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Mathematica Slovaca

DOI: 10.2478/s12175-008-0100-5
Math. Slovaca **58** (2008), No. 6, 653–670

DEDUCTIVE SYSTEMS OF A CONE ALGEBRA – I: SEMI-ℓg-CONES

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(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. A semi-ℓg-cone is an algebra $(C; *, :, \cdot)$ of type $(2, 2, 2)$ satisfying the equations $(a*a)*b = b = b:(a:a)$; $a*(b:c) = (a*b):c$; $a:(b*a) = (b:a)*b$ and $(ab)*c = b*(a*c)$. An ℓ-group cone is a semi-ℓg-cone and a bounded semi-ℓg-cone is term equivalent to a pseudo MV-algebra. Also, a subset A of a semi-ℓg-cone C is an ideal of C if and only if it is a deductive system of its reduct $(C; *, :)$.

Introduction

The concepts of a brick and a cone algebra are due to B. Bosbach [2] and the notion of a pseudo MV-algebra is due to G. Georgescu and A. Iorgulescu [7] and, in an equivalent form due to J. Rachůnek [10], under the name "generalized MV-algebra". We have shown recently ([12]) that a pseudo MV-algebra is term equivalent to a brick; and we show in this part, in a more general context, that a subset A of a pseudo MV-algebra C is an *ideal* of C ([11, p. 156]) if and only if A is a *deductive system* ([13, p. 17]) of the brick equivalent to C.

We have found that the comparison of the ideals of a pseudo MV-algebra with the deductive systems of its equivalent brick is made easier by introducing the class of semi-ℓg-cones, which contains as special cases, ℓ-group cones and equivalents of pseudo MV-algebras. A semi-ℓg-cone is a generalization of a pseudo MV-algebra within the frame work of Bosbach's cone algebra, and is

2000 Mathematics Subject Classification: Primary 03B50, 06D35.
Keywords: cone algebra, semi-ℓg-cone, pseudo MV-algebra, deductive system, ℓ-group, ideal.

equivalent to an integral GMV-algebra of Galatos and Tsinakis [5], where a GMV-algebra is a generalization of an MV-algebra within the context of residuated lattices. Also, if $(A; *, :, \cdot)$ is a semi-ℓg-cone, then $(A; \cdot, \to, \rightsquigarrow, 1)$ is a Wajsberg pseudo-hoop ([8]) if we define $a \to b = b : a$, $a \rightsquigarrow b = a * b$ and 1 is the least element of the reduct $(A; *, :)$ (see Definition 2.1 and Lemma 1.2). Conversely, if $(A; \odot, \to, \rightsquigarrow, 1)$ is a Wajsberg pseudo-hoop, then $(A; *, :, \odot)$ is a semi-ℓg-cone, if we put $a * b = a \rightsquigarrow b$ and $a : b = b \to a$. Hence, semi-ℓg-cones and Wajsberg pseudo-hoops are term equivalent.

In Section 1, we introduce the concept of a precone algebra, by picking up the common axioms from the lists presented by Bosbach [2] for cone algebras and bricks separately; and show that the class of cone algebras is a proper subclass of the class of precone algebras (Example 1.7). In Theorem 1.5, we characterize the class of cone algebras within the class of precone algebras.

In Section 2, we introduce the class of semi-ℓg-cones and show that every semi-ℓg-cone has a cone algebra reduct and obtain the equivalence of an *ideal* of a semi-ℓg-cone with a deductive system of its cone algebra reduct (Theorems 2.8 and 2.13 are crucial).

Section 3 characterizes those special cases of a semi-ℓg-cone, which include ℓ-group cones and pseudo MV-algebras, which are important for our purpose.

1. Precone algebras

We begin with the following definition:

Definition 1.1. An algebra $(C; *, :)$ of type $(2,2)$ is called a *precone algebra* if and only if the following equations are satisfied:

(1) $(a * a) * b = b = b : (a : a)$

(2) $a * (b : c) = (a * b) : c$ and

(3) $a : (b * a) = (b : a) * b$.

If $(C; *, :)$ is a precone algebra, then so is $(C; \circledast, \odot)$ if we define $a \circledast b = b : a$ and $a \odot b = b * a$. Hence any equation, valid in a precone algebra, remains valid if written in the reverse order with $*$ and $:$ interchanged. This is the *principle of duality* for a precone algebra, which we frequently use.

A precone algebra $(C; *, :)$ satisfying the equation

$$a * b = b : a$$

is said to be *symmetric*.

Although B o s b a c h has stated, without proof, the contents of the following lemma ([2, p. 59]) in the context of cone algebras, he has proved them ([2, p. 65]), using only the axioms (BR1) thorough (BR4) of a brick, which are precisely the axioms of a precone algebra.

Lemma 1.2 (Bosbach). *Let $(C; *, :)$ be a precone algebra; then*

(1) $a * a = b : b =: 0$

(2) $0 * a = a = a : 0$

(3) $a * 0 = 0 = 0 : a$

(4) $a * b = 0 \iff b : a = 0$

(5) $a \leq b \iff a : b = 0$ *defines a partial order on* C

(6) $b * (a * c) = 0 \iff (c : b) : a = 0$

(7) $a * (b * a) = 0$ *and* $(a : b) : a = 0$

(8) $b \leq c \implies c * a \leq b * a$ *and* $a : c \leq a : b$

(9) $b \leq c \implies a * b \leq a * c$ *and* $b : a \leq c : a$

(10) $a : (b * a) = \inf(a, b) =: a \wedge b$

(11) $a : (b * a) = b : (a * b) = (b : a) * b = (a : b) * a$ *and*

(12) $a \wedge b = 0 \iff a * b = b \iff a : b = a.$

Observe that we are using 0 in place of 1 used by B o s b a c h .

