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TOPOLOGICAL PROPERTIES OF THE MULTIFUNCTION SPACE $L(X)$ OF CUSCO MAPS

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ABSTRACT. A set-valued mapping F from a topological space X to a topological space Y is called a cusco map if F is upper semicontinuous and $F(x)$ is a nonempty, compact and connected subset of Y for each $x \in X$. We denote by $L(X)$, the space of all subsets F of $X \times \mathbb{R}$ such that F is the graph of a cusco map from the space X to the real line $\mathbb{R}$. In this paper, we study topological properties of $L(X)$ endowed with the Vietoris topology.

1. Introduction

There have recently been many papers devoted to the study of topologies and convergences on spaces of set-valued maps. Without question graph convergence (i.e. Painleve-Kuratowski convergence of graphs) is the most studied convergence of set-valued maps. It has been used in a number of books and papers ([1], [2], [6], [10], [30]). In particular, graph convergence has found many applications to variational and optimization problems, differential equations and approximation theory. However for many purposes graph convergence is too weak (see [6]).

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As for topologies on spaces of set-valued maps, there are mainly two approaches in the literature — hyperspace topologies and function space topologies. There has been interest in studying extensions of natural topologies on the space of continuous functions to the space of densely continuous forms, and to the spaces of usco and minimal usco maps ([15], [14], [17], [20], [25]). Hyperspace topologies on set-valued maps with closed graphs were studied in [12], [18], [27], [29], [26], in which multifunctions are identified with their graphs and are considered as elements of a hyperspace.

A classical problem of approximations of relations by continuous functions leads to the study of special class of set-valued maps, called the cusco maps. For this problem, let X be a Hausdorff space, let $C(X)$ be the space of all continuous real-valued functions defined on X and let $CL(X \times \mathbb{R})$ be the hyperspace of all nonempty closed subsets of $X \times \mathbb{R}$, where $\mathbb{R}$ is the space of real numbers. It is known (see [4], [18], [19]) that if X is a locally connected, locally compact metric space without isolated points and $F \in CL(X \times \mathbb{R})$, then F can be approximated by continuous functions in the Hausdorff metric if and only if F is the graph of a cusco map. The fundamental result needed to prove the above theorem is due to Cellina [7].

In [21] the following analogy for the locally finite topology was proved. If X is a locally connected, countably paracompact normal q-space without isolated points and if $F \in CL(X \times \mathbb{R})$, then F can be approximated by continuous functions in the locally finite topology if and only if F is the graph of a cusco map. It was shown in [22] that if X is a countably paracompact normal space without isolated points and $F \in CL(X \times \mathbb{R})$ is the graph of a cusco map, then F can be approximated by continuous functions in the locally finite topology, and also in the Vietoris topology. The cusco maps and minimal cusco maps are also important tools in convex analysis (see [5]).

In our paper we will study topological properties of cusco maps equipped with the Vietoris topology.

2. Preliminaries

We refer to Beer [3] and Engelking [13] for basic notions. If X and Y are nonempty sets, a set-valued mapping or multifunction from X to Y is a mapping that assigns to each element of X a (possibly empty) subset of Y. If T is a set-valued mapping from X to Y, then its graph is $\{\langle x, y\rangle : y \in T(x)\}$.

If F is a subset of $X \times Y$ and $x \in X$, define $F(x) = \{y \in Y : \langle x, y\rangle \in F\}$. We assign to each subset F of $X \times Y$ a set-valued mapping which takes the value $F(x)$ at each point $x \in X$. Then F is the graph of the set-valued mapping. In this paper, we identify mappings with their graphs.

Let X and Y be topological spaces, and let T be a set-valued mapping from X to Y. Then T is called *upper semicontinuous* (*usc*) if for each $x \in X$ and any open set V containing $T(x)$, there exists a neighbourhood U_x of x such that $T(z) \subseteq V$ for all $z \in U_x$. (In the literature there is also a weaker notion of c-upper semicontinuity ([16]), which is closely related to the notion of closedness of graph.) Following Christensen [8] we say that T is a usco map if T is a usc map such that $T(x)$ is a nonempty compact set for all $x \in X$. Similarly, we say that T is cusco if it is usco and $T(x)$ is connected for all $x \in X$. In the literature, the notation cusco ([5]) is also used for usco maps with convex values in a topological vector space. Since we are working only with multifunctions with values in $\mathbb{R}$, both of these notations coincide in our case.

To describe the hypertopologies that we are using in this paper, we need to introduce the following notation. Let (X, τ) be a topological space and $CL(X)$ be the hyperspace of all nonempty closed subsets of X. For $U \subseteq X$, define

$$U^+ = \{A \in CL(X) : \ A \subseteq U\} \qquad \text{and} \qquad U^- = \{A \in CL(X) : \ A \cap U \neq \emptyset\}.$$

If $\mathscr{U}$ is a family of sets in X, define $\mathscr{U}^- = \bigcap\{U^- : \ U \in \mathscr{U}\}$.

A subbase for the Vietoris (resp., locally finite) topology on $CL(X)$ (see [3]) are the sets of the form U^+ with $U \in \tau$ and of the form $\mathscr{U}^-$ with $\mathscr{U} \subseteq \tau$ finite (resp., locally finite). The lower Vietoris topology τ_{V^-} on $CL(X)$ is generated by all subcollections of the form G^-, where $G \in \tau$; similarly the upper Vietoris topology τ_{V^+} is generated by all G^+, where $G \in \tau$. The supremum $\tau_{V^+} \vee \tau_{V^-}$ is the Vietoris topology τ_V. Also note that a set-valued mapping T from a space X to a space Y, such that $T(x)$ is a nonempty closed subset of Y for each $x \in X$, is usc if and only if it is continuous considered as a map from X to the space $(CL(Y), \tau_{V^+})$.

3. Some basic results

In the sequel we will always denote by X a Hausdorff space. Let $C(X)$ be the space of all continuous real-valued functions defined on X and let $CL(X \times \mathbb{R})$ be the hyperspace of all nonempty closed subsets of $X \times \mathbb{R}$. Denote by $L(X)$ the space of all graphs of cusco maps with values in $\mathbb{R}$. Note that $L(X)$ is a subset of $CL(X \times \mathbb{R})$ and hence can be endowed with any hyperspace topology inherited from $CL(X \times \mathbb{R})$. The spaces $CL(X \times \mathbb{R})$, $L(X)$, $CL(X)$ and $C(X)$ all are endowed with the Vietoris topology τ_V, unless otherwise mentioned.

We state some basic facts about cusco maps in the following lemma whose proof is left to the reader.

