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SEQUENTIAL CONVERGENCES ON CYCLICALLY ORDERED GROUPS WITHOUT URYSOHN'S AXIOM

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ABSTRACT. In this paper we investigate sequential convergences on a cyclically ordered group G which are compatible with the group operation and with the relation of cyclic order; we do not assume the validity of the Urysohn's axiom. The system $\operatorname{conv} G$ of convergences under consideration is partially ordered by means of the set-theoretical inclusion. We prove that $\operatorname{conv} G$ is a Brouwerian lattice.

1. Introduction

The notion of a cyclically ordered group goes back to Rieger [13]. A fundamental theorem on the structure of cyclically ordered groups was proved by Swierczkowski [14]; this theorem is presented with a full proof in the monograph [3] by Fuchs. For further results, cf. [1], [4], [6], [9], [10]–[12], [15], [16].

For references concerning sequential convergences cf. the expository article [2]. Systems of sequential convergences on lattice ordered groups were studied in several papers; cf. [5], [7], [8] and the quotations in these articles. Analogous questions for cyclically ordered groups were dealt with in [4].

When investigating sequential convergences on a cyclically ordered group we have to distinguish between the case when the Urysohn's axiom is assumed to be valid, and the case when this axiom is not supposed to hold. An analogous

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situation occurs when dealing with sequential convergences on lattice ordered groups (cf. the above quotations).

Let G be a cyclically ordered group. We define a system $\operatorname{conv} G$ of sequential convergences on G which are compatible with the group operation and with the relation of cyclic order; the Urysohn's axiom is not assumed to be valid. The system $\operatorname{conv} G$ is partially ordered by the set-theoretical inclusion. (For the detailed definition, cf. Section 2 below.)

Another definition of sequential convergence on a cyclically ordered group G was given by Harminc [4]. In his definition, the Urysohn's axiom was applied. Let us denote the corresponding system of sequential convergences on G by $\operatorname{Conv} G$. In [4] it was proved that $\operatorname{Conv} G$ is either a one-element set or a two-element set.

Let us denote by Cycl the class of all cyclically ordered groups. We show that there exists a proper class K_1 of mutually nonisomorphic elements of Cycl such that for each $G \in K_1$, $\operatorname{conv} G$ is a one-element set. Further, there exists a proper class K_2 of mutually nonisomorphic cyclically ordered group G such that $\operatorname{conv} G$ if infinite. For each $G \in \mathrm{Cycl}$ we have $\operatorname{Conv} G \subseteq \operatorname{conv} G$ and $\operatorname{conv} G$ is a Brouwerian lattice.

2. Preliminaries

We start by recalling the definition of a cyclically ordered group.

Let G be a group. The group operation will be written additively, the commutativity of this operation will not be assumed. Suppose that there is given a subset C of G^3 such that the following axioms are satisfied:

I. If x, y and z are distinct elements of G, then either $(x, y, z) \in C$ or $(z, y, x) \in C$.

II. If $(x, y, z) \in C$, then $(y, z, x) \in C$.

III. If $(x, y, z) \in C$ and $(y, u, z) \in C$, then $(x, u, z) \in C$.

IV. If $(x, y, z) \in C$, then the elements x, y and z are mutually distinct.

V. If $(x, y, z) \in C$ and $u, v \in G$, then $(u + x + v, u + y + v, u + z + v) \in C$ and $(-z, -y, -x) \in C$.

Under these assumptions we say that G is a *cyclically ordered group*; in more detail, we denote it by $(G; +, C)$. We sometimes write $[x, y, z]$ if $(x, y, z) \in C$. Then the ternary relation under consideration is the relation of cyclic order on G. Each subgroup of G is cyclically ordered by the relation of cyclic order induced from the cyclic order in G.

In accordance with [15] we will apply the following terminology and notation.

Let G be a cyclically ordered group. For $x \in G$ we put

$$|x| = \begin{cases} x & \text{if either } x = -x \text{ or } (-x, 0, x) \in C \\ -x & \text{otherwise.} \end{cases}$$

The set

$$P^u(G) = \{x \in G : x = |x|\}$$

is the *positive cone* of G.

If $\operatorname{card} G < 3$, then the set C must be empty; this case is trivial. In what follows we assume that $\operatorname{card} G \geqq 3$. In this case we have $P^u(G) \neq \emptyset$ and $G \setminus P^u \neq \emptyset$.

Theorem 2.1. (Cf. [15].) *The positive cone of a cyclically ordered group uniquely determines the corresponding cyclic order.*

Further, we denote by $P(G)$ the set of all $x \in P^u(G)$ such that the relation $x = -x$ implies $x = 0$. We often write P instead of $P(G)$.

We define by induction a system S of finite sequences $(x_1, x_2, \ldots, x_n)$ of elements of G (with $n \geqq 3$) as follows:

1) $(x_1, x_2, x_3) \in S$ iff $(x_1, x_2, x_3) \in C$;
2) let $n > 3$; $(x_1, x_2, \ldots, x_n)$ belongs to S iff $(x_1, x_2, \ldots, x_{n-1}) \in S$ and $(x_{n-1}, x_n, x_1) \in S$.

The following assertions 2.1.1–2.1.4 are easy to verify; the proofs will be omitted.

2.1.1. Let $0 \neq x_1 \in P$, $x_2 \notin P$. Then $(0, x_1, x_2) \in S$.

2.1.2. Let $(x_1, x_2, \ldots, x_n) \in S$, $m \in \mathbb{N}$, $1 \leqq m < n$, $(x_m, y, x_{m+1}) \in S$. Then $(x_1, x_2, \ldots, x_m, y, x_{m+1}, \ldots, x_n) \in S$.

The meaning of a subsequence of the sequence $(x_1, x_2, \ldots, x_n)$ is obvious.

2.1.3. Let $(x_1, x_2, \ldots, x_n) \in S$ and let $(y_1, y_2, \ldots, y_m)$ (with $m \geqq 3$) be a subsequence of $(x_1, x_2, \ldots, x_n)$. Then $(y_1, y_2, \ldots, y_m)$ belongs to S.

