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DIFFERENCE POSETS AS COMMUTATIVE DIRECTOIDS WITH SECTIONAL ANTITONE INVOLUTIONS

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ABSTRACT. It is shown that every difference poset can be converted into a total algebra in a manner similar to that which is used for difference lattices. As a tool applied here we have commutative directoids and posets with sectional antitone involutions.

The concept of a *difference poset* (or briefly a D-poset) was introduced by F. Kôpka [6] in 1992 as a powerful tool for investigations of the so-called quantum structures (see e.g. [4]). This was firstly applied to fuzzy set models of quantum mechanics. A long series of papers devoted to D-posets was published by F. Chovanec and F. Kôpka, see e.g. [1], [2], [6], [7] and a complete list of references in [4]. As a source for our treaty, we refer the compendium [4].

Definition 1. By a *D-poset* is meant a structure $\mathscr{P} = (P; \leq, 0, 1, \ominus)$ such that $(P; \leq)$ is a poset with a least element 0 and a greatest element 1 and a partial binary operation $\ominus$ satisfying the following conditions

(D1) $b \ominus a$ is defined if and only if $a \leq b$;

(D2) if $a \leq b$ then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$;

(D3) if $a \leq b \leq c$ then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

If the underlying poset $(P; \leq)$ is a lattice then the corresponding D-poset is called a *D-lattice*. It is well-known that every D-lattice (which is a partial algebra with respect to the operation $\ominus$) can be converted into a total algebra

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replacing the partial operation $\ominus$ by a new operation $-$ defined as follows:

$$x - y = x \ominus (x \wedge y).$$

The aim of this paper is to show that, similarly, every D-poset can be converted into a total algebra, however, this way is not unique. For this, we need several new concepts. Let $\mathscr{P} = (P; \leq, 0)$ be a poset with a least element 0.

For $a \in P$, the interval $[0, a]$ will be called a *section*. A mapping $f_a\colon [0, a] \to [0, a]$ is called a *sectional mapping*. Instead of $f_a(x)$, we will write briefly x^a. A sectional mapping $x \mapsto x^a$ on $[0, a]$ is called an *antitone involution* if $x^{aa} = x$ and $x \leq y \implies y^a \leq x^a$ for $x, y \in [0, a]$.

Definition 2. A poset $\mathscr{P} = (P; \leq, 0, 1)$ with 0 and 1 is said to be with *strict sectional antitone involutions* (with *strict SAI*, for brief) if for each $a \in P$ there exists a sectional antitone involution $x \mapsto x^a$ on $[0, a]$ and, moreover,

$$a \leq b \leq c \quad \text{implies} \quad (b^c)^{(a^c)} = a^b. \tag{$*$}$$

The fact that $\mathscr{P}$ is a poset with SAI will be expressed by the notation $\mathscr{P} = (P; \leq, 0, 1, ({}^a)_{a \in P})$.

Theorem 1.

(a) *Let $\mathscr{P} = (P; \leq, 0, 1, ({}^a)_{a \in P})$ be a poset with strict SAI. Define $b \ominus a = a^b$. Then $(P; \leq, 0, 1, \ominus)$ is a D-poset.*

(b) *Let $\mathscr{P} = (P; \leq, 0, 1, \ominus)$ be a D-poset. For every $a \in P$ and $x \in [0, a]$ define $x^a = a \ominus x$. Then $(P; \leq, 0, 1, ({}^a)_{a \in P})$ is a poset with strict SAI.*

Proof.

(a): (D1) follows directly from the definition of sectional involutions. If $a \leq b$ then $b \ominus a = a^b \in [0, b]$ and hence $b \ominus a \leq b$. Since $x \mapsto x^b$ is an involution, we have $b \ominus (b \ominus a) = a^{bb} = a$ proving (D2). Since this involution is antitone, we have for $a \leq b \leq c$ also $c \ominus b = b^c \leq a^c = c \ominus a$. Due to $(*)$ we conclude $(c \ominus a) \ominus (c \ominus b) = a^c \ominus b^c = (b^c)^{(a^c)} = a^b = b \ominus a$ proving (D3).