Lemma 3.1. *For a Hausdorff space X, the following statements are equivalent.*

(a) *$F \subset X \times \mathbb{R}$ is the graph of a cusco map.*

(b) *F is a closed, locally bounded subset of $X \times \mathbb{R}$ with $F(x)$ nonempty and connected for each $x \in X$.*

(c) *There exist real-valued functions f and g on X with $f \le g$ and f and g lower and upper semicontinuous respectively such that $F(x) = [f(x), g(x)]$ for each $x \in X$.*

Proposition 3.2. *For a Hausdorff space X, the space $CL(X)$ (with the upper Vietoris, lower Vietoris, Vietoris topology) is embeddable in $L(X)$ (with upper Vietoris, lower Vietoris, Vietoris topology, respectively).*

Proof. Let $E \in CL(X)$. Define $F_E = E \times [0,1] \cup (X \setminus E) \times \{0\} \in L(X)$ and let $\mathscr{F} = \{F_E : E \in CL(X)\} \subseteq L(X)$. Define a map $\phi\colon CL(X) \to \mathscr{F}$ by $\phi(E) = F_E$ for every $E \in CL(X)$. Clearly ϕ is one-one.

We show that ϕ is a homeomorphism from $(CL(X), \tau_{V^+})$ to $(L(X), \tau_{V^+})$. Let $A \in CL(X)$ and let W^+ be an open neighbourhood of F_A. Let U be an open subset of X such that $A \times [0,1] \subseteq U \times [0,1] \subseteq W$. Now it is easy to verify that for each $B \in U^+ \cap CL(X)$, $\phi(B) = F_B \in W^+$. This shows that ϕ is continuous. Similarly, it can be shown that ϕ is an open map.

In order to show that ϕ is a homeomorphism from $(CL(X), \tau_{V^-})$ to $(L(X), \tau_{V^-})$, let $A \in CL(X)$ and let W^- be an open neighbourhood of F_A in $(L(X), \tau_{V^-})$. Let $\langle x, t\rangle \in W \cap F_A$. If $t = 0$, then $\phi(CL(X)) \subseteq W^-$. So let $t \neq 0$ and choose some open neighbourhood U of x and an open interval V of t such that $\langle x, t\rangle \in U \times V \subseteq W$. Now it can be easily seen that if $B \in U^-$, then $F_B \in W^-$. Similarly, if U^- is an open neighbourhood of some $A \in CL(X)$, then $(U \times (1/2, 2))^- \subseteq \phi(U^-)$. Therefore, ϕ is a homeomorphism from $(CL(X), \tau_{V^-})$ to $(L(X), \tau_{V^-})$. Consequently, ϕ is a homeomorphism from $CL(X)$ to $L(X)$ with the Vietoris topology. $\square$

Proposition 3.3. *Let X be a regular space. Then the space $CL(X)$ is embeddable as a closed subspace of $L(X)$.*

Proof. We shall show that the subspace $\mathscr{F} = \{F_A = A \times [0,1] \cup (X \setminus A) \times \{0\} : A \in CL(X)\}$ is closed in $L(X)$. Let $F \in L(X) \setminus \mathscr{F}$. If F is zero function, that is, $F(x) = \{0\}$ for all $x \in X$, then $(X \times (-1,1))^+$ is an open neighbourhood of F containing no member of $\mathscr{F}$. So let us suppose that F is not a zero function. Since $F \notin \mathscr{F}$, there exists some $x \in X$ such that $F(x) \neq [0,1]$ and $F(x) \neq \{0\}$. If $a \in F(x) \setminus [0,1]$, then we can find an open interval V containing a such that $V \cap [0,1] = \emptyset$. Let U be an open neighbourhood of x. Then $F \in (U \times V)^-$ but no $F_E \in (U \times V)^-$ for each $E \in CL(X)$.

Now if for each $z \in X$, $F(z) \subseteq [0,1]$, then obviously $F(x) \subset [0,1]$. We shall show that there exists an open neighbourhood of F in $L(X)$ that contains no member of $\mathscr{F}$. Let us consider the case when $F(x) = [0,a]$ where $0 < a < 1$. Since F is closed, we can find an open subset U containing x such that $1 \notin F(y)$ for all $y \in U$. By regularity of X, we can find some open subset U_0 of X such that $x \in U_0 \subseteq \overline{U_0} \subseteq U$. Define $G = U \times (-1/2, 1) \cup (X \setminus \overline{U_0}) \times \mathbb{R}$. Then $F \in G^+ \cap (U_0 \times (0,1))^-$ but $F_E \notin G^+ \cap (U_0 \times (0,1))^-$ for all $E \in CL(X)$.

Now if $F(x) = [a,b]$ where $0 < a \leq b \leq 1$, then G^+ where $G = X \times (\mathbb{R} \setminus \{0\})$, is an open set containing F such that $\mathscr{F} \cap G^+ = \emptyset$.

Hence $\mathscr{F}$ is a closed subset of $L(X)$. □

The previous proposition shows that for a regular space X, the space $CL(X)$ and consequently X can be considered as a closed subset of $L(X)$. Also, as was mentioned above, for a binormal (countably paracompact, normal) space X without isolated points the space $C(X)$ of all real-valued continuous functions is dense in $L(X)$. So in this paper, we shall study to what extent the topological properties of $C(X)$ and $CL(X)$ can be extended to the space $L(X)$.

Lemma 3.4. *For any Hausdorff spaces X and Y,*

$$hd(X)hd(Y) \leq hd(X \times Y) \leq hd(X)hd(Y)\min\{w(X), w(Y)\}.$$

Consequently, $hd(X \times \mathbb{R}) = hd(X)$.

Proof. Since X and Y can be embedded in $X \times Y$, $hd(X) \leq hd(X \times Y)$ and $hd(Y) \leq hd(X \times Y)$.

Now in order to prove the second inequality, first note that due to symmetry, it suffices to show that $hd(X \times Y) \leq hd(X)w(Y)$. Let F be a nonempty subset of $X \times Y$ and let $\mathscr{B}$ be a base for Y with $|\mathscr{B}| = w(Y)$. For each $U \in \mathscr{B}$, define the set

$$A(F,U) = \{x \in X : \ F(x) \cap U \neq \emptyset\}.$$

Let $D(F,U)$ be a dense subset of $A(F,U)$ with $|D(F,U)| \leq hd(X)$ and let $D = \bigcup\{D(F,U) : \ U \in \mathscr{B}\}$. Now for each $x \in D$, let D_x be a dense subset of $F(x)$ with cardinality less than or equal to $w(Y)$. Define the set

$$D_0 = \bigcup_{x \in D} \{x\} \times D_x.$$

It is clear that $|D_0| \leq hd(X)w(Y)$. We claim that D_0 is a dense subset of F. Let $G \times H$ be a basic open subset of $X \times Y$ such that $G \times H \cap F \neq \emptyset$. Let $\langle x, t\rangle \in G \times H \cap F$. Choose some $U \in \mathscr{B}$ such that $t \in U \subseteq H$. Now, since $x \in A(F,U) \cap G$, we can find some $z \in G \cap D(F,U)$. Further, we can find some $s \in D_z \cap U$ and consequently, $\langle z, s\rangle \in D_0 \cap G \times H$, showing that D_0 is a dense subset of F. □

Note that for the inequalities given in the above lemma, we can find spaces such that the equalities are attained. Consider the space X given in [13, Example 1.6.19]. Then $\aleph_0 = hd(X) = hd(X \times X) < hd(X)w(X)$; and for the Sorgenfrey line S, we have $\aleph_0 = hd(S) < hd(S \times S) = hd(S)w(S)$.

