which $PSL_2(q)$ acts semiregularly on $\binom{V}{k}$, indicated by the symbol $\surd$ in the table. An "x" indicates a value of k for which not all G-orbits on k-subsets are regular.

Some (q, k) pairs for which $\mathbf{PSL_2(q)}$ acts semiregularly on $\binom{V}{k}$

	•	• •	$\emptyset$	k									
v	q	$(q-1)/2$	$(q+1)/2$	5	6	7	8	9	10	11	12	13	14
20	19	9	10	x	x	x	x	x	x				
24	23	11	12	$\surd$	x	$\surd$	x	x	x	x	x	x	
28	27	13	14	$\surd$	x	x	x	x	x	$\surd$	x	x	x
32	31	15	16	x	x	x	x	x	x	x	x	x	x
44	43	21	22	x	x	x	x	x	x	x	x	x	x
48	47	23	24	$\surd$	x	$\surd$	x	x	x	$\surd$	x	$\surd$	x
60	59	29	30	x	x	$\surd$	x	x	x	$\surd$	x	$\surd$	x
68	67	33	34	x	x	x	x	x	x	x	x	x	x
72	71	35	36	x	x	x	x	x	x	x	x	$\surd$	x
80	79	39	40	x	x	x	x	x	x	x	x	x	x
84	83	41	42	$\surd$	x	x	x	x	x	$\surd$	x	$\surd$	x
104	103	51	52	x	x	x	x	x	x	x	x	x	x
108	107	53	54	$\surd$	x	$\surd$	x	x	x	$\surd$	x	$\surd$	x

Thus, for each pair (q, k) marked by $\surd$ in the table above, we can construct a large set of $[PSL_2(q)]$-invariant 3-$(q + 1, k, \lambda)$ designs. The obvious question that arises is: Are there infinitely many pairs (q, k) for which the action of G on $\binom{V}{k}$ is regular? The following proposition answers this question.

Theorem 7.1. (Cusack and Magliveras, 1999 [19]) *For each prime k there exists an infinite family of Large Sets of* 3-$(q + 1, k, \lambda)$ *designs where q is a prime power, $q \equiv 3 \pmod 4$.*

In composing elements to construct $[G]$-invariant large sets, it is not necessary for all G-orbits of k-sets to have equal lengths. The following theorem gives precise information on collecting and composing $[G]$-invariant t-(v, k, λ_i) designs to form a $\mathrm{LS}[N](t, k, v)$.

The splicing theorem:

Theorem 7.2. (Laue, Wassermann, and Magliveras, 2001, [43]) *Let $t < k < v$ be natural numbers, and V a set of v points. Suppose that for natural numbers λ and N, $\lambda N = \binom{v-t}{k-t}$. Let P be a partition of $\binom{V}{k}$ into n disjoint t-designs such that, for $j = 1, \ldots, n$, there are exactly a_j designs with parameter λ_j in P. Let $A = (a_{ij})$ be an $m \times n$ integer matrix such that $0 \le a_{ij} \le a_j$, and for each $i = 1, \ldots, m$:*

$$\sum_{j=1}^{n} a_{ij}\lambda_j = \lambda. \qquad (3)$$

Then, each integer solution vector $(N_1, \ldots, N_m)$ to the diophantine system:

$$(N_1, \ldots, N_m)A = (a_1, \ldots, a_n) \qquad (4)$$

determines a large set $\mathrm{LS}[N](t,k,v)$ *by selecting* N_i t-(v,k,λ) *designs which correspond to the ith row* $(a_{i1},\ldots,a_{in})$*. In such a solution* a_{ij} *designs have parameters* t-(v,k,λ_j)*.*

The following example appeared in [43], and illustrates the above theorem.

Example 7.1. Let $G = P\Gamma L(2,27)$ act on the projective line V of $v = 28$ points. Let $k = 11$ and $t = 3$. Since this group is 3-homogeneous, each k-orbit is a 3-design. These orbits form our partition P. $\binom{V}{11}$ partitions into $a_1 = 343$ designs with $\lambda_1 = 2970$, $a_2 = 33$ designs with $\lambda_2 = 1485$, and $a_3 = 14$ designs with $\lambda_3 = 990$. Note that 495 divides each of the λ_i. We have $\binom{v-t}{k-t} = \binom{25}{8} = 495 \cdot 5 \cdot 19 \cdot 23$. Let us search for an LS[5](3, 11, 28) so that $\lambda = 495 \cdot 19 \cdot 23$ using the above theorem. We can simplify our first equation

$$\sum_{j=1}^{n} a_{ij}\lambda_j = \lambda$$

by dividing both sides by 495 to get

$$6 \cdot a_{11} + 3 \cdot a_{12} + 2 \cdot a_{13} = 437 = 19 \cdot 23.$$

A first solution vector is $(a_{11}, a_{12}, a_{13}) = (72, 1, 1)$. A second solution is $(a_{21}, a_{22}, a_{23}) = (55, 29, 10)$. Then $(N_1, N_2) = (4, 1)$ solves the diophantine system (4) so that we have to combine 4 designs of the first kind with 1 design of the second kind to get the desired LS[5](3, 11, 28).

8. Challenges

We close the paper by mentioning some challenges that seem to resist solution attempts by a number of researchers over a rather long period.

- Does there exist a large set of 3-(v,4,1) designs?
- If $|V| = n^2 + n + 1$, where n is the order of a projective plane, does $\binom{V}{n+1}$ partition into a large set of mutually disjoint projective planes?
- For $|V| = 24$ does $\binom{V}{8}$ split into 969 mutually disjoint $S(5,8,24)$?
- Does there exist a 4-(21, 5, 1) design? Preferably resolvable into 57 projective planes of order 4?
- For $|V| = 21$ does $\binom{V}{5}$ resolve into 17 such 4-(21, 5, 1) designs?
- Do there exist large sets of Steiner systems for $t \geq 6$?

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