2.1.4. Let $(0, t_i, b) \in C$, $(0, t'_i, t_i) \in C$ for $i = 1, 2$, and $(0, t_1 + t_2, b) \in C$. Then $(0, t'_1 + t'_2, b) \in C$.

In the sequel, we will apply 2.1.1–2.1.4 without quotation.

For distinct elements x and y of G we put

$$\operatorname{in}(x, y) = \{z \in G : (x, z, y) \in C\}.$$

Let $\mathbb{N}$ be the set of all positive integers. The elements of $G^{\mathbb{N}}$ will be denoted as $(x_n)_{n \in \mathbb{N}}$ or simply (x_n). We say that (x_n) is a *sequence* in G. The notion of subsequence of (x_n) has the usual meaning. If $(x_n) \in G^{\mathbb{N}}$, $x \in G$ and $x_n = x$ for each $n \in \mathbb{N}$, then we write $(x_n) = \operatorname{const} x$.

Let α be a nonempty subset of $G^{\mathbb{N}} \times G$. The relation $((x_n), x) \in \alpha$ will be expressed also by writing $x_n \to_\alpha x$.

Consider the following conditions for the set α:

(i) If $x_n \to_\alpha x$ and (y_n) is a subsequence of (x_n), then $y_n \to_\alpha x$.

(ii) If $x_n \to_\alpha x$ and $y_n \to_\alpha y$, then $x_n + y_n \to_\alpha x + y$ and $-x_n \to_\alpha -x$.

(iii) If each subsequence (y_n) of (x_n) has a subsequence (z_n) such that $z_n \to_\alpha x$, then $x_n \to_\alpha x$.

(iv) If $(x_n) = \operatorname{const} x$, then $x_n \to_\alpha x$.

(v) If $x_n \to_\alpha x$ and $x_n \to_\alpha y$, then $x = y$.

(vi) If $x_n \to_\alpha x$ and $(0, |y_n - x|, |x_n - x|) \in C$ for each $n \in \mathbb{N}$, then $y_n \to_\alpha x$.

(vi') If $x_n \to_\alpha x$ and for each $n \in \mathbb{N}$ either $(0, |y_n - x|, |x_n - x|) \in C$ or $|y_n - x| \in \{0, |x_n - x|\}$, then $y_n \to_\alpha x$.

(vii) If $((x_n), x) \in G^{\mathbb{N}} \times G$ and if there exists $m \in \mathbb{N}$ such that $x_{m+n} \to_\alpha x$, then $x_n \to_\alpha x$.

(viii) For each $(x_n) \in G^{\mathbb{N}}$, $x_n \to_\alpha 0$ iff $|x_n| \to_\alpha 0$.

Recall that (iii) is the well-known Urysohn's condition.

Notation 2.2.1. (Cf. [4].) The system of all $\alpha \subseteq G^{\mathbb{N}} \times G$, $\alpha \neq \emptyset$, satisfying the conditions (i)–(vi) will be denoted by $\operatorname{Conv} G$.

Notation 2.2.2. The system of all $\alpha \subseteq G^{\mathbb{N}} \times G$, $\alpha \neq \emptyset$, satisfying the conditions (i), (ii), (iv), (v), (vi'), (vii) and (viii) will be denoted by $\operatorname{conv} G$.

The elements of $\operatorname{conv} G$ will be called sequential convergences on G.

Proposition 2.3. $\operatorname{Conv} G \subseteq \operatorname{conv} G$ *for each cyclically ordered group* G.

Proof. Let $\alpha \in \operatorname{Conv} G$. We have to verify that α satisfies the conditions (vi'), (vii) and (viii).

First, let us deal with the condition (viii). Assume that $x_n \to_\alpha 0$ and let (z_n) be a subsequence of the sequence $(|x_n|)$. Then either

a) there is a subsequence (t_n) of (z_n) such that (t_n) is a subsequence of (x_n),

or

b) there is a subsequence (t'_n) of (z_n) such that (t'_n) is a subsequence of $(-x_n)$.

In the case a) it suffices to apply (iii); in the case b) we have to apply (iii) and (ii). We obtain that the relation $|x_n| \to_\alpha 0$ is valid. Conversely, assume that $|x_n| \to_\alpha 0$. By an analogous argument we get that $x_n \to_\alpha 0$. Therefore (viii) is valid.

Let the assumptions of (vi') be satisfied. Suppose that (z_n) is a subsequence of (y_n). Then there exists a subsequence (t_n) of (z_n) such that some of the following conditions is valid:

a) for each $n \in \mathbb{N}$, $(0, |t_n - x|, |x_n - x|) \in C$;
b) for each $n \in \mathbb{N}$, $|t_n - x| \in \{0, |x_n - x|\}$.

Further, if b) holds, then there is a subsequence (q_n) of (t_n) such that either $|q_n - x| = 0$ for each $n \in \mathbb{N}$, or $|q_n - x| = |x_n - x|$ for each $n \in \mathbb{N}$. By applying (vi), (iv), (ii), (viii) and (iii) we obtain that (vi') holds. Finally, (vii) is a consequence of (iii). □

We remark that below we will write simply "in view of 2.3" instead of "in view of Proposition 2.3", and analogously for other quotations.

For $\alpha \in \operatorname{conv} G$ we set

$$\alpha_0 = \{(x_n) \in G^{\mathbb{N}} : \ x_n \to_\alpha 0\}.$$

Next, we denote

$$\operatorname{conv}_0 G = \{\alpha_0 : \ \alpha \in \operatorname{conv} G\}, \qquad \operatorname{Conv}_0 G = \{\alpha_0 : \ \alpha \in \operatorname{Conv} G\}.$$

Each of the systems $\operatorname{conv} G, \operatorname{Conv} G, \operatorname{conv}_0 G, \operatorname{Conv}_0 G$ is partially ordered by the set-theoretical inclusion.