We give a similar result for π-weight in the next lemma which can be proved in a similar way as Lemma 3.4.

Lemma 3.5. *For Hausdorff spaces X and Y,*

$$h\pi w(X)h\pi w(Y) \leq h\pi w(X \times Y) \leq h\pi w(X)h\pi w(Y) \min\{w(X), w(Y)\}.$$

Consequently, $h\pi w(X \times \mathbb{R}) = h\pi w(X)$.

4. Cardinal functions on $L(X)$

Here we study cardinal functions on $L(X)$ with the Vietoris, lower Vietoris and upper Vietoris topologies. We suppose that all cardinal functions are greater than or equal to $\aleph_0$.

In order to study the character of $L(X)$ with the lower Vietoris, upper Vietoris and Vietoris topologies, we need the following basic lemma.

Lemma 4.1. *Let $F \in L(X)$ and W be an open set containing F. Then there exists an open subset G of $X \times \mathbb{R}$ such that $F \subseteq G \subseteq W$ and $G(x)$ is connected for each $x \in X$.*

Proof. Since W is an open set containing F and for each $x \in X$, $F(x)$ is a compact interval in $\mathbb{R}$, for each $x \in X$, we can find some open neighbourhood U_x of x and an open interval V_x such that $\{x\} \times F(x) \subseteq U_x \times V_x \subseteq W$. Also by upper semicontinuity of F, we can assume that $F(U_x) \subseteq V_x$ for each $x \in X$. Define $G = \bigcup\{U_x \times V_x : x \in X\}$. Clearly G is an open set and $F \subseteq G \subseteq W$. Also for each $x \in X$, $G(x) = \bigcup\{V_z : x \in U_z\}$ is a connected set. $\square$

Proposition 4.2. *For a Hausdorff space X, the following hold.*

(a) $\chi((L(X), \tau_{V^+})) = \sup\{\chi(A, X \times \mathbb{R}) : A \in L(X)\}$.

(b) $\chi((L(X), \tau_{V^-})) = hd(X) \cdot \chi(X) = \chi((CL(X), \tau_{V^-}))$.

(c) $\chi(L(X)) = \chi((L(X), \tau_{V^+})) \cdot \chi((L(X), \tau_{V^-}))$.

Proof. Part (a) is immediate. The second equality in part (b) follows from [23, Theorem 2.2]. Now we shall prove the first equality in part (b).

Since $CL(X)$ can be considered as a subspace of $L(X)$ with the lower Vietoris topology,

$$\chi((L(X), \tau_{V^-})) \geq \chi((CL(X), \tau_{V^-})) = hd(X) \cdot \chi(X).$$

Also, by Lemma 3.4,

$\chi((L(X),\tau_{V^-}))\le\chi((CL(X\times\mathbb{R}),\tau_{V^-}))=hd(X\times\mathbb{R})\cdot\chi(X\times\mathbb{R})=hd(X)\cdot\chi(X)$.

Hence $\chi((L(X),\tau_{V^-}))=hd(X).\chi(X)$.

Now we prove part (c). By [9, Proposition 2.1], we have

$$\chi(L(X))\le\chi((L(X),\tau_{V^+}))\chi((L(X),\tau_{V^-})).$$

Now by using part (b) and Proposition 3.2, we have

$$\chi((L(X),\tau_{V^-}))\le\chi(CL(X))\le\chi(L(X)).$$

We need to show that $\chi((L(X),\tau_{V^+}))\le\chi(L(X))$. Let $F\in L(X)$ and $\mathscr{B}$ be a base of open neighbourhoods at F with $|\mathscr{B}|\le\chi(L(X))$. By using Lemma 4.1, we can assume that $\mathscr{B}$ is of the following form:

$$\mathscr{B}=\bigl\{W_i^+\cap\mathscr{F}_i^-:\ i\in\mathscr{I}\bigr\},$$

where W_i is an open subset of $X\times\mathbb{R}$ containing F with $W_i(x)$ connected for each $x\in X$ and $\mathscr{F}_i$ is a finite family of open sets such that $F\in\mathscr{F}_i^-$. We shall show that $\mathscr{B}'=\bigl\{W_i^+:\ i\in\mathscr{I}\bigr\}$ forms a base for $(L(X),\tau_{V^+})$ at F. Let W be an open subset of $X\times\mathbb{R}$ such that $F\in W^+$. Since $\mathscr{B}$ forms a base at F in $(L(X),\tau_V)$, we can find some $i\in\mathscr{I}$ with $W_i^+\cap\mathscr{F}_i^-\subseteq W^+$. We claim that $W_i\subseteq W$. Suppose by way of contradiction, there exists some $\langle x,t\rangle\in W_i\setminus W$. Let $t>\max F(x)$. The case when $t<\min F(x)$ is similar. Now since $W_i(x)$ is connected, the set $F'=F\cup\bigl(\{x\}\times[\max F(x),t]\bigr)\in W_i^+\cap\mathscr{F}_i^-\setminus W^+$, which is a contradiction. Hence $W_i\subseteq W$. $\square$

Lemma 4.3. *Let X be a binormal space. Then for every lower (upper) semicontinuous function f, there exists a family $\mathscr{F}$ of continuous functions such that $|\mathscr{F}|\le\sup\bigl\{\psi(A,X):\ A\in CL(X)\bigr\}$, $g<f$ $(g>f)$ for each $g\in\mathscr{F}$ and for each $x\in X$, $f(x)=\sup\bigl\{g(x):\ g\in\mathscr{F}\bigr\}$ $\bigl(f(x)=\inf\bigl\{g(x):\ g\in\mathscr{F}\bigr\}\bigr)$.*

P r o o f. Let $\sup\bigl\{\psi(A,X):\ A\in CL(X)\bigr\}=\gamma$. Let f be a lower semicontinuous function which takes only a finite number of values $r_1,r_2,\dots,r_k$ $(r_1<r_2<\dots<r_k)$. Then for each $i=1,\dots,k$, the set $A_i=f^{-1}(r_i)$ is the intersection of a closed and an open subset of X and hence can be expressed as a union of a family $\mathscr{A}_i$ of closed sets where $|\mathscr{A}_i|\le\gamma$. Let $\mathscr{A}=\bigcup\limits_{i=1}^{k}\mathscr{A}_i$. Since X is normal, by [13, Problem 2.7.4(b)], for each $F\in\mathscr{A}$ and for each $n\in\mathbb{N}$, we can find some continuous function $f_{F,n}$ such that $f_{F,n}<f$ and $f(x)-f_{F,n}(x)\le 1/n$ for all $x\in F$.