The precone algebras are term equivalent to the so called commutative pseudo BCK-algebras considered by K ü h r [9], who has shown that ([9, Theorem 4.2]) an algebra $(A; \to, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is a commutative pseudo BCK-algebra if and only if it satisfies the following identities:

(i) $(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x = (x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x$

(ii) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$

(iii) $x \to x = 1 = x \rightsquigarrow x$ and

(iv) $1 \to x = x = 1 \rightsquigarrow x$

Clearly, the equations (iii) and (iv) together imply the equations: $(x \to x) \to y = y = (x \rightsquigarrow x) \rightsquigarrow y$. Writing $x * y$, $y : x$ and 0 respectively for $x \to y$, $x \rightsquigarrow y$ and 1, it is clear that if $(A; \to, \rightsquigarrow, 1)$ is a commutative pseudo BCK-algebra, then $(A; *, :)$ is a precone algebra. Conversely, if $(A; *, :)$ is a precone algebra, then, by Lemma 1.2 and the above theorem of K ü h r , $(A; \to, \rightsquigarrow, 1)$ is a commutative pseudo BCK-algebra, where $a \to b$, $a \rightsquigarrow b$ and 1 stand for $a * b$, $b : a$ and 0 (see Lemma 1.2) respectively. Hence precone algebras and commutative pseudo BCK-algebras are term equivalent. Also observe that it follows from the above that in [9, Theorem 4.2], the equations (i) can be replaced by the single equation:

$$(x \to y) \rightsquigarrow y = (y \rightsquigarrow x) \to x.$$

We now present an alternative characterization of commutative pseudo BCK-algebras in the following theorem:

Theorem 1.3. *An algebra* $(A; \to, \rightsquigarrow, 1)$ *of type* $(2,2,0)$ *is a commutative pseudo BCK-algebra if and only if it satisfies the following identities:*

(α) $(y \to x) \rightsquigarrow x = (x \rightsquigarrow y) \to y$

(β) $1 \to x = x = 1 \rightsquigarrow x$

(γ) $(x \rightsquigarrow y) \rightsquigarrow [(y \rightsquigarrow z) \to (x \rightsquigarrow z)] = 1$ *and*

(δ) $(x \to y) \to [(y \to z) \rightsquigarrow (x \to z)] = 1.$

Proof. By Lemma 1.2(5) and the discussion in [9, Section 3], every commmutative pseudo BCK-algebra satisfies the equations (α) through (δ). Conversely, assume that $(A; \to, \rightsquigarrow, 1)$ is an algebra of type $(2,2,0)$ satisfying the equations (α) through (δ). Then we have:

(1) $z \rightsquigarrow z = z \to z = 1$ by taking $x = y = 1$ in (γ) and (δ) and using (β).

(2) $a \rightsquigarrow 1 = 1 = a \to 1$.

For, by (1), $1 = a \rightsquigarrow a = (1 \to a) \rightsquigarrow a$ (by β) $= (a \rightsquigarrow 1) \to 1$ by (α) and hence $a \rightsquigarrow 1 = a \rightsquigarrow [(a \rightsquigarrow 1) \to 1] = (1 \rightsquigarrow a) \rightsquigarrow [(a \rightsquigarrow 1) \to (1 \rightsquigarrow 1)] = 1$ by (γ). Similarly, $a \to 1 = 1$ by using (δ).

(3) $a \to b = 1 \iff a \rightsquigarrow b = 1$.

For, assuming $a \to b = 1$, we have $b = 1 \rightsquigarrow b = (a \to b) \rightsquigarrow b = (b \rightsquigarrow a) \to a$ and hence $a \rightsquigarrow b = ((b \rightsquigarrow a) \to 1) \to [(1 \to a) \rightsquigarrow ((b \rightsquigarrow a) \to a)] = 1$ and hence $a \to b = 1 \implies a \rightsquigarrow b = 1$. The reverse implication follows similarly.

We now define $a \le b \iff a \to b = 1 \iff a \rightsquigarrow b = 1$. Then $a \le b$ and $b \le a \implies a = b$ by (3) and (α). Also, taking $x = 1$ in (γ) and (δ) respectively and using (3), we have $y \le (y \rightsquigarrow z) \to z$ and $y \le (y \to z) \rightsquigarrow z$. Hence $(A; \le, \to, \rightsquigarrow, 1)$ is a pseudo BCK-algebra. It was proved in [6] that pseudo BCK-algebras satisfy $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$. Hence by Kühr's theorem ([9, Theorem 4.2]), $(A; \to, \rightsquigarrow, 1)$ is a commutative pseudo BCK-algebra. □

Lemma 1.4. *Let* $(C; *, :)$ *be a precone algebra and let* $a, b, c \in C$*; and assume that* $(C; \le)$ *is directed above. Then*

(1) $(a * b) * (a * c) = (b * a) * (b * c)$ *and*

(2) $(c : a) : (b : a) = (c : b) : (a : b)$.