If $((x_n), x) \in G^{\mathbb{N}} \times G$ and $\alpha \in \operatorname{conv} G$, then

$$x_n \to_\alpha x \iff x_n - x \to_\alpha 0;$$

thus we have $\operatorname{conv} G \simeq \operatorname{conv}_0 G$. Analogously, $\operatorname{Conv} G \simeq \operatorname{Conv}_0 G$.

Let us consider the following condition for a sequence (x_n) in G:

(p) There exists $m \in \mathbb{N}$ such that $x_{m+n} \in P$ for each $n \in \mathbb{N}$.

Let $\alpha_0 \in \operatorname{conv}_0 G$. We denote by α_0^+ the set of all $(x_n) \in \alpha_0$, which satisfy the condition (p). We put

$$\operatorname{conv}_0^+ G = \{\alpha_0^+ : \ \alpha_0 \in \operatorname{conv}_0 G\}.$$

Similarly as the systems considered above, $\operatorname{conv}_0^+ G$ is partially ordered by the set-theoretical inclusion. In view of (viii) we obtain

$$\operatorname{conv}_0^+ G \simeq \operatorname{conv}_0 G.$$

Therefore we have

$$\operatorname{conv}_0^+ G \simeq \operatorname{conv} G. \tag{+}$$

The relation (+) and Theorem 2.1 are a motivation for a more detailed investigation of the partially ordered system $\operatorname{conv}_0^+ G$.

3. Basic properties of $\operatorname{conv} G$ and $\operatorname{conv}_0^+ G$

For a cyclically ordered group G we apply the notation as above. In this section there is given an internal characterization of the system $\operatorname{conv}_0^+ G$. We prove that whenever $x_n \to_\alpha x$, $y_n \to_\alpha y$, $z_n \to_\alpha z$, where x, y, z are mutually distinct and $(x_n, y_n, z_n) \in C$ for each $n \in \mathbb{N}$, then $(x, y, z) \in C$. Further, we show that $\operatorname{conv} G$ is a complete lattice.

Lemma 3.1. *Let $\alpha \in \operatorname{conv} G$. Then α_0^+ satisfies the following conditions:*

(i_1) *If $(x_n) \in \alpha_0^+$, then each subsequence of (x_n) belongs to α_0^+.*

(ii_1) *If (x_n) and (y_n) belong to α_0^+, then $(x_n + y_n)$ belongs to α_0^+ as well.*

(iii_1) *Let $x \in G$. Then $\operatorname{const} x \in \alpha_0^+$ if and only if $x = 0$.*

(iv_1) *Let $(x_n) \in \alpha_0^+$ and $(y_n) \in G^{\mathbb{N}}$. If either $y_n \in \{0, x_n\}$ or $(0, y_n, x_n) \in C$ for each $n \in \mathbb{N}$, then $(y_n) \in \alpha_0^+$.*

(v_1) *If $(x_n) \in \alpha_0^+$ and $y \in G$, then $(-y + x_n + y) \in \alpha_0^+$.*

(vi_1) *Let $(x_n) \in G^{\mathbb{N}}$. If there exists $m \in \mathbb{N}$ such that $(x_{m+n}) \in \alpha_0^+$, then $(x_n) \in \alpha_0^+$.*

P r o o f. We start by dealing with the condition (v_1). Let $(x_n) \in \alpha_0^+$ and $y \in G$. For each $n \in \mathbb{N}$ we put $z_n = -y + x_n + y$. Since $x_n \to_\alpha 0$, in view of (ii) and (iv) we obtain $z_n \to_\alpha 0$. There exists $m \in \mathbb{N}$ such that $x_n \in P$ for each $n > m$. Thus for each $n > m$ we have either $x_n = 0$ or $(-x_n, 0, x_n) \in C$. Therefore for each $n > m$ either $z_n = 0$ or $(-z_n, 0, z_n) \in C$. This yields that $(z_n) \in \alpha_0^+$. We verified that the condition (v_1) is valid.

Let $(x_n), (y_n) \in \alpha_0^+$. In proving that ($\mathrm{ii}_1$) holds, it suffices to deal with the case when $(x_n) \in P$ and $(y_n) \in P$ for each $n \in \mathbb{N}$. Put $z_n = x_n + y_n$. Then according to (ii), $z_n \to_\alpha 0$. By way of contradiction, let us assume that there is a subsequence (t_n) of (z_n) such that $t_n \notin P$ for each $n \in \mathbb{N}$. We have $P \neq \emptyset$ and hence there is $0 \neq a \in P$. Thus $(t_n, 0, a)$ for each $n \in \mathbb{N}$. We obtain $(0, a, t_n)$ for each $n \in \mathbb{N}$. Since $t_n \to_\alpha 0$, the condition (vi) yields $\operatorname{const} a \to_\alpha 0$, which is a contradiction. Thus α_0^+ satisfies (ii_1).

The condition (i_1) is a consequence of (i). From (iv) we get that (iii_1) is valid. Further, (iv_1) follows from (vi'). Finally, in view of (vii), the condition (vi_1) holds. □

Let β be a nonempty subset of $G^{\mathbb{N}}$. Assume that the conditions (i_1)–(vi_1) are satisfied if α_0^+ is replaced by β. Further, suppose that for each $(x_n) \in \beta$ the condition (p) is valid.

By means of β, we define a subset γ of $G^{\mathbb{N}} \times G$ as follows. First, we put $((x_n), 0) \in \gamma$ if there exists $(y_n) \in \beta$ such that $x_n \in \{y_n, -y_n, 0\}$ for each $n \in \mathbb{N}$.

Thus we have $((x_n), 0) \in \gamma$ for each $(x_n) \in \beta$. Moreover, if $((x_n), 0) \in \gamma$, then $((-x_n), 0) \in \gamma$.

Further, for $((x_n), x) \in G^{\mathbb{N}} \times G$, $x \neq 0$, we set $((x_n), x) \in \gamma$ if $((x_n - x), 0) \in \gamma$.

The relation $((x_n), x) \in \gamma$ will be expressed by writing $x_n \to_\gamma x$.