Let f be a bounded lower semicontinuous function such that $0<f<1$. For each $j,i\in\mathbb{N}$, $0\le j<i$, define $C_{j,i}=\bigl\{x\in X:\ j/i<f(x)\le(j+1)/i\bigr\}$. Now define $g_i\colon X\to\mathbb{R}$ by $g_i(x)=j/i$ for all $x\in C_{j,i}$. Then clearly g_i is a lower semicontinuous function, $g_i<f$ and $f(x)=\lim\limits_{n\to\infty}g_n(x)$ for all $x\in X$. As shown

above, for each $n \in \mathbb{N}$, we can find a family $\mathscr{F}_n$ of continuous functions with cardinality at most γ such that $g_n = \sup\{h : h \in \mathscr{F}_n\}$. Let $\mathscr{F} = \bigcup\{\mathscr{F}_n : n \in \mathbb{N}\}$. Clearly $|\mathscr{F}| \le \gamma$ and $f = \sup\{h : h \in \mathscr{F}\}$.

Now let f be any lower semicontinuous function. Then the function $g = (\arctan(f) + \pi/2)/\pi$ is a lower semicontinuous function such that $0 < g < 1$. Then as shown above, we can find a family $\mathscr{F}$ with cardinality at most γ such that $g(x) = \sup\{h(x) : h \in \mathscr{F}\}$ for all $x \in X$. Also, since X is binormal and $0 < g < 1$, by [13, Problem 5.5.20(a)], we can assume that $0 < h < g < 1$ for each $h \in \mathscr{F}$. Now, let $\mathscr{F}' = \{\tan(\pi h - \pi/2) : h \in \mathscr{F}\}$. Then clearly $k < f$ for all $k \in \mathscr{F}'$ and $f(x) = \sup\{k(x) : k \in \mathscr{F}'\}$. □

Before giving the next result, we introduce the following notation:

$$M(X) = \{\langle f, g\rangle \in C(X) \times C(X) : f(x) < g(x) \text{ for all } x \in X\},$$

and for every $\langle f, g\rangle \in M(X)$,

$$M_{f,g} = \{\langle x, t\rangle \in X \times \mathbb{R} : f(x) < t < g(x)\}.$$

Note that for each $\langle f, g\rangle \in M(X)$, $M_{f,g}$ is an open set in $X \times \mathbb{R}$ such that

$$\overline{M_{f,g}} = \{\langle x, t\rangle \in X \times \mathbb{R} : f(x) \le t \le g(x)\}.$$

Proposition 4.4. *Let X be a binormal space and D be a dense subset of $C(X)$. Then the set*

$$\mathscr{B} = \{M^+_{f,g} : \langle f, g\rangle \in (D \times D) \cap M(X)\}$$

forms a base for $(L(X), \tau_{V^+})$.

Proof. The proof follows from Lemma 4.1 and [21, Lemma 4.1] and the result that the Vietoris topology on $C(X)$ coincides with the graph (upper Vietoris) topology. □

Theorem 4.5. *For a binormal space X,*

$$\psi(L(X)) = \sup\{\psi(A, X) : A \in CL(X)\} \cdot h\pi w(X).$$

Proof. By [23, Theorem 2.17] and Proposition 3.2,

$$\sup\{\psi(A, X) : A \in CL(X)\} h\pi w(X) = \psi(CL(X)) \le \psi(L(X)).$$

So we only need to prove that $\psi(L(X)) \le \lambda \cdot \mu$, where we take $\lambda = \sup\{\psi(A, X) : A \in CL(X)\}$ and $\mu = h\pi w(X)$. Let $F \in L(X)$ with f and g its respective lower and upper boundaries. Then, by Lemma 3.1, f is lower semicontinuous and g is upper semicontinuous. Hence, by Lemma 4.3, we can find families $\mathscr{F}$ for f and $\mathscr{G}$ for g which satisfy the conditions of the lemma. Note that $\mathscr{F} \times \mathscr{G} \subseteq M(X)$. Define the set

$$\mathscr{W} = \{M^+_{f,g} : \langle f, g\rangle \in \mathscr{F} \times \mathscr{G}\}.$$

Clearly $|\mathscr{W}| \le \lambda$.

Now let $\mathscr{U}$ be a family of open subsets of $X \times \mathbb{R}$ which forms a π-base for F (considered as a subspace of $X \times \mathbb{R}$), with $|\mathscr{U}| \leq h\pi w(X \times \mathbb{R}) = h\pi w(X) = \mu$ (see Lemma 3.5). Now, define the family

$$\mathscr{B} = \{W^+ \cap U_1^- \cap \cdots \cap U_n^- : \ W \in \mathscr{W}, \ U_1, \ldots, U_n \in \mathscr{U}, \ n \in \mathbb{N}\}.$$

We claim that $\{F\} = \bigcap \mathscr{B} = \bigcap\{G : \ G \in \mathscr{B}\}$. The inclusion $\{F\} \subseteq \bigcap \mathscr{B}$ is obvious. We need to show the reverse inclusion. Let $E \in L(X)$ such that $E \neq F$. Let $\langle x, t\rangle \in E \setminus F$. Then, we can find some $f_1 \in \mathscr{F}$ and $g_1 \in \mathscr{G}$ such that $\langle x, t\rangle \notin M_{f_1, g_1}$. Hence $E \notin \bigcap \mathscr{B}$. So $E \subset F$. Let $\langle x, t\rangle \in F \setminus E$. Since $\mathscr{U}$ forms a π-base for F, there exists some $U \in \mathscr{U}$ such that $U \cap F \subseteq E^c \cap F$. Since $E \subset F$, $U \cap E = \emptyset$ and consequently, $E \notin U^-$. Hence $E \notin \bigcap \mathscr{B}$, which proves the required equality.

Since $|\mathscr{B}| \leq \lambda \cdot \mu$, $\psi(L(X)) \leq \lambda \cdot \mu$. □

The following theorem shows that for a compact space X, the character of $L(X)$ is equal to the character of $CL(X)$.