P r o o f. Since $(C;\le)$ is directed above, there exists $s\in C$ such that $a\le s$, $b\le s$ and $c\le s$. Hence $a*b=a*(b\wedge s)=a*(s:(b*s))=(a*s):(b*s)$. Hence

$$\begin{aligned}(a*b)*(a*c) &= [(a*s):(b*s)]*[(a*s):(c*s)]\\ &= [((a*s):(b*s))*(a*s)]:(c*s)\\ &= [((b*s):(a*s))*(b*s)]:(c*s) \qquad \text{by Lemma 1.2(11)}\\ &= (b*a)*(b*c)\end{aligned}$$

The other equation follows by duality. □

B o s b a c h has defined a *cone algebra* as a precone algebra satisfying both the equations (1) and (2) of the above Lemma 1.4. Hence by that lemma, every precone algebra which is directed above, is a cone algebra and in particular, every precone algebra bounded above, is a cone algebra. We recall that B o s b a c h has defined a *brick* ([2, p. 64]) as an algebra $(C;*,:,1)$ — we are using 1 where B o s b a c h has used 0 — of type $(2,2,0)$ where $(C;*,:)$ is a precone algebra and the equation $1:(a*1)=a$ is satisfied. By Lemma 1.2(10), this amounts to asserting that $(C;*,:)$ is a bounded precone algebra. Hence a brick is a cone algebra.

B o s b a c h has proved that ([2, Statement (1.12), p. 60]) in a cone algebra the following are identities:

$$(a*b)\wedge(b*a)=0 \qquad \text{and} \qquad (a:b)\wedge(b:a)=0.$$

Now we show below that each of these identities also characterizes a cone algebra within the class of precone algebras.

Following K ü h r [9], we say that a precone algebra C has the *relative cancellation property* (RCP, for short) if and only if for $a,b,c\in C$,

$$(c\le a\wedge b \ \text{ and } \ c*a=c*b) \implies a=b.$$

Assume now that a precone algebra $(C;*,:)$ has RCP, $a,b,c\in C$, $c\le a\wedge b$ and $a:c=b:c$; then $a:c=b:c\le a\wedge b$ and $(a:c)*a=c=(b:a)*b$. Since C has RCP, we have $a=b$. Thus if C has RCP, then the dual of C also has RCP and since the dual of the dual of C is C, it follows that C has RCP if and only if dual of C has RCP (see [9, p. 12]). We now prove:

Theorem 1.5. *Let $(C;*,:)$ be a precone algebra, then:*

(α) *C has RCP*

if and only if

(β) *C satisfies* any one of *the following identities:*

(β1) $(a * b) \wedge (b * a) = 0$

(β2) $c * (a \wedge b) = (c * a) \wedge (c * b)$ *and*

(β3) $(a * b) * (a * c) = (b * a) * (b * c)$

or *the dual of any of the identities* (β).

We first prove a lemma.

Lemma 1.6. *Let* $(C; *, :)$ *be a precone algebra; then the following equations are valid in* C.

$$(a \wedge b) * b = a * b \qquad \textit{and} \qquad b : (a \wedge b) = b : a.$$

Proof. By Lemma 1.2(7), $a * b \leq b$ and hence $(a \wedge b) * b = (b : (a * b)) * b = b \wedge (a * b) = a * b$. The other equality is dual. $\square$

Proof of Theorem 1.5. It is enough to prove (α) $\Longrightarrow$ (β1) $\Longrightarrow$ (β2) $\Longrightarrow$ (β3) $\Longrightarrow$ (α) since C has RCP if and only if dual of C has RCP. Kühr [9] has proved (α) $\Longrightarrow$ (β1) by invoking an embedding theorem due to Dvurečenskij and Vetterlein [4] — see [9, Theorem 6.7, p. 14]; however, we will present an elementary proof. Also, Dvurečenskij and Vetterlein have proved ([4]) (α) $\Longrightarrow$ (β2) for a pseudo ŁBCK-algebra which is precisely the dual of a commutative pseudo BCK-algebra with RCP. We obtain this result from (α) $\Longrightarrow$ (β1) $\Longrightarrow$ (β2).

Now assume that C has RCP, i.e. (α), and let $a, b \in C$. Put $c = a \wedge b$ so that by Lemma 1.6, $(a*b) \wedge (b*a) = (c*b) \wedge (c*a) = (c*b) : ((c*a)*(c*b)) = c*x$ where $x = b : ((c*a)*(c*b)) = b : ((b*a)*(a*b))$. Similarly, $(a*b) \wedge (b*a) = c*y$ where $y = a : ((a*b)*(b*a))$. Now $c = a \wedge b = b : (a*b) \leq b : ((b*a)*(a*b)) = x$ and similarly, $c \leq y$. Hence $x = y$ because of RCP so that $c \leq x = x \wedge y \leq a \wedge b = c$ (since $x \leq b$ and $y \leq a$). Hence $c = x$ so that $(a * b) \wedge (b * a) = c * x = 0$. Hence ($\alpha$) $\Longrightarrow$ (β1).

Assume (β1); then by Lemma 1.2(9),

$$a * (b \wedge c) \leq (a * b) \wedge (a * c) \tag{\#}$$

Hence,

$$\begin{aligned}
& (a * (b \wedge c)) * ((a * b) \wedge (a * c)) \\
\leq\ & [(a * (b \wedge c)) * (a * b)] \wedge [(a * (b \wedge c)) * (a * c)] \\
=\ & [((a * b) : (c * b)) * (a * b)] \wedge [((a * c) : (b * c)) * (a * c)] \\
=\ & (a * b) \wedge (c * b) \wedge (a * c) \wedge (b * c) \\
\leq\ & (c * b) \wedge (b * c) = 0 \qquad \text{by } (\beta 1).
\end{aligned}$$

Hence $(a*b)\wedge(a*c)\le a*(b\wedge c)$ so that by (#), $a*(b\wedge c)=(a*b)\wedge(a*c)$. Hence $(\beta 1)\implies(\beta 2)$.