Lemma 3.2. *$\gamma \in \operatorname{conv} G$ and $\beta = \gamma_0^+$.*

Proof. For proving the relation $\gamma \in \operatorname{conv} G$ we have to verify that the conditions (i), (ii), (iv), (v), (vi'), (vii) and (viii) from the definition of $\operatorname{conv} G$ are satisfied (cf. 2.2.2).

The condition (i) is a consequence of (i_1).

Assume that $x_n \to_\gamma x$ and $y_n \to_\gamma y$. Hence $x_n - x \to_\gamma 0$ and $y_n - y \to_\gamma 0$. In view of (v_1) we obtain

$$-x + (x_n - x) + x \to_\gamma 0,$$

hence $-x + x_n \to_\gamma 0$. Thus (ii_1) yields

$$(-x + x_n) + (y_n - y) \to_\gamma 0.$$

Applying (v_1) again we get

$$x + (-x + x_n + y_n - y) - x \to_\gamma 0,$$

$$(x_n + y_n) - (x + y) \to_\gamma 0.$$

Again, suppose that $x_n \to_\gamma x$. Hence $x_n - x \to_\gamma 0$ and this yields $x - x_n \to_\gamma 0$. We obtain

$$-x + x - x_n + x \to_\gamma 0,$$

hence $-x_n + x \to_\gamma 0$ and so $-x_n \to_\gamma -x$. Therefore the condition (ii) is satisfied.

Let $(x_n) = \operatorname{const} x$, $(x_n) \in \beta$. Then according to (iii_1), $x = 0$. Thus (iv) holds.

Assume that $x_n \to_\gamma x$ and $x_n \to_\gamma y$. Hence $x_n - x \to_\gamma 0$ and $x_n - y \to_\gamma 0$. Then the definitions of β and γ yield $-(x_n - x) \to_\gamma 0$, i.e., $x - x_n \to_\gamma 0$. We get

$$(x - x_n) + (x_n - y) \to_\gamma 0.$$

Thus $t_n \to_\gamma 0$, where $t_n = \operatorname{const}(x - y)$. Therefore $x - y = 0$. Hence the condition (v) is satisfied.

The condition (vii) is a consequence of (vi_1).

The validity of the condition (viii) follows immediately from the definitions of β and γ. In view of (iv_1) and (viii), the condition (vi') is satisfied. We verified that $\gamma \in \operatorname{conv} G$.

It remains to verify that $\beta = \gamma_0^+$ is valid.

Let $(t_n) \in \beta$. Then $t_n \to_\gamma 0$. Further (t_n) satisfies the condition (p), hence there is $m \in \mathbb{N}$ such that $t_{m+n} \in P$ for each $n \in \mathbb{N}$. Thus $(t_n) \in \gamma_0^+$. Therefore $\beta \subseteq \gamma_0^+$.

Conversely, assume that (t_n) belongs to γ_0^+. Thus $t_n \to_\gamma 0$ and there is $m \in \mathbb{N}$ such that $t_{m+n} \in P$ for each $n \in \mathbb{N}$. In view of the definition of the relation $t_n \to_\gamma 0$ we obtain that there exists $(y_n) \in \beta$ such that $t_n \in \{y_n, -y_n, 0\}$ for each $n \in \mathbb{N}$. There is $m_1 \in \mathbb{N}$ such that $y_{m_1+n} \in P$ for each $n \in \mathbb{N}$. Put $m_2 = \max\{m_1, m\}$. For each $n \in \mathbb{N}$ we must have $t_{m_2+n} \in \{y_{m_2+n}, 0\}$. In view of ($\mathrm{iv}_1$) we obtain $(t_{m_2+n}) \in \beta$. Applying (vi_1) we get $(t_n) \in \beta$. Hence $\gamma_0^+ \subseteq \beta$. Summarizing, $\beta = \gamma_0^+$. □

From 3.1 and 3.2 we obtain the following proposition.

Proposition 3.3. *Let β be a nonempty subset of $G^{\mathbb{N}}$. Then β belongs to $\operatorname{conv}_0^+ G$ if and only if the conditions* (i_1)–(vi_1) *from* 3.1 *and the condition* (p) *are satisfied.*

Lemma 3.4. *Let $\alpha \in \operatorname{conv} G$, $x_n \to_\alpha 0$. Assume that $(a, 0, b) \in C$. Then there exists $m \in \mathbb{N}$ such that $(a, x_{m+n}, b) \in C$ for each $n \in \mathbb{N}$.*

Proof. Suppose that there exists a subsequence (x'_n) of (x_n) such that $x_n \in P$ and $(0, x'_n, b) \notin C$ for each $n \in \mathbb{N}$. Then we have $(0, b, x'_n) \in C$ for each $n \in \mathbb{N}$. Since $x'_n \to_\alpha 0$, in view of (vi') and (iv) we obtain $b = 0$ which is a contradiction. Thus there exists only a finite number of $n \in \mathbb{N}$ such that x'_n has the mentioned property.

Similarly, there is only a finite number of $n \in \mathbb{N}$ such that $x_n \notin P$ and $(a, x_n, 0) \notin C$. This completes the proof. □

Lemma 3.5. *Let $\alpha \in \operatorname{conv} G$, $x_n \to_\alpha x$. Assume that $(a', x, b') \in C$. Then there is $m \in \mathbb{N}$ such that $(a', x_{m+n}, b') \in C$ for each $n \in \mathbb{N}$.*

Proof. It suffices to consider the sequence (y_n) with $y_n = x_n - x$ and to apply Lemma 3.4. □

Proposition 3.6. *Let $\alpha \in \operatorname{conv} G$, $x_n \to_\alpha x$, $y_n \to_\alpha y$, $z_n \to_\alpha z$. Assume that $(x_n, y_n, z_n) \in C$ for each $n \in \mathbb{N}$ and that x, y and z are mutually distinct. Then $(x, y, z) \in C$.*

Proof. By way of contradiction, suppose that (x, y, z) does not belong to C. Then $(x, z, y) \in C$. Consider the sets

$$\operatorname{in}(x, z), \qquad \operatorname{in}(z, y), \qquad \operatorname{in}(y, x). \tag{$*$}$$

We distinguish the following cases.