Theorem 4.6. *Let X be a compact space. Then*

(a) $\chi((L(X), \tau_{V^+})) = \sup\{\psi(A, X) : \ A \in CL(X)\} = \chi((CL(X), \tau_{V^+}))$

(b) $\chi(L(X)) = hd(X) \cdot \sup\{\psi(A, X) : \ A \in CL(X)\} = \chi(CL(X))$

(c) $\psi(L(X)) = \chi(L(X))$.

Proof. We shall prove part (a). Part (b) will then easily follow from Proposition 4.2, [23, Theorem 2.2] and the result that for a compact space X, $\psi(A, X) = \chi(A, X)$, for any $A \in CL(X)$.

Since $(CL(X), \tau_{V^+})$ can be considered as a subspace of $(L(X), \tau_{V^+})$,

$$\chi((L(X), \tau_{V^+})) \geq \chi((CL(X), \tau_{V^+})) = \sup\{\chi(A, X) : \ A \in CL(X)\}.$$

We shall prove the reverse inequality. Let $E \in L(X)$ and let f, g be its lower and upper boundaries respectively. By Lemma 4.3, we can find families $\mathscr{F}$ and $\mathscr{G}$ of continuous functions with cardinality at most $\sup\{\chi(A, X) : \ A \in CL(X)\}$ such that for each $h \in \mathscr{F}$ and $k \in \mathscr{G}$, $h < f$ and $k > g$ and $f(x) = \sup\{h(x) : \ h \in \mathscr{F}\}$ and $g(x) = \inf\{k(x) : \ k \in \mathscr{G}\}$ for all $x \in X$. Let $\mathscr{F}'$ and $\mathscr{G}'$ be the set of all finite nonempty subsets of $\mathscr{F}$ and $\mathscr{G}$ respectively and for each $F \in \mathscr{F}'$ and $G \in \mathscr{G}'$, let $f_F = \max\{h : \ h \in F\}$ and $g_G = \min\{k : \ k \in G\}$. Note that if $F_1, F_2 \in \mathscr{F}'$ such that $F_1 \subseteq F_2$, then $f_{F_1} \leq f_{F_2}$ and for $G_1, G_2 \in \mathscr{G}'$ with $G_1 \subseteq G_2$, $g_{G_1} \geq g_{G_2}$. Then $(f_F)_{F \in \mathscr{F}'}$ and $(g_G)_{G \in \mathscr{G}'}$ are nets in $C(X)$ such that $f_F < f$ for all $F \in \mathscr{F}'$ and $g_G > g$ for all $G \in \mathscr{G}'$. Also $f(x) = \sup\{f_F(x) : F \in \mathscr{F}'\}$ and $g(x) = \inf\{g_G(x) : \ G \in \mathscr{G}'\}$ for each $x \in X$. Now consider the family $\mathscr{B} = \{M^+_{f_F, g_G} : \ F \in \mathscr{F}', \ G \in \mathscr{G}'\}$. Clearly $|\mathscr{B}| \leq \sup\{\chi(A, X) : A \in CL(X)\}$. We shall show that $\mathscr{B}$ forms a base for E in $(L(X), \tau_{V^+})$. Let

$E \in W^+$, for some open set W. Since X is binormal, we can assume that W^+ is of the form $M^+_{a,b}$, where $\langle a, b\rangle \in M(X)$.

The assumption that for each $F \in \mathscr{F}'$, there exists some x_F such that $f_F(x_F) \leq a(x_F)$ leads to a contradiction. Since X is compact, the net $(x_F)_{F\in\mathscr{F}'}$ has a cluster point x in X. Since $a < f$ and due to our choice of f_F, we can choose some $F_0 \in \mathscr{F}'$ such that $a(x) < f_{F_0}(x) < f(x)$. Now the set $U = \bigl\{y \in X : \ a(y) < f_{F_0}(y)\bigr\}$ is an open neighbourhood of x. Since x is a cluster point of (x_F), there exists some $F_1 > F_0$ such that $x_{F_1} \in U$, that is, $f_{F_1}(x_{F_1}) \leq a(x_{F_1}) < f_{F_0}(x_{F_1})$. This is a contradiction to the fact that $f_{F_0} \leq f_{F_1}$. Hence there exists some $F \in \mathscr{F}'$ such that $a(x) < f_F(x)$ for all $x \in X$. Similarly, there exists some $G \in \mathscr{G}'$ such that $b(x) > g_G(x)$ for all $x \in X$. Therefore, for $\langle f_F, g_G\rangle$, $E \in M^+_{f_F,g_G} \subseteq W^+$.

Part (c) follows from part (b) and the inequality that for any topological space Z, $hd(Z) \leq h\pi w(Z)$. □

The next theorem can be proved by using the same technique as in the proof of Theorem 4.6.

Theorem 4.7. *Let X be a countably compact perfectly normal space. Then the following hold.*

(a) $(L(X), \tau_{V^+})$ *is first countable.*

(b) $\chi(L(X)) = hd(X) = \psi(L(X))$.

Theorem 4.8. *Let X be a binormal space. Then $t(L(X)) = \chi(L(X))$.*

P r o o f. By Lemma 3.4, [23, Theorem 2.2, Proposition 2.6], we have $hd(X \times \mathbb{R})\cdot \chi(X \times \mathbb{R}) = hd(X) \cdot \chi(X) \leq \chi(CL(X)) = t(CL(X))$. Since $CL(X)$ can be considered as a subspace of $L(X)$, $t(CL(X)) \leq t(L(X))$. So let $t(L(X)) = \gamma$. Let $F \in L(X)$. Let D be a dense subset of F such that $|D| \leq \gamma$ and for each $x \in D$, let B_x be a base at x with $|B_x| \leq \gamma$. Let $\mathscr{B} = \bigcup\bigl\{B_x : \ x \in D\bigr\}$. Obviously, $|\mathscr{B}| \leq \gamma$. Define a set $G = \bigl\{\overline{M_{f,g}} : \ \langle f, g\rangle \in M(X) \text{ and } F \subseteq M_{f,g}\bigr\}$.

We claim that $F \in \overline{G}$. Let $W^+ \cap W_1^- \cap \cdots \cap W_n^-$ be an open neighbourhood of F. Then by binormality of X and by Proposition 4.4, there exists some $\langle f, g\rangle \in M(X)$ such that $F \subseteq M_{f,g} \subseteq \overline{M_{f,g}} \subseteq W$. Also, since $F \subseteq M_{f,g}$, $\overline{M_{f,g}} \in W_1^- \cap \cdots \cap W_n^-$, thus showing that $F \in \overline{G}$.