Now assume $(\beta 2)$; then by Lemma 1.6,

$$(a*b)*(a*c)=((a*b)\wedge(a*c))*(a*c)=(a*(b\wedge c))*(a*c)$$

(by assumption $(\beta 2)$) $=((a*c):(b*c))*(a*c)=(a*c)\wedge(b*c)$. Similarly, $(b*a)*(b*c)=(b*c)\wedge(a*c)$. Hence $(\beta 2)\implies(\beta 3)$.

Finally, assume $(\beta 3)$; and let $a,b,c\in C$, $c\le a\wedge b$ and $c*a=c*b$. Then $a*b=0*(a*b)=(a*c)*(a*b)=(c*a)*(c*b)=0$ so that $a*b=0$. By symmetry, $b*a=0$ and hence $a=b$. Thus $(\beta 3)\implies(\alpha)$. $\square$

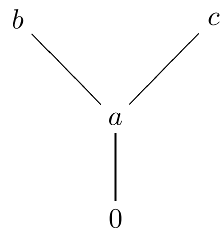
We conclude this section with a couple of examples showing that

(1) the class of precone algebras is *larger* than the class of cone algebras and

(2) a cone algebra need not be directed above.

Example 1.7. Let A be the set consisting of four distinct elements $0,a,b,c$ and define the operation $*$ by the following table and : by the equation $x*y=y:x$.

$*$	0	a	b	c
0	0	a	b	c
a	0	0	a	a
b	0	0	0	a
c	0	0	a	0

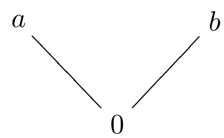


Since $(a*c)*(a*b)=0\neq a=(c*a)*(c*b)$, $(A;*,:)$ is *not* a cone algebra and it is a routine verification to show that $(A;*,:)$ is a precone algebra.

[9, Example 6.1] is equivalent to the above one and illustrates the same point within the framework of BCK-algebras.

Example 1.8. Let $C=\{0,a,b\}$ where $0,a,b$ are distinct and define the operation $*$ by the following table and : by the equation $x*y=y:x$.

$*$	0	a	b
0	0	a	b
a	0	0	b
b	0	a	0



Then $(C;*,:)$ is a symmetric cone algebra, which is not directed above, since the elements a and b have *no* common upper bound.

2. Semi-ℓg-cones

In this section, we present a generalization of a pseudo MV-algebra of Georgescu and Iorgulescu [7] (or equivalently, of a generalized MV-algebra of Rachůnek [10]) within the framework of Bosbach's cone algebras ([2]), by means of the following definition:

Definition 2.1. An algebra $(C;*, :, \cdot)$ of type $(2,2,2)$ is called a *semi-ℓg-cone* if and only if $(C;*, :)$ is a precone algebra and the following identity holds in C:

$$(ab) * c = b * (a * c).$$

A related concept, which will be convenient for us in the sequel, is introduced by the following definition:

Definition 2.2. An algebra $(C;*, :, +)$ of type $(2,2,2)$ is called an *ℓg-cone* if and only if $(C;*, :)$ is a precone algebra and the following are valid in C:

$$a * (a + b) = b = (b + a) : a.$$

Clearly, the duality principle of a precone algebra extends to ℓg-cones also (by the defining equations above) and we will show below (Lemma 2.3) that it extends to semi-ℓg-cones also.

We now assume that $(C;*, :, \cdot)$ is a semi-ℓg-cone and $a, b, c, \ldots \in C$. We now prove:

Lemma 2.3. *We have $c : (ab) = (c : b) : a$.*

Proof. If $u \in C$, then $c : (ab) \le u \iff (u * c) : (ab) = 0 \iff (ab) * (u * c) = 0 \iff b * (a * (u * c)) = 0 \iff (ua) * c \le b \iff (ua) * (c : b) = 0 \iff a * (u * (c : b)) = 0 \iff ((c : b) : a) : u = 0$ (by Lemma 1.2(6)) $\iff (c : b) : a \le u$. Hence $c : (ab) = (c : b) : a$. □

Observe that a semi-ℓg-cone can be as well defined by means of the equation $c : (ab) = (c : b) : a$.

Corollary 2.4. $c \le ab \iff a * c \le b \iff c : b \le a$.

Corollary 2.5. $a \le ab$ *and* $b \le ab$.

Proof. $a * a = 0 \le b \implies a \le ab$ and $b : b = 0 \le a \implies b \le ab$. □

Now by Corollary 2.5, the reduct $(C;*, :)$ of a semi-ℓg-cone $(C;*, :, \cdot)$ is a precone algebra, which is directed above. Hence by Lemma 1.4, we get:

Theorem 2.6. *If $(C;*, :, \cdot)$ is a semi-ℓg-cone, then $(C;*, :)$ is a cone algebra.*

However, *not* every cone algebra is the reduct of some semi-ℓg-cone; for instance, the cone algebra of Example 1.8 *cannot* be the reduct of a semi-ℓg-cone, since this cone algebra is not directed above and the reduct $(C;*,:)$ of a semi-ℓg-cone must be directed above by Corollary 2.5.