1) Assume that all sets $(*)$ are nonempty. Then there exist elements c_1, c_2 and c_3 in G such that

$$c_1 \in \operatorname{in}(x, z), \quad c_1 \in \operatorname{in}(z, y), \quad c_3 \in \operatorname{in}(y, x).$$

We obtain

$$(x, c_1, z, c_2, y, c_3) \in S.$$

Hence in view of 3.5, there is $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ we have

$$(c_3, x_{m+n}, c_1) \in C, \quad (c_1, z_{m+n}, c_2) \in C, \quad (c_2, y_{m+n}, c_3) \in C,$$
$$(c_3, x_{m+n}, c_1, z_{m+n}, c_2, y_{m+n}) \in S.$$

Thus $(x_{m+n}, z_{m+n}, y_{m+n}) \in C$, which is a contradiction.

2) Assume that one of the sets $(*)$ is empty and that the remaining two are nonempty. Without loss of generality we can take

$$\operatorname{in}(x,z) = \emptyset, \quad \operatorname{in}(z,y) \neq \emptyset, \quad \operatorname{in}(y,x) \neq \emptyset.$$

Let c_2 and c_3 be as in 1). We have

$$(x, z, c_2, y, c_3, x) \in S.$$

From 3.5 and from the relation $\operatorname{in}(x,z) = \emptyset$ we infer that there is $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ we have

$$(c_1, z_{m+n}, c_2) \in C, \qquad (c_2, y_{m+n}, c_3) \in C,$$

thus either $x_{m+n} = x$ or $(c_3, y_{m+n}, x) \in C$. From this we obtain

$$(z_{m+n}, y_{m+n}, x_{m+n}) \in C,$$

which is impossible.

3) Assume that two of the sets in $(*)$ are empty and the remaining one is nonempty. Without loss of generality we can put

$$\operatorname{in}(x,z) = \emptyset, \quad \operatorname{in}(z,y) = \emptyset, \quad \operatorname{in}(y,x) \neq \emptyset.$$

Let c_3 be as in 1). Then in view of 3.5 there is $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ we have $z_{m+n} = z$ (since $\operatorname{in}(x,y) = \{z\}$ and $(x, z_{m+n}, y) \in C$); further, the following conditions are satisfied:

(i) either $\quad y_{m+n} = y \quad$ or $\quad (y, y_{m+n}, c_3) \in C$,

(ii) either $\quad x_{m+n} = x \quad$ or $\quad (c_3, x_{m+n}, x) \in C$.

In all these cases we have $(x_{m+n}, z_{m+n}, y_{m+n}) \in C$, which is a contradiction. $\square$

We denote by $\alpha_{\min}$ the set of all $((x_n), x) \in G^{\mathbb{N}} \times G$ such that there exists $m \in \mathbb{N}$ (depending on (x_n)) with $(x_{m+n}) = \operatorname{const} x$. In view of condition (iv) in Section 2 we obtain:

Lemma 3.7. *$\alpha_{\min}$ is the least element in both $\operatorname{conv} G$ and $\operatorname{Conv} G$. Consequently, $(\alpha_{\min})_0$ is the least element of $\operatorname{conv}_0 G$ and of $\operatorname{Conv}_0 G$.*

Lemma 3.8. *Let I be a nonempty set and for each $i \in I$ let β_i be an element of $\operatorname{conv}_0^+ G$. Put $\beta = \bigcap\limits_{i \in I} \beta_i$. Then $\beta \in \operatorname{conv}_0^+ G$.*

Proof. Let $\alpha_{\min}$ be as in 3.7; put $\beta_0 = (\alpha_{\min})_0^+$. In view of 3.7 we have $\beta_0 \subseteq \beta_i$ for each $i \in I$. Thus $\beta_0 \subseteq \beta$ and hence $\beta \neq \emptyset$.

Let us consider the conditions (i_1)–(vi_1) from 3.1 where α_0^+ is replaced by β. All these conditions are obviously valid. Thus $\beta \in \operatorname{conv}_0^+ G$. $\square$

Under the notation as in 3.8, β is the greatest lower bound of the system $\{\beta_i\}_{i\in I}$ in $\operatorname{conv}_0^+ G$. Hence we write, in the usual way, $\beta = \bigwedge\limits_{i\in I} \beta_i$.

We denote by β_1 the set of all $(x_n) \in \operatorname{conv}_0^+ G$ which satisfy the following condition:

(c$_1$) Whenever $(a, 0, b) \in C$, then there exists $m \in \mathbb{N}$ such that $(a, x_{m+n}, b) \in C$ for each $n \in \mathbb{N}$.

Lemma 3.9. *β_1 is an element of* $\operatorname{conv}_0^+ G$.

Proof. Again, we consider the conditions (i$_1$)–(vi$_1$) from 3.1; we have to verify that these conditions hold for β_1.

Let us deal with the condition (ii$_1$) (the validity of the remaining conditions is easy to verify). Let $(a, 0, b) \in C$.

1) Suppose that there exists $b_1 \in P$, $b_1 \neq 0$ such that $\operatorname{in}(0, b_1) = \emptyset$. Then we have either $b_1 = b$ or $(0, b_1, b)$. There exists $m \in \mathbb{N}$ such that $(a, x_{m+n}, b_1) \in C$ and $(a, y_{m+n}, b_1) \in C$ for each $n \in \mathbb{N}$. This yields that $x_{m+n} = 0 = y_{m+n}$, hence $x_{m+n} + y_{m+n} = 0$ for each $n \in \mathbb{N}$. Therefore $(a, x_{m+n} + y_{m+n}, b)$ for each $n \in \mathbb{N}$.