Hence we can find some subset G' of G such that $|G'| \leq \gamma$ and $F \in \overline{G'}$. Now take $\mathscr{G} = \bigl\{W^+ \cap U_1^- \cap \cdots \cap U_n^- : \ \overline{W} \in G', \ U_1, \ldots, U_n \in \mathscr{B}\bigr\}$. Clearly $|\mathscr{G}| \leq \gamma$ and $\mathscr{G}$ forms a base for F in $L(X)$. Therefore $\chi(L(X)) \leq t(L(X))$ and consequently $t(L(X)) = \chi(L(X))$. □

Proposition 4.9. *For a Hausdorff space X, the following statements hold.*

(a) $w((L(X),\tau_{V^-}))=w(X)$.

(b) $w(L(X))=w((L(X),\tau_{V^+}))\cdot w((L(X),\tau_{V^-}))$.

Proof. The proof of part (a) easily follows from the fact that if $\mathscr{B}$ is a base for X, then $\mathscr{B}'=\{(B\times(p,q))^- :\ B\in\mathscr{B},\ p,q \text{ rationals } (p<q)\}$ forms a subbase for $(L(X),\tau_{V^-})$.

For part (b), it is easy to verify that $w(L(X))\le w((L(X),\tau_{V^+}))\cdot w((L(X),\tau_{V^-}))$. Also, by part (a) and Proposition 3.2, $w((L(x),\tau_{V^-}))=w(X)\le w(L(X))$. The proof of the inequality $w((L(X),\tau_{V^+}))\le w(L(X))$ is similar to the proof of (c) in Proposition 4.2. □

The next theorem gives the weight of $L(X)$ with the upper Vietoris and Vietoris topologies, for a binormal space X. But before stating the theorem, we would like to give the following lemma.

Lemma 4.10. *Let X be a Tychonoff space. Then $w(X)\le w(C(X))$.*

Proof. Let $\mathscr{B}$ be a base for $C(X)$ with $|\mathscr{B}|\le w(C(X))$. Without loss of generality, we can assume that $\mathscr{B}$ is of the following form:

$$\mathscr{B}=\{W_\alpha^+ : W_\alpha \text{ is an open set such that for each } x\in X,\ W_\alpha(x) \text{ is connected}\}.$$

Now for each $W_\alpha^+\in\mathscr{B}$, define

$$U_\alpha=\{x\in X:\ \langle x,0\rangle\in W_\alpha\}.$$

Clearly U_α is an open set. We claim that the family $\mathscr{U}=\{U_\alpha\}$ forms a base for X. Let $x\in X$ and U be an open neighbourhood of x. Let $f\in C(X)$ such that $0\le f\le 1$, $f(x)=0$ and $f(X\setminus U)=\{1\}$. Now define an open set $W=U\times(-1,\infty)\cup X\times(1/2,\infty)$. Clearly $f\in W^+$. Since $\mathscr{B}$ forms a base for $C(X)$, we can find some W_α^+ such that $f\in W_\alpha^+\subseteq W^+$.

We claim that $U_\alpha\subseteq U$. Suppose by way of contradiction, there exists some $x_0\in U_\alpha\setminus U$. Now since $f\in W_\alpha^+$ and $f(x_0)=1$, $1\in W_\alpha(x_0)$. Due to our choice of W_α, $[0,1]\subseteq W_\alpha(x_0)$. Hence we can find some open neighbourhood V of x_0 such that $V\times[0,1]\subseteq W_\alpha$. Choose a continuous function g on X such that $0\le g\le 1$, $g(x_0)=0$ and $g(X\setminus V)=\{1\}$. Then the continuous function h defined by $h(x)=f(x)g(x)$ for all $x\in X$ belongs to $W_\alpha^+\setminus W^+$, which is a contradiction to our choice of W_α. □

We end this section with the result that shows, for a binormal space X, the cellularity, the density and the weight of $L(X)$ with the Vietoris and upper Vietoris topologies are equal.

Theorem 4.11. *Let X be a binormal space. Then the following hold.*

(a) $c(L(X)) = d(L(X)) = w(L(X)) = w(C(X))$

(b) $c((L(X), \tau_{V^+})) = d((L(X), \tau_{V^+})) = w((L(X), \tau_{V^+})) = w(C(X))$.

Proof. We only prove part (a). The proof of part (b) is similar. It is well-known that $c(Z) \leq d(Z) \leq w(Z)$ for any topological space Z. So we need to prove $w(L(X)) \leq c(L(X))$.

By Proposition 4.4, $w((L(X), \tau_{V^+})) \leq d(C(X)) \leq w(C(X))$. Also since the Vietoris topology on $C(X)$ coincides with the graph topology, $C(X)$ can be considered as a subspace of $(L(X), \tau_{V^+})$, and hence $w((L(X), \tau_{V^+})) = w(C(X))$. So, by Proposition 4.9, we get that $w(L(X)) = w(C(X)) \cdot w(X)$. Hence by Lemma 4.10, we have $w(L(X)) = w(C(X))$.

Now since X is a binormal space, by [21, Lemma 4.1], $C(X)$ is a dense subspace of $(L(X), \tau_{V^+})$. Therefore, $c(C(X)) = c((L(X), \tau_{V^+})) \leq c(L(X))$. Also by [11, Theorem 2.9, Theorem 2.11] and the fact that the Vietoris topology on $C(X)$ coincides with the graph topology, we have $c(C(X)) = d(C(X)) = w(C(X))$. Now since $c(C(X)) \leq c(L(X))$ and $w(L(X)) = w(C(X))$, we have $c(L(X)) = d(L(X)) = w(L(X)) = w(C(X))$. $\square$

5. Metrizability and countability properties of $L(X)$

In this section, we study some basic topological properties of $L(X)$ such as metrizability, complete metrizability and various countability properties in terms of topological properties of X.

Lemma 5.1. *If for a Hausdorff space X, $L(X)$ is first countable, then X is countably compact, perfectly normal and hereditarily separable.*

Proof. Suppose X is not countably compact. So there is an infinite set $\{x_n : n \in \mathbb{N}\}$ without an accumulation point. Let $\{\mathscr{G}_n : n \in \mathbb{N}\}$ be a base of open neighbourhoods of the function F_0 identically equal to zero in $L(X)$. Without loss of generality we can assume that every $\mathscr{G}_n$ is of the following form:

$$\mathscr{G}_n = W_n^+ \cap \mathscr{F}_n^-,$$

where W_n is an open set in $X \times \mathbb{R}$ such that $F_0 \subseteq W_n$ and $\mathscr{F}_n$ is a finite family of open sets in $X \times \mathbb{R}$ with $F_0 \in \mathscr{F}_n^-$.