Lemma 2.7. *If $(C;*,:,\cdot)$ is a semi-ℓg-cone, then $(C;\cdot)$ is a semigroup with identity element 0.*

Proof. By Corollary 2.4, $u \le a0 \iff u = u : 0 \le a$ and dually, $u \le 0a \iff u = 0 * u \le a$. Hence $a0 = 0a = a$.

Further, $u \le (ab)c \iff u : c \le ab \iff a * (u : c) \le b \iff (a * u) : c \le b \iff a * u \le bc \iff u \le a(bc)$. Hence $(ab)c = a(bc)$. □

The following theorem is very important for our purpose.

Theorem 2.8. *Let $(C;*,:)$ be a precone algebra and let A be a nonempty subset of C; then the following statements are equivalent:*

(1) (α) $0 \in A$ *and*
 (β) $a \in A$ *and* $a * b \in A \implies b \in A$
(2) (α) $0 \in A$ *and*
 (β) $a \in A$ *and* $b : a \in A \implies b \in A$.

Further, if $(C;*,:)$ is the reduct of a semi-ℓg-cone $(C;*,:,\cdot)$, then each of the above is equivalent to

(3) (α) $a \in A, b \in A \implies ab \in A$ *and*
 (β) $a \in A$ *and* $b \le a \implies b \in A$.

Proof. Assume (1); then if $a \in A$ and $b \le a$, we have $a * b = 0 \in A$ and hence $b \in A$. Thus A is convex. Now let $a \in A$ and $b : a \in A$; then $(b : a) * b = b \wedge a \le a$ and hence $(b : a) * b \in A$. Since $b : a \in A$, we get $b \in A$ by (1)(β). Thus (1) $\implies$ (2); and (2) $\implies$ (1) by duality.

Now assume that $(C;*,:,\cdot)$ is a semi-ℓg-cone and assume (1). Then (3)(β) is true by the first part of the proof. Now let $a \in A$ and $b \in A$; then $a * ab \le b \in A$ (since $ab \le ab$) and so $a * ab \in A$. Since $a \in A$, this implies $ab \in A$ by (1)(β). Hence (1) $\implies$ (3).

Conversely assume (3); and suppose $a \in A$ and $a * b \in A$. Since $a * b \le a * b$, we have $b \le a(a * b) \in A$; and hence $b \in A$ by (3)(β). Also, $0 \in A$ since A is nonempty. Hence (3) $\implies$ (1). □

We now introduce the following definition:

Definition 2.9.

(i) If $(C; *, :)$ is a precone algebra, then a subset A of C is called a *deductive system* (or simply, d.s.) if and only if it satisfies either of the conditions (1) and (2) of the above Theorem 2.8; and

(ii) if $(C; *, :, \cdot)$ is a semi-ℓg-cone, then a nonempty subset A of C satisfying condition (3) of the above Theorem 2.8, is called an *ideal*.

Hence, by Theorem 2.8, the ideals of a semi-ℓg-cone $(C; *, :, \cdot)$ are precisely the deductive systems of its reduct cone algebra $(C; *, :)$. Clearly, an ideal of a semi-ℓg-cone $(C; *, :, \cdot)$ is a convex submonoid of $(C;\ \cdot, 0)$; and we will next show that an ideal A is ℓ-submonoid of C, i.e., $(A; \leq)$ is a sublattice of $(C; \leq)$. First we prove:

Lemma 2.10. *Let $(C; *, :, \cdot)$ be a semi-ℓg-cone; then $(C; \leq)$ is a lattice, in which $a \vee b = a(a * b) = b(b * a) = (b : a)a = (a : b)b$.*

P r o o f. By Corollary 2.5, $(C; *, :)$ is a directed precone algebra and hence a cone algebra. Now, if $a, b, c \in C$, then $(a(a * b)) * c = (a * b) * (a * c) = (b * a) * (b * c) = (b(b * a)) * c$. Taking $c = a(a * b)$ and $b(b * a)$ respectively, we get $a(a * b) = b(b * a)$ by Lemma 1.2(5). Hence by Corollary 2.5, $a \leq a(a * b)$ and $b \leq a(a * b)$. Suppose now that $a \leq c$ and $b \leq c$; then by Lemma 1.2(9), $a * b \leq a * c$ and hence $a * (a(a * b)) \leq a * b \leq a * c$ so that by Corollary 2.5, $a(a*b) \leq a(a*c) = c(c*a) = c0 = c$. Hence $a(a*b) = b(b*a) = \sup(a, b) =: a \vee b$. By duality, we also have $a \vee b = (a : b)b = (b : a)a$.

We already know that $\inf(a, b) = a : (b*a)$ (by Lemma 1.2(10)). Hence $(C; \leq)$ is a lattice. □

Lemma 2.11. *Let $(C; *, :, \cdot)$ be a semi-ℓg-cone and A an ideal of C; then $(A; \leq)$ is a sublattice of $(C; \leq)$. Hence every ideal is a convex ℓ-submonoid of C.*

P r o o f. Let $a, b \in A$; since A is convex and $a * b \leq b$ and $a : b \leq a$, $(A; *, :)$ is a subalgebra of $(C; *, :)$. Hence, if $a, b \in A$, then $a \wedge b = a : (b * a) \in A$ and $a \vee b = a(a * b) \in A$. Hence $(A; \leq)$ is a sublattice of $(C; \leq)$. □

Lemma 2.12. *Let $(C; *, :, \cdot)$ be a semi-ℓg-cone, then*

(α) *$a(b \wedge c) = (ab) \wedge (ac)$ and $(b \wedge c)a = (ba) \wedge (ca)$ for all $a, b, c \in C$; and*

(β) *$(C; \leq)$ is a distributive lattice.*

P r o o f.