2) Now suppose that there does not exist any $b_1 \in P$, $b_1 \neq 0$, with $\operatorname{in}(0, b_1) = \emptyset$. Hence for each $b_1 \in P$ with $(a, 0, b_1) \in C$ there exists $b_2 \in G$ such that $(0, b_2, b_1) \in C$. Put $b_3 = b_1 - b_2$. Then $b_3 \in P$, $b_3 \neq 0$ and $(0, b_3, b_1) \in C$. There exists $b_4 \in G$ such that $(0, b_4, b_2) \in C$ and $(0, b_4, b_3) \in C$. Further, there exists $m \in \mathbb{N}$ such that

$$(0, x_{m+n}, b_4) \in C, \qquad (0, y_{m+n}, b_4) \in C$$

for each $n \in \mathbb{N}$. We obtain

$$(0, x_{m+n} + y_{m+n}, b_4 + b_4, b_2 + b_3, b) \in S$$

for each $n \in \mathbb{N}$. Hence

$$(a, x_{m+n} + y_{m+n}, b) \in C$$

for each $n \in \mathbb{N}$, completing the proof. □

Lemma 3.10. *β_1 is the greatest element of* $\operatorname{conv}_0^+ G$.

Proof. This is a consequence of 3.9 and 3.4. □

In view of 3.7, 3.8 and 3.10 we obtain:

Proposition 3.11. $\operatorname{conv}_0^+ G$ *is a complete lattice.*

Since $\operatorname{conv} G \simeq \operatorname{conv}_0^+ G$ we have:

Corollary 3.12. $\operatorname{conv} G$ *is a complete lattice.*

4. The classes K_1 and K_2

In this section we prove the following assertions:

(A_1) There exists a proper class K_1 of mutually nonisomorphic cyclically ordered groups such that for each $G \in K_1$, $\operatorname{conv} G$ is a one-element set.

(A_2) There exists a proper class K_2 of mutually nonisomorphic cyclically ordered groups such that for each $G \in K_2$, $\operatorname{conv} G$ is infinite.

We prove (A_1) and (A_2) by using linearly ordered groups.

For linearly ordered groups we apply the terminology and the notation as in [3] with the distinction that the group operation is written additively.

Let G be a linearly ordered group having more than two elements. We denote by C the set of all triples (x, y, z) of elements of G such that some of the following relations is valid:

$$x < y < z, \quad y < z < x, \quad z < x < y. \tag{1}$$

It is well-known that C determines a cyclic order on G and that $(G; +, C)$ is a cyclically ordered group. The linearly ordered group $(G; +, \leqq)$ can be reconstructed from $(G; +, C)$. (Cf. [3].)

In this sense we consider each linearly ordered group G as to be cyclically ordered; hence we have $G \in \mathrm{Cycl}$.

In accordance with this we conclude that the number of convergences on the linearly ordered group G is the same as the number of convergences on G when we consider G as a cyclically ordered group.

If I is a linearly ordered set and for each $i \in I$, G_i is a linearly ordered group, then

$$\Gamma_{i \in I} G_i$$

denotes the lexicographic product of the system $(G_i)_{i \in I}$. (Cf., e.g., [3].)

Let R be the additive group of all reals with the natural linear order.

For each infinite cardinal $\mathbf{m}$ let $\omega(\mathbf{m})$ be the first ordinal having the cardinality $\mathbf{m}$. Further, let $I(\mathbf{m})$ be the linearly ordered set dual to $\omega(\mathbf{m})$. We denote

$$G(\mathbf{m}) = \Gamma_{i \in I(\mathbf{m})} G_i,$$

where $G_i = R$ for each $i \in I(\mathbf{m})$.

Lemma 4.1. *Let $\mathbf{m}$ be a cardinal, $\mathbf{m} > \aleph_0$. Then $\operatorname{conv}_0^+ G(\mathbf{m})$ is a one-element set.*

Proof. By way of contradiction, assume that there exists $\beta \in \operatorname{conv} G(\mathbf{m})$ such that $\beta \neq \alpha_{\min}$. Hence there exists $(x_n) \in \beta$ such that $x_n \neq x_m$ whenever $n, m \in \mathbb{N}$, $n \neq m$ and $x_n \neq 0$, $x_n \in P$. We have $x_n \to_\beta 0$.

For $0 < x \in G(\mathbf{m})$ and $i \in I(\mathbf{m})$ let x^i be the component of x in G_i. There exists $i(x) \in I(\mathbf{m})$ such that $x^{i(x)} \neq 0$ and $x^i = 0$ for each $i < i(x)$. Then we must have $x^{i(x)} > 0$.

Consider the system $(i(x_n))_{n\in\mathbb{N}}$. In view of the definition of $I(\mathbf{m})$ there exists $i_0 \in I(\mathbf{m})$ such that $i_0 < i(x_n)$ for each $n \in I(\mathbf{m})$. Further, there exists $x \in G(\mathbf{m})$ such that $x^{i_0} = 1$, $x^i = 0$ for each $i \in I(\mathbf{m})$, $i \neq i_0$. Then we have

$$(-x, 0, x) \in C,$$
$$(-x, x_n, x) \notin C \qquad \text{for each} \quad n \in \mathbb{N}.$$

Thus in view of 3.4, the relation $x_n \to_\beta 0$ does not hold; we arrived at a contradiction. □

If $\mathbf{m}_1$ and $\mathbf{m}_2$ are distinct infinite cardinals then it is easy to verify that $G(\mathbf{m}_1)$ is not isomorphic to $G(\mathbf{m}_2)$.

Let K_1 be the class of all $G(\mathbf{m})$, where $\mathbf{m}$ runs over all cardinals larger that $\aleph_0$. We verified that K_1 has the properties desired in (A_1). Hence (A_1) is valid.

Let $\mathbf{m}$ and $I(\mathbf{m})$ be as above. Put $J(\mathbf{m}) = \{t\} \cup I(\mathbf{m})$, where t is any element not belonging to $I(\mathbf{m})$. We define a linear order on $J(\mathbf{m})$ as follows: t is the least element of $J(\mathbf{m})$; for elements of $I(\mathbf{m})$ we consider the linear order as above. Again, for each $j \in J(\mathbf{m})$ let $G_j = R$ and

$$H(\mathbf{m}) = \Gamma_{j\in J(\mathbf{m})} G_j.$$

We denote by K_2 the class of all $H(\mathbf{m})$, where $\mathbf{m}$ runs over the class of all infinite cardinals. If $\mathbf{m}_1$ and $\mathbf{m}_2$ are distinct infinite cardinals, then $H(\mathbf{m}_1)$ is not isomorphic to $H(\mathbf{m}_2)$.