For every $n \in \mathbb{N}$, there is an open set $O(x_n)$ in X with $x_n \in O(x_n)$ and an open interval (a_n, b_n) containing zero with

$$O(x_n) \times (a_n, b_n) \subseteq W_n.$$

For every $n \in \mathbb{N}$, let $c_n \in (a_n, b_n)$ be such that $0 < c_n$. Consider the set

$$W = X \times \mathbb{R} \setminus \{\langle x_n, c_n \rangle :\ n \in \mathbb{N}\}.$$

Then W is an open set in $X \times \mathbb{R}$ and $F_0 \in W^+$, but there is no $n \in \mathbb{N}$ with $\mathscr{G}_n \subseteq W^+$.

Indeed for each $n \in \mathbb{N}$, define $F_n \in L(X)$ as $F_n(x_n) = [0, c_n]$ and $F_n(x) = 0$ for $x \neq x_n$. Then $F_n \in W_n^+ \cap \mathscr{F}_n^- = \mathscr{G}_n$, but of course $F_n \notin W^+$.

The perfect normality of X follows from Proposition 3.2 and [23, Theorem 2.3] and the hereditary separability of X follows from Theorem 4.7. □

Theorem 5.2. *For a Hausdorff space X, the following statements are equivalent.*

(a) *$L(X)$ is first countable.*

(b) *X is countably compact, perfectly normal and hereditarily separable.*

(c) *X is countably compact, normal, hereditarily separable and every $F \in L(X)$ is a G_δ-set in $X \times \mathbb{R}$.*

(d) *X is hereditarily separable and every $F \in L(X)$ has a countable base of neighbourhoods in $X \times \mathbb{R}$.*

Proof. The implication (a) $\Longrightarrow$ (b) follows from Lemma 5.1 and (b) $\Longrightarrow$ (c) follows from Theorem 4.7. The implication (d) $\Longrightarrow$ (a) follows from Proposition 4.2.

(c) $\Longrightarrow$ (d): Let $F \in L(X)$ and $\{G_n :\ n \in \mathbb{N}\}$ be a decreasing sequence of open sets in $X \times \mathbb{R}$ such that $F = \bigcap\{G_n :\ n \in \mathbb{N}\}$. As X is binormal (and hence $X \times \mathbb{R}$ is normal), we may assume that in fact $F = \bigcap\{\overline{G_n} :\ n \in \mathbb{N}\}$. Since X is countably compact, there exists some $m \in \mathbb{N}$ such that $F \subseteq X \times (-m, m)$. We may assume that for each $n \in \mathbb{N}$, $G_n \subseteq X \times (-m, m)$. Now since $X \times [-m, m]$ is countably compact and normal, the family $\{G_n :\ n \in \mathbb{N}\}$ forms a base at F in $X \times \mathbb{R}$. □

Proposition 5.3. *If X is compact and metrizable, then the space $L(X)$ is completely metrizable.*

Proof. Since X is compact and metrizable, $K(X\times\mathbb{R})$, the space of all nonempty compact subsets of $X \times \mathbb{R}$, is completely metrizable and consequently, the space $L(X)$ being subspace of it, is metrizable. Also, $\overline{L(X)}$ is a G_δ-set of $K(X \times \mathbb{R})$, that is, $\overline{L(X)} = \bigcap\limits_{n=1}^{\infty} G_n$, where G_n are open in $K(X \times \mathbb{R})$. Also note that $\overline{L(X)} \subseteq \{F \in K(X \times \mathbb{R}) :\ F(x) \neq \emptyset \text{ for all } x \in X\}$. If $L(X)$ is closed in $K(X \times \mathbb{R})$, then clearly $L(X)$ is a G_δ-set of $K(X \times \mathbb{R})$. So let $\overline{L(X)} \neq L(X)$. Let $\{B_n\}$ be a countable base for X. Let for each $n \in \mathbb{N}$, $\mathscr{B}_n = \{\langle A, B \rangle :\ A, B \text{ are open in } X \text{ such that } A \cap B = \emptyset,\ A \cup B = B_n\}$ and let

$S = \big\{\langle n,m\rangle \in \mathbb{N}\times\mathbb{N} :\ \overline{B_m} \subseteq B_n\big\}$. Clearly $\langle\emptyset, B_n\rangle, \langle B_n, \emptyset\rangle \in \mathscr{B}_n$ and S is countable. For each $s = \langle n,m\rangle \in S$ and p, q rationals such that $p < q$, define an open set in $K(X\times\mathbb{R})$ by $U(s,p,q) = \bigcup\big\{\big(A\times(-\infty,q)\cup B\times(p,\infty)\cup(X\setminus\overline{B_m}\,)\times\mathbb{R}\big)^{+} :\ \langle A,B\rangle \in \mathscr{B}_n\big\} \cup \big(B_n\times(p,q)\big)^{-}$.

Now let $F \in L(X)$. If $F \in \big(B_n\times(p,q)\big)^{-}$, then obviously $F \in U(s,p,q)$. So let us assume that $F\cap B_n\times(p,q) = \emptyset$. Then for each $x \in B_n$, either $F(x) \subseteq (-\infty,p] \subseteq (-\infty,q)$ or $F(x) \subseteq [q,\infty) \subseteq (p,\infty)$. Let $A = \big\{x \in B_n :\ F(x) \subseteq (-\infty,q)\big\}$ and $B = \big\{x \in B_n :\ F(x) \subseteq (p,\infty)\big\}$. It can be easily verified that $\langle A,B\rangle \in \mathscr{B}_n$. Then we have $F \in \big(A\times(-\infty,q)\cup B\times(p,\infty)\cup \big(X\setminus\overline{B_m}\,\big)\times\mathbb{R}\big)^{+} \subseteq U(s,p,q)$. Then the set $\mathscr{M} = \bigcap\big\{U(s,p,q) :\ s \in S,\ p,q$ rationals, $p<q\big\} \cap \bigcap\limits_{n=1}^{\infty} G_n$ is a G_δ-subset of $K(X\times\mathbb{R})$ contained in $\overline{L(X)}$ and containing $L(X)$. Now let $E \in \overline{L(X)}\setminus L(X)$. Obviously $E(x) \neq \emptyset$ for all $x \in X$. Since $E \notin L(X)$, we can find some $x \in X$ such that $E(x)$ is not connected, that is, there exist rational numbers p', q' ($p' < q'$) such that $E(x)\cap(-\infty,p') \neq \emptyset$, $E(x)\cap(q',\infty) \neq \emptyset$ and $E(x)\cap(p',q') = \emptyset$. Since E is closed in $X\times\mathbb{R}$, we can find some open neighbourhood V of x and rational numbers p, q with $p' < p < q < q'$ such that $E(V)\cap(p,q) = \emptyset$. Choose some $n,m \in \mathbb{N}$ such that $x \in B_m \subseteq \overline{B_m} \subseteq B_n \subseteq V$. Let $\langle A,B\rangle \in \mathscr{B}_n$. Since $x \in B_m$ and $E(x)$ is neither contained in $(-\infty,q)$ nor contained in (p,∞), $E \notin \big(A\times(-\infty,q)\cup B\times(p,\infty)\cup\big(X\setminus\overline{B_m}\,\big)\times\mathbb{R}\big)^{+} \cap \big(B_n\times(p,q)\big)^{-}$. Hence $E \notin \mathscr{M}$. Therefore $L(X) = \mathscr{M}$. Consequently $L(X)$ is a G_δ-subset of $K(X\times\mathbb{R})$ and hence completely metrizable. □

Note that $L(X)$ with the lower Vietoris topology is always separable. However, this is not the case for $L(X)$ with the Vietoris and upper Vietoris topologies, as shown by the following lemma.