(α) $u \leq a(b \wedge c) \iff a * u \leq b \wedge c \iff a * u \leq b$ and $a * u \leq c \iff u \leq ab$ and $u \leq ac \iff u \leq (ab) \wedge (ac)$. The other equality follows by duality. Hence (α) holds.

(β) We know that $(C;\leq)$ is a lattice (Lemma 2.10). Now

$$\begin{aligned} a\vee(b\wedge c) &= a(a*(b\wedge c)) && \text{(Lemma 2.10)}\\ &= a((a*b)\wedge(a*c)) && \text{(by Theorem 1.5)}\\ &= (a(a*b))\wedge(a(a*c)) && \text{(by } (\alpha))\\ &= (a\vee b)\wedge(a\vee c). \end{aligned}$$

Hence $(C;\leq)$ is a distributive lattice.

□

Theorem 2.13. *Let $(C;*,:)$ be a precone algebra and let A be a deductive system of C; then the following are equivalent:*

(1) *$c:(c:a)\in A$ and $(a*c)*c\in A$ for all $a\in A$ and $c\in C$;*

(2) *If $a,b\in C$, then $a*b\in A \iff b:a\in A$.*

Further, if $(C;*,:)$ is the reduct of a semi-ℓg-cone $(C;*,:,\cdot)$, then each of the above is equivalent to:

(3) *$cA=Ac$ for all $c\in C$.*

Proof. Assume (1); then $a*b\in A \implies b:a=b:(a\wedge b)=b:(b:(a*b))\in A$ and $b:a\in A \implies a*b=(a\wedge b)*b=((b:a)*b)*b\in A$. Hence (1) $\implies$ (2).

Assume (2); and let $c\in C$ and $a\in A$. Then $(c:a)*c=c\wedge a\leq a\in A$ and since A is convex, $(c:a)*c\in A$. Hence by (2), $c:(c:a)\in A$. Dually, $(a*c)*c\in A$. Hence (2) $\implies$ (1).

Now assume that $(C;*,:,\cdot)$ is a semi-ℓg-cone; then A is an ideal of C. Further, assume (2); and let $a\in A$ and $c\in C$. Then $c*(ca)\leq a$ and hence $c*(ca)\in A$. Hence by (2), $(ca):c\in A$. Now $ca=(ca)\vee c=(ca:c)c\in Ac$ so that $cA\subseteq Ac$. By duality, $Ac\subseteq cA$ so that $Ac=cA$. Hence (2) $\implies$ (3).

Finally, assume (3); and let $a*b\in A$. Then $a(a*b)\in aA=Aa$ and hence $a(a*b)=ua$ for some $u\in A$. Hence $0=b:(a\vee b)=b:(a(a*b))=b:(ua)=(b:a):u$ so that $b:a\leq u$. Since $u\in A$, it follows $b:a\in A$. The reverse implication follows by duality. Hence (3) $\implies$ (2). □

Definition 2.14. A deductive system of a precone algebra, satisfying either of the conditions (1) and (2) of Theorem 2.13 above is said to be *normal*; and an ideal of a semi-ℓg-cone satisfying (3) of Theorem 2.13, is called a *normal ideal.*

Now we conclude this section with a brief introduction to *polars* in a precone algebra. Let $(C;*,:)$ be a precone algebra and let $A\subseteq C$; and write

$$A^{\perp}=\{c\in C:\ c\wedge a=0 \ \text{for all}\ a\in A\}.$$

Lemma 2.15. *Let $(C;*,:)$ be a precone algebra and $a,b,c \in C$; then*

$$(a*b)*(a*c) \le b*c \qquad \text{and} \qquad (c:a):(b:a) \le c:b.$$

P r o o f. $(a*c):(b*c) = a*(b\wedge c) \le a*b$; and hence $0 = (a*b)*((a*c):(b*c)) = ((a*b)*(a*c)):(b*c)$. So $(a*b)*(a*c) \le b*c$. The second inequality follows by duality. □

We now prove:

Lemma 2.16. *$A^{\perp}$ is a deductive system of C.*

P r o o f. Clearly, $0 \in A^{\perp}$; and now assume that $c \in A^{\perp}$ and $c*d \in A^{\perp}$. Let $a \in A$; then $c\wedge a = 0 = (c*d)\wedge a$ so that by Lemma 1.2(12), $c*a = a = (c*d)*a$. Hence, by Lemma 2.15, $a = (c*d)*(c*a) \le d*a \le a$ and hence $a = d*a$ so that by Lemma 1.2(12), $d\wedge a = 0$. Hence $d \in A^{\perp}$. □

Lemma 2.17.

(α) *$A \subseteq B \implies B^{\perp} \subseteq A^{\perp}$ and*

(β) *$A^{\perp} = A^{\perp\perp\perp}$ for all $A, B \subseteq C$.*

P r o o f. Routine. □

Subsets of C of the form $A^{\perp}$ are called *polars* and so, by Lemma 2.16, every polar is a deductive system; and deductive systems which are polars are called *polar deductive systems*.