(b): By (D1) and (D2), $x \mapsto x^a$ is a sectional mapping which is an involution. By (D3) we have that this involution is antitone and satisfies $(*)$ since

$$(b^c)^{(a^c)} = (c \ominus a) \ominus (c \ominus b) = b \ominus a = a^b.$$

□

The following concept was introduced in [5].

Definition 3. By a *commutative directoid* is meant a grupoid $(D; \sqcap)$ satisfying the following identities

(C1) $x \sqcap x = x$;

(C2) $x \sqcap y = y \sqcap x$;

(C3) $x \sqcap ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z$.

It was shown in [5] that the relation $\leq$ defined by

$$x \leq y \quad \text{if and only if} \quad x \sqcap y = x$$

is an order, the so-called *induced order* of $(D; \sqcap)$. Denote by $L(x, y) = \{a \in D : a \leq x \text{ and } a \leq y\}$ the set of all lower bounds of $\{x, y\}$. It is easy to see that $x \sqcap y \in L(x, y)$ for any $x, y \in D$.

Also conversely, let $\mathscr{P} = (P; \leq)$ be a downward directed poset. A mapping $\varphi\colon P \times P \to P$ will be called an *L-mapping* if $\varphi(x, y) = \varphi(y, x)$ and $\varphi(x, y) \in L(x, y)$, and $x \leq y$ yields $\varphi(x, y) = x$.

Let φ be an L-mapping on $\mathscr{P} = (P; \leq)$. Define $x \sqcap y = \varphi(x, y)$. It is an easy exercise to check that $(P; \sqcap)$ is a commutative directoid.

Definition 4. By a *D-directoid* is meant a commutative directoid $(D; \sqcap)$ which has a least element 0 and a greatest element 1 with respect to the induced order $\leq$ and such that $(D; \leq, 0, 1, \ominus)$ is a D-poset. A D-directoid will be denoted by $\mathscr{D} = (D; \sqcap, 0, 1, \ominus)$.

With respect to the foregoing remark, every D-poset can be recognized as a D-directoid when an arbitrary L-mapping φ is chosen and $x \sqcap y = \varphi(x, y)$. Hence, it make sense to classify D-directoids.

Lemma. *Let $\mathscr{D} = (D; \sqcap, 0, 1, \ominus)$ be a D-directoid. Define $x - y = x \ominus (y \sqcap x)$. Then $-$ is an everywhere defined operation on D satisfying the following properties:*

(1) *if $a \leq b$ then $b - a = b \ominus a$;*

(2) *$b - a \leq b$;*

(3) *$a - (a - b) = a \sqcap b$;*

(4) *$b \leq a$ implies $b - a = 0$.*

Proof. Since $a \sqcap b \leq b$, it is plain that the operation $-$ is everywhere defined and $b - a = b \ominus (a \sqcap b) = b \ominus a$ for $a \leq b$. For (2), we have by (D2) that $b - a = b \ominus (a \sqcap b) \leq b$. To prove (3) we apply (D2) and (2) as follows

$$\begin{aligned} a - (a - b) &= a \ominus ((a - b) \sqcap a) = a \ominus (((a \ominus (a \sqcap b)) \sqcap a) \\ &= a \ominus (a \ominus (a \sqcap b)) = a \sqcap b. \end{aligned}$$

(4): Suppose $b \leq a$. Then $a \sqcap b = b$ and $b - a = b \ominus (a \sqcap b) = (b \ominus b) = 0$. $\square$

We are going to get an axiomatization of D-directoids.

Theorem 2.

(a) *Let* $\mathscr{D} = (D; \sqcap, 0, 1, \ominus)$ *be a D-directoid,* $\leq$ *its induced order. Then the operation* $x - y = x \ominus (y \sqcap x)$ *satisfies the following conditions*

(A1) $a - 0 = a$;

(A2) $a - (a - b) = b - (b - a)$;

(A3) $a - b \leq a$;

(A4) $a \leq b \leq c$ *implies* $c - b \leq c - a$ *and* $(c - a) - (c - b) = b - a$;

(b) *Let* $\mathscr{P} = (P; \leq, 0, 1, -)$ *be an ordered set with a least element* 0 *and a greatest element* 1 *and with a binary operation* $-$ *satisfying* (A1)–(A4). *Define* $x \sqcap y = x - (x - y)$, *and for* $y \leq x$ *define* $x \ominus y = x - y$. *Then* $\mathscr{D}(P) = (P; \sqcap, 0, 1, \ominus)$ *is a D-directoid whose induced order coincides with* $\leq$.