Proposition 5.4. *If $L(X)$ with Vietoris (upper Vietoris) topology is separable, then X is countably compact.*

Proof. Note that if $(L(X),\tau_V)$ is separable, then $(L(X),\tau_{V^+})$ is also separable. So assume that $(L(X),\tau_{V^+})$ is separable. Let $\big\{F_n :\ n\in\mathbb{N}\big\}$ be a dense subset of $(L(X),\tau_{V^+})$. Suppose X is not countably compact. Then there is an infinite set $\big\{x_n :\ n\in\mathbb{N}\big\}$ with no accumulation point. Since $\mathbb{R}$ is uncountable, there is a $t\in\mathbb{R}$ such that for every $n\in\mathbb{N}$, $F_n(x_n) \neq \{t\}$. Then for every $n\in\mathbb{N}$, let $t_n \in F_n(x_n)\setminus\{t\}$. Define

$$W = (X\times\mathbb{R})\setminus\big\{\langle x_n,t_n\rangle :\ n\in\mathbb{N}\big\},$$

which is an open subset of $X\times\mathbb{R}$. Now W^{+} is nonempty because it contains F defined by $F(x) = \{t\}$ for all $x\in X$. However, no F_n is in W^{+}, which contradicts $\big\{F_n :\ n\in\mathbb{N}\big\}$ being dense in $L(X)$. □

Proposition 5.5. *Let X be a normal space and let the space $L(X)$ with the Vietoris (upper Vietoris) topology satisfy the countable chain condition. Then X is compact and metrizable.*

Proof. Note that if $(L(X), \tau_V)$ satisfies the countable chain condition, then $(L(X), \tau_{V^+})$ also satisfies the countable chain condition. So assume that $(L(X), \tau_{V^+})$ satisfies the countable chain condition.

Consider the family $\mathscr{W}$ of open subsets of $L(X)$ of the form W^+, where $\overline{W} \subseteq X \times (0, \infty)$. Then by Zorn's Lemma, we can find a maximal subfamily $\mathscr{W}'$ of $\mathscr{W}$ consisting of pairwise disjoint open subsets. Since $(L(X), \tau_{V^+})$ satisfies the countable chain condition, $\mathscr{W}'$ is countable, say $\mathscr{W}' = \{W_n^+ : n \in \mathbb{N}\}$. We shall show that X is countably compact. Suppose by way of contradiction, X is not countably compact, that is, there exists a countably infinite set $\{x_n : n \in \mathbb{N}\}$ of X without an accumulation point. Since for each $W_n^+ \in \mathscr{W}'$, $\overline{W_n} \subseteq X \times (0, \infty)$, for each $n \in \mathbb{N}$, we can find some $t_n > 0$ such that $\langle x_n, t\rangle \notin \overline{W_n}$ for each $t \in (0, t_n]$. Since $\{x_n : n \in \mathbb{N}\}$ is a closed discrete set and X is normal, we can find positive continuous functions f and g on X such that $f(x_n) = t_n$ and $g < f$. Now $\langle g, f\rangle \in M(X)$. Obviously, since $\frac{1}{2}(f + g) \in M_{g,f}^+$, $M_{g,f}^+$ is a nonempty open set in $(L(X), \tau_{V^+})$. It can be easily verified that $\overline{M_{g,f}} \subseteq X \times (0, \infty)$ and $M_{g,f}^+ \cap W_n^+ = \emptyset$ for all $W_n^+ \in \mathscr{W}'$ which is a contradiction to the maximality of $\mathscr{W}'$. Hence X is countably compact.

Now since X is binormal, by Theorem 4.11, we have $C(X)$ is second countable and hence, by Lemma 4.10, X is compact and metrizable. □

Theorem 5.6. *For a Hausdorff space X, the following statements are equivalent.*

(a) *$L(X)$ is second countable.*
(b) *$L(X)$ has a countable network.*
(c) *$L(X)$ is first countable and separable.*
(d) *$L(X)$ is first countable and satisfies the countable chain condition.*
(e) *$L(X)$ is developable.*
(f) *$L(X)$ is metrizable.*
(g) *$L(X)$ is completely metrizable.*
(h) *X is compact and metrizable.*

Proof. The implications (a) $\Longrightarrow$ (b), (a) $\Longrightarrow$ (c), (c) $\Longrightarrow$ (d), (g) $\Longrightarrow$ (f) $\Longrightarrow$ (e) are immediate. The implication (h) $\Longrightarrow$ (g) follows from Proposition 5.3 and (d) $\Longrightarrow$ (h) follows from Lemma 5.1 and Proposition 5.5.

(h) $\implies$ (a): Since X is compact, $L(X) \subseteq K(X \times \mathbb{R})$, the space of all nonempty compact subsets of $X \times \mathbb{R}$. Thus by [28, Proposition 4.5], $L(X)$ is second countable.

(b) $\implies$ (h): Since $L(X)$ has countable network, $L(X)$ is separable. Hence by Proposition 5.4, X is countably compact. Also since $L(X)$ has a countable network and X can be considered as a subspace of $L(X)$ (by Proposition 3.2), X has a countable network. Therefore, X is also Lindelöf. Consequently X is compact with a countable base and hence compact and metrizable.

(e) $\implies$ (h): Since $L(X)$ is developable, by Proposition 3.2, $CL(X)$ is also developable. Hence by [24, Theorem 3.3], X is compact and metrizable. $\square$

Theorem 5.7. *Let X be a normal space. Then the following statements are equivalent.*

(a) *$L(X)$ is separable.*

(b) *$L(X)$ has countable chain condition.*

(c) *X is compact and metrizable.*

Proof.

(a) $\implies$ (b): This is immediate.

The implication (c) $\implies$ (a) follows from Theorem 5.6 and (b) $\implies$ (c) follows from Proposition 5.5. $\square$

Remark 5.8. All the results of this section are also true for $L(X)$ with the locally finite topology. Also note that the locally finite topology on $L(X)$ coincides with the Vietoris topology when X is countably compact.

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