Lemma 2.18. *Let $(C;*,:)$ be a precone algebra; and let A be a deductive system of C; then A is a polar deductive system if and only if $A = A^{\perp\perp}$.*

P r o o f. Routine. □

3. Special semi-ℓg-cones

In this section, we describe three special cases of a semi-ℓg-cone $(C;*,:,\cdot)$ where

(1) the monoid $(C;\cdot,0)$ is cancellative,

(2) $(C;\cdot,0)$ is a monoid of idempotents and

(3) $(C;\le)$ is bounded

and identify them with objects which are fairly well familiar in the literature. Now let $(C;*,:,\cdot)$ be a semi-ℓg-cone and $a,b,c,\ldots \in C$.

Lemma 3.1. *$b \le c \implies ab \le ac$ and $ba \le ca$.*

Proof. $a * ab \le b \le c$ and hence $ab \le ac$; and $ba : a \le b \le c$ and hence $ba \le ca$. □

Lemma 3.2. *$ab = a(a * ab) = (ab : b)b$.*

Proof. By Lemma 2.10, $a(a * ab) = a \vee ab = ab$ and $(ab : b)b = ab \vee b = ab$ by Corollary 2.5. □

Corollary 3.3. *$(C; \cdot, 0)$ is a cancellative monoid if and only if $a * ab = b$ and $ab : b = a$, for all $a, b \in C$, and hence in this case, $(C; *, :, \cdot)$ is an ℓg-cone.*

Proof. Routine; also see Definition 2.2. □

Lemma 3.4. *Let $(C; *, :, +)$ be an ℓg-cone; then*

(1) *$a \le a + b$ and $b \le a + b$*

(2) *$(C; *, :, +)$ is a cancellative semi-ℓg-cone; i.e.,*
  (α) *$(a + b) * c = b * (a * c)$ and*
  (β) *the monoid $(C; +, 0)$ is cancellative.*

Proof.
(1) $a = (a + b) : b \le a + b$ and $b = a * (a + b) \le a + b$.
(2) By (1) and Lemma 1.4, $(C; *, :)$ is a cone algebra. Hence $(a + b) * c = 0 * ((a + b) * c) = ((a + b) * a) * ((a + b) * c) = (a * (a + b)) * (a * c) = b * (a * c)$. Hence $(C; *, :, +)$ is a semi-ℓg-cone. Finally, $(C; +, 0)$ is a cancellative monoid by Corollary 3.3. □

Combining Corollary 3.3 and Lemma 3.4, we get the following theorem.

Theorem 3.5. *Let $(C; *, :, \cdot)$ be a semi-ℓg-cone; then $(C; \cdot, 0)$ is cancellative if and only if $(C; *, :, \cdot)$ is an ℓg-cone.*

Now, let $(G; +, \le)$ be a (not necessarily abelian) lattice ordered group and G^+ be its positive cone; then it is easily verified that $(G^+; *, :, +)$ is an ℓg-cone if we define $*$ and $:$ by

$$a * b = (-a + b) \vee 0 \qquad \text{and} \qquad a : b = (a - b) \vee 0.$$

On the other hand, let $(C; *, :, +)$ be an ℓg-cone; then $(C; *, :)$ is a cone algebra and given $a, b \in C$, there exists $x \in C$ (namely, $a + b$) such that $a * x = b$ and $x * a = 0$. We now recall the following result due to Bosbach ([2, Statement (1.20), p. 61]):

Theorem $\mathfrak{A}$ (Bosbach)**.** *Let R be a cone algebra. Then R is the cone algebra of some ℓ-group cone if it satisfies, in addition, the law:*

(AC) *given $a, b \in R$, there exists $x \in R$ such that*

$$a * x = b \qquad \text{and} \qquad x * a = 0.$$

Hence, $(C; *, :)$ is the cone algebra of some ℓ-group cone $(C; *, :, \cdot)$ and so has RCP.

Hence we have:

Lemma 3.6. *Let $(C; *, :)$ be a cone algebra and let $a, x, y \in C$. If $a * x = a * y$ and $x * a = y * a = 0$, then $x = y$.*

Now since $a + b$ and ab both satisfy the equations $a * x = 0$ and $x * a = 0$, by the above Lemma 3.6, $a + b = ab$ for all $a, b \in C$. Then $(C; *, :, +)$ is the cone algebra of some ℓ-group cone. Consequently, the cancellative semi-ℓg-cones are ℓ-group cones.[1]

We now assume that $(C; *, :, \cdot)$ is a semi-ℓg-cone such that $a^2 = a$ for all $a \in C$. Then $a*b = a^2*b = a*(a*b)$ and hence by Lemma 1.2(12), $a \wedge (a*b) = 0$. Also, $a \vee b = a(a*b) = a(a*(a*b)) = a \vee (a*b)$ and dually, we obtain $a \wedge (b : a) = 0$ and $a \vee b = a \vee (b : a)$. Now the lattice $(C; \leq)$ is distributive (Lemma 2.12(β)) and hence $a * b = b : a$ is the complement of a in the interval $[0, a \vee b]$. Hence $(C; \leq)$ is a sectionally complemented (i.e., each interval $[0, a]$ is complemented) distributive lattice.

Now ab is an upper bound of a and b (Corollary 2.5) and hence $a \vee b \leq ab$. On the other hand, $a * ab \leq b$ and hence $a * (ab) = a^2 * (ab) = a * (a * ab) \leq a * b$ (Lemma 1.2(9)) so that $ab = a(a * ab)$ (Lemma 3.2) $\leq a(a * b)$ (Lemma 3.1) $= a \vee b$. Hence $ab = a \vee b$. Thus we have proved the following theorem.