For proving (A_2) it remains to verify that $\operatorname{conv} H(\mathbf{m})$ is infinite for each infinite cardinal $\mathbf{m}$.

First, we deal with certain sequential convergences on R. We consider R as a cyclically ordered group in view of the cyclic order mentioned above. Put $R = G$.

Let $n(1) \in \mathbb{N}$. Let $(x_n) \in G^{\mathbb{N}}$. Assume that for each $k \in \mathbb{N}$ there exists $m(k) \in \mathbb{N}$ such that

$$0 \leqq k x_{m(k)+n} < \frac{1}{n^{n(1)}} \qquad \text{for each} \quad n \in \mathbb{N}.$$

The set of all sequences (x_n) in G satisfying this condition will be denoted by $\beta_{n(1)}$.

Lemma 4.2. *For each $n(1) \in \mathbb{N}$, $\beta_{n(1)}$ belongs to $\operatorname{conv}_0^+ G$.*

P r o o f. If suffices to verify that conditions (i_1)–(vi_1) from 3.1 are valid for $\beta_{n(1)}$; the detailed steps will be omitted. □

Lemma 4.3. *Let $n(1) \in \mathbb{N}$, $n(1) > 1$. Put $x_n = \frac{1}{n^{n(1)-1}}$. Then (x_n) does not belong to $\beta_{n(1)}$.*

Proof. Let $k \in \mathbb{N}$. By way of contradiction, assume that there exists $m(k) \in \mathbb{N}$ such that

$$kx_{m(k)+n} < \frac{1}{n^{n(1)}} \qquad \text{for each} \quad n \in \mathbb{N}.$$

Thus for each $n \in \mathbb{N}$ we have

$$\frac{k}{(m(k)+n)^{n(1)-1}} < \frac{1}{n^{n(1)}},$$

$$k\left(\frac{n}{m(k)+n}\right)^{n(1)-1} < \frac{1}{n}. \tag{1}$$

Since

$$\lim_{n\to\infty} \frac{n}{m(k)+n} = 1,$$

we obtain

$$\lim_{n\to\infty} k\left(\frac{n}{m(k)+n}\right)^{n(1)-1} = k > 0.$$

Hence in view of (1) we arrived at a contradiction. □

On the other hand, we clearly have $\left(\frac{1}{n^{n(1)}}\right) \in \beta_{n(1)}$. Therefore we get:

Corollary 4.4. *For each $n(1) > 1$, $\beta_{n(1)-1} \neq \beta_{n(1)}$.*

From this we easily obtain that all $\beta_{n(1)}$ $(n(1) \in \mathbb{N})$ are mutually distinct. Hence the set $\operatorname{conv}_0^+ G$ is infinite.

Let $H(\mathbf{m})$ be as above. For $x \in H(\mathbf{m})$ and $j \in J(\mathbf{m})$ we denote by x^j the component of x in G_j. Also, let t be as above and $G = R$.

Assume that $\beta \in \operatorname{Conv}_0^+ G$. We denote by β' the system of all sequences (x_n) of elements of $H(\mathbf{m})$ such that the following conditions are satisfied:

a) there exists $m \in \mathbb{N}$ (depending on (x_n)) such that $x^j_{m+n} = 0$ for each $j \in I(\mathbf{m})$ and each $n \in \mathbb{N}$;

b) $(x^t_{m+n}) \in \beta$.

By applying 4.2 and 3.3 we obtain:

Lemma 4.5. *For each $\beta \in \operatorname{Conv}_0^+ G$, β' is an element of $\operatorname{conv}_0^+(H(\mathbf{m}))$.*

If β_1 and β_2 are distinct elements of $\operatorname{conv}_0^+ G$, then clearly $\beta_1' \neq \beta_2'$. Thus from 4.5 and from the fact that $\operatorname{conv}_0^+ G$ is infinite we conclude that $\operatorname{conv}_0^+ H(\mathbf{m})$ is infinite. Therefore the assertion (A_2) is valid.

5. Further properties of the lattice $\operatorname{conv} G$

In this section we give a constructive description of the operation of join in the complete lattice $\operatorname{conv}_0^+ G$. Next, we prove that for each cyclically ordered group G, the lattice $\operatorname{conv} G$ is Brouwerian. We apply the isomorphism between $\operatorname{conv} G$ and $\operatorname{conv}_0^+ G$.

Lemma 5.1. *Let x_1, x_2 and x be nonzero elements of P such that $x = x_1 + x_2$. Then $(0, x_1, x) \in C$ and $(0, x_2, x) \in C$.*

Proof. Since $x_2 \neq 0$ we get $x_1 \neq x$. By way of contradiction, assume that $(0, x_1, x)$ does not belong to C. Then $(0, x, x_1) \in C$. This yields $(-x_1, x_2, 0) \in C$. Since $-x_1 \notin P$, we obtain $(-x_1, 0, x_2) \in C$, which is a contradiction. For proving the relation $(0, x_2, x) \in C$ we proceed analogously. □

Lemma 5.2. *Let $x_1, x_2 \in G$. Assume that $x_1 + x_2 \in P$, $(0, y, x_1 + x_2) \in C$. Then there are elements y_1 and y_2 in G such that $y = y_1 + y_2$ and*

$$\text{either} \quad (0, y_1, x_1) \in C \quad \text{or} \quad y_1 \in \{0, x_1\},$$
$$\text{either} \quad (0, y_2, x_2) \in C \quad \text{or} \quad y_2 \in \{0, x_2\}.$$

Proof. If $x_1 = 0$, then we put $y_1 = 0$ and $y_2 = y$. The case $x_2 = 0$ is analogous. Next, suppose that $x_1 \neq 0 \neq x_2$. In view of 5.1, we have $(0, x_1, x) \in C$ and $(0, x_2, x) \in C$. In the case $y = x_1$ we set $y_1 = x_1$, $y_2 = 0$. Similarly, in the case $y = x_2$ we set $y_1 = 0$ and $y_2 = x_2$.