Proof.

(a): (A1) follows by (1), (A3) is (2) and (A2) follows by (3) and (A4) follows by (D3).

(b): By (A3) we have $a - (a - b) \leq a$ and, by (A2), also $a - (a - b) = b - (b - a) \leq b$ thus

$$a \sqcap b = a - (a - b) \in L(a, b).$$

If $a \leq b$ then, by (A2), (A4) and (A1),

$$a \sqcap b = a - (a - b) = b - (b - a) = (b - 0) - (b - a) = a - 0 = a.$$

Hence, $(P; \sqcap)$ is a commutative directoid. Moreover, if $a \sqcap b = a$ then, by (A3), also $a = b \sqcap a = b - (b - a) \leq b$ thus its induced order coincides with $\leq$.

Let $b \in P$ and $x \in [0, b]$. Define $x^b = b - x$. By (A2) we conclude that $x \mapsto x^b$ is a sectional mapping. By (A3) it is an involution and, due to (A4), it is antitone and satisfies $(*)$ of Definition 2. Hence, $(P; \leq, 0, 1, (^a)_{a \in P})$ is a poset with strict SAI. Altogether, $a \ominus b = a - b$ for $b \leq a$ and, by the Lemma, $\mathscr{D}(P) = (P; \sqcap, 0, 1, \ominus)$ is a D-directoid. $\square$

As already mentioned, every D-poset can be converted into a D-directoid, i.e. an algebra with two binary operations $\sqcap$ and $-$ and two nullary operations 0, 1 which are everywhere defined. It means that also every D-poset can be treated similarly as a D-lattice, see e.g. [2] and [3] for the details.

The method applied in the previous theorems is demonstrated by the following example.

Example. Let $D = \{0, a, b, c, d, 1\}$ and the operation $\ominus$ is given as follows ($-$ means that it is not defined):

$\ominus$	0	a	b	c	d	1
0	0	$-$	$-$	$-$	$-$	$-$
a	a	0	$-$	$-$	$-$	$-$
b	b	$-$	0	$-$	$-$	$-$
c	c	b	a	0	$-$	$-$
d	d	a	b	$-$	0	$-$
1	1	d	c	b	a	0

The induced ordered set is visualized in Fig. 1.

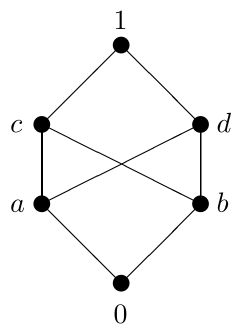

FIGURE 1

It is evident that $(D; \leq)$ is not a lattice. Define an L-mapping as follows: $\varphi(d, c) = \varphi(c, d) = b$ and $\varphi(x, y) = x \wedge y$ in the remaining cases. Take $x \sqcap y = \varphi(x, y)$. Then $(D; \sqcap)$ is a commutative directoid and the total operation $-$ is as follows

$-$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	a	0	a	0	0	0
b	b	b	0	0	0	0
c	c	b	a	0	a	0
d	d	a	b	b	0	0
1	1	d	c	b	a	0

(since e.g.

$$a - b = a \ominus (a \sqcap b) = a \ominus 0 = a,$$
$$c - d = c \ominus (c \sqcap d) = c \ominus b = a,$$
$$d - c = d \ominus (c \sqcap d) = d \ominus b = b).$$

Hence, our original D-poset $(D; \leq, 0, 1, \ominus)$ can be described as a total algebra $(D; \sqcap, 0, 1, -)$. Let us note that the original D-poset can be completed into a total algebra by three possible manners since there exist just three L-mappings on $(D; \leq)$. They differs only on elements c, d, so the other alternatives are: $\varphi(d, c) = \varphi(c, d) = a$ and $\varphi(d, c) = \varphi(c, d) = 0$.

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