Assume that $x_1 \neq y \neq x_2$. The elements $0, x_1$ and y are mutually distinct, hence we have either $(0, y, x_1)$ or $(0, x_1, y)$. In the first case we put $y_1 = y$ and $y_2 = 0$. In the second case we get

$$(0, x_1, y, x_1 + x_2) \in S,$$

hence $(x_1, y, x_1 + x_2) \in C$. This yields

$$(0, -x_1 + y, x_2) \in C.$$

Now it suffices to put $y_1 = x_1$ and $y_2 = -x_1 + y$. □

From 5.2 we obtain by the obvious induction:

Lemma 5.3. *Let $x_1 + x_2 + \dots + x_k \in P$, $(0, y, x_1 + \dots + x_k) \in C$. Then there are elements $y_1, \dots, y_k$ in G such that for each $j \in \{1, 2, \dots, k\}$ either $(0, y_j, x_j) \in C$, or $y_j \in \{0, x_j\}$, and $y = y_1 + \dots + y_k$.*

For elements (x_n) and (y_n) of $G^{\mathbb{N}}$ we write, as usual, $(x_n) + (y_n) = (x_n + y_n)$.

Let I be a nonempty set and for each $i \in I$ let $\beta_i \in \operatorname{conv}_0^+ G$. We denote by β the set of all sequences (x_n) of elements of G which have the following property: there exist $i(1), \dots, i(k) \in I$ and sequences $(x_n^1) \in \beta_{i(1)}, \dots, (x_n^k) \in \beta_{i(k)}$ such that

$$(x_n) = (x_n^1) + \dots + (x_n^k). \tag{1}$$

Lemma 5.4. *$\beta \in \operatorname{conv}_0^+ G$.*

P r o o f. In view of 3.3, we have to verify that conditions (i_1)–(vi_1) from 3.1 are satisfied for β. The case of (i_1), (ii_1) and (iii_1) is clear.

Let us consider the condition (iv_1). Assume that $(x_n) \in \beta$, $(y_n) \in G^{\mathbb{N}}$, and for each $n \in \mathbb{N}$, either $(0, y_n, x_n) \in C$ or $y_n \in \{0, x_n\}$. If $(0, y_n, x_n) \in C$, then we get $(0, y_n, x_n^1 + \cdots + x_n^k) \in C$; thus in view of 5.3 there exist $y_n^1, \ldots, y_n^k$ in G such that $y_n = y_n^1 + \cdots + y_n^k$ and either $(0, y_n^j, x_n^j) \in C$ or $y_n^j \in \{0, x_n^j\}$ for $j = 1, 2, \ldots, k$. We apply condition (iv_1) for $\beta_{i(j)}$ and we obtain $(y_n^j) \in \beta_{i(j)}$ for $j = 1, 2, \ldots, k$. According to the definition of β we get $(y_n) \in \beta$.

Let $(x_n) \in \beta$ and $y \in G$. We have

$$(-y + x_n + y) = (-y + x_n^1 + y) + \cdots + (-y + x_n^k + y).$$

Each $\beta_{i(j)}$ satisfies (v_1), thus $(-y + x_n^1 + y) \in \beta_{i(j)}$; hence $(-y + x_n + y)$ belongs to β. Therefore (v_1) holds for β.

The validity of conditions (vi_1) follows from the fact that this condition is valid for each β_i $(i \in I)$. $\square$

Proposition 5.5. *Let β_i $(i \in I)$ and β be as above. Then $\beta = \bigvee_{i \in I} \beta_i$ in the lattice $\operatorname{conv}_0^+ G$.*

P r o o f. Let us now write $\leqq$ and $\geqq$ instead of $\subseteq$ or $\supseteq$, respectively. In view of the definition of β, we have $\beta_i \subseteq \beta$ for each $i \in I$. According to 5.4, $\beta \in \operatorname{conv}_0^+ G$. Let $\gamma \in \operatorname{conv}_0^+ G$, $\gamma \leqq \beta_i$ for each $i \in I$. Let (x_n) be as in (1). Since $(x_n^1) \in \beta_{i(1)}, \ldots, (x_n^k) \in \beta_{i(x)}$, and since γ is closed with respect to the operation $+$, we obtain $(x_n) \in \gamma$. Thus $\beta \subseteq \gamma$. Therefore $\beta = \bigvee_{i \in I} \beta_i$. $\square$

Let β_i $(i \in I)$ and β be as above. Further, let γ be any element of $\operatorname{conv}_0^+ G$. We will deal with the elements

$$\delta_1 = \gamma \wedge \Bigl(\bigvee_{i \in I} \beta_i\Bigr), \qquad \delta_2 = \bigvee_{i \in I} (\gamma \wedge \beta_i).$$

Lemma 5.6. *The relation $\delta_1 = \delta_2$ is valid.*

P r o o f. We have clearly $\delta_1 \geqq \delta_2$. Let $(x_n) \in \delta_1$. Then $(x_n) \in \gamma$ and $(x_n) \in \bigvee_{i \in I} \beta_1$. From the last relation and from 5.5 we conclude that (x_n) can be expressed in the form (1).

Now, let us deal with the relation $(x_n) \in \gamma$. Let $j \in \{1, 2, \ldots, k\}$. Consider the element (x_n^j) of $G^{\mathbb{N}}$. From (1), 5.3 and (iv_1) we obtain that (x_n^j) belongs to γ.

Hence (x_n^j) is an element of $\gamma \wedge \beta_i$. Then (1) and 5.5 yield $(x_n) \in \delta_2$. Therefore $\delta_1 \leqq \delta_2$. Summarizing, we obtain $\delta_1 = \delta_2$. $\square$

It is well-known that a complete lattice L is Brouwerian if and only if for any $x \in L$ and $\{y_n\}_{i\in I} \subseteq L$ the relation $x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I} (x \wedge y_i)$ is valid.

Hence from 3.11 and 5.6 we obtain:

COROLLARY 5.7. conv G *is a Brouwerian lattice.*

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