Theorem 3.7. *Let $(C; *, :, \cdot)$ be a semi-ℓg-cone in which every element is idempotent; then*

(1) *$(C; \leq)$ is a distributive lattice in which each interval $[0, a]$ is complemented,*

(2) *$a * b = b : a$ is the complement of a in the interval $[0, a \vee b]$, and*

(3) *$ab = a \vee b$ for all $a, b \in C$.*

Hence, in this case, $(C; *, :)$ is a symmetric cone algebra and $(C; \cdot, 0)$ is an abelian monoid.

Conversely, let $(C; \leq)$ be a sectionally complemented distributive lattice and define $*$ and $:$ by $a*b = b : a =$ the complement of a in the interval $[0, a \vee b]$; then it is a routine verification to show that $(C; *, :, \vee)$ is a semi-ℓg-cone in which every element is idempotent.

[1] The referee has kindly pointed out that this result also follows from [1, Theorem 6.2].

Remark 3.8. The distributive lattices described in Theorem 3.7 above, are well known to be *term equivalent* to Boolean rings. If $(C;\leq)$ is a sectionally complemented distributive lattice, then $(C;+,\cdot)$ is a Boolean ring if we define $a+b=(a*b)\vee(b*a)$ and $ab=a\wedge b$. In the reverse direction, if $(C;+,\cdot)$ is a Boolean ring, then $(C;\leq)$ is a sectionally complemented distributive lattice if we define $a\leq b \iff a=ab$. For this reason, we shall call a semi-ℓg-cone, in which every element is idempotent, a *Boolean cone.*

Finally, we now consider a *bounded* semi-ℓg-cone and show that it is *term equivalent* to a pseudo MV-algebra of Georgescu and Iorgulescu [7]. We now recall the following definition of a pseudo MV-algebra as quoted in Dvurečenskij [3]:

Definition 3.9 (Georgescu and Iorgulescu)**.** By a pseudo MV-algebra is meant an algebra $(A;\oplus,{}^-,{}^\sim,0,1)$ of type $(2,1,1,0,0)$, which, together with an additional binary operation $\odot$, defined by

(A0) $b\odot a=\left(\bar{a}\oplus\bar{b}\right)^{\sim}$,

satisfies the following axioms:

(A1) $(x\oplus y)\oplus z=x\oplus(y\oplus z)$

(A2) $x\oplus 0=x=0\oplus x$

(A3) $x\oplus 1=1=1\oplus x$

(A4) $\widetilde{1}=0=\bar{1}$

(A5) $(\bar{x}\oplus\bar{y})^{\sim}=(\widetilde{x}\oplus\widetilde{y})^{-}$

(A6) $x\oplus(\widetilde{x}\odot y)=y\oplus(\widetilde{y}\odot x)=(x\odot\bar{y})\oplus y=(y\odot\bar{x})\oplus x$

(A7) $x\odot(\bar{x}\oplus y)=(x\oplus\widetilde{y})\odot y$

(A8) $(\bar{x})^{\sim}=x.$

We also need the following lemma from [12], which is a simple consequence of the above Definition 3.9. (Also, see [4].)

Lemma 3.10. ([12, Lemma 2.2]) *Let* $(A;\oplus,{}^-,{}^\sim,0,1)$ *be a pseudo MV-algebra; then*

(1) $(\bar{x})^{\sim}=(\widetilde{x})^{-}=x$

(2) $\bar{0}=1=\widetilde{0}$

(3) $x\odot 0=0\odot x=0$*;* $x\odot 1=1\odot x=x$

(4) $x\odot\bar{x}=\widetilde{x}\odot x=0$

(5) $(x \odot y)^{-} = \bar{y} \oplus \bar{x}$; $(x \odot y)^{\sim} = \widetilde{y} \oplus \widetilde{x}$

(6) $(x \odot y) \odot z = x \odot (y \odot z)$

(7) $(\widetilde{x} \odot x)^{\sim} \odot y = y = y \odot (x \odot \bar{x})^{-}$

(8) $x \odot (\widetilde{y} \odot x)^{-} = (y \odot \bar{x})^{\sim} \odot y$.

Lemma 3.11. *Let $(A; \oplus, ^{-}, ^{\sim}, 0, 1)$ be a pseudo MV-algebra and define $a * b = \tilde{a} \odot b$ and $a : b = a \odot \bar{b}$; then $(A; *, :, \oplus)$ is a bounded semi-ℓg-cone.*

Proof. The equations (7) and (8) of the above Lemma 3.10, translate into the equations $(a * a) * b = b = b : (a : a)$ and $a : (b * a) = (b : a) * b$ respectively; and $a * (b : c) = \tilde{a} \odot (b \odot \bar{c}) = (\tilde{a} \odot b) \odot \bar{c}$ (by the equation (6) of the above Lemma 3.10) $= (a * b) : c$. Also, $(a \oplus b) * c = (a \oplus b)^{\sim} \odot c = (\tilde{b} \odot \tilde{a}) \odot c$ (by equation (1) of Lemma 3.10 and the axiom (A0) of the Definition 3.9) $= b * (a * c)$. Hence $(A; *, :, \oplus)$ is a semi-ℓg-cone and so $(A; *, :)$ is a cone algebra. Finally, if $a \in A$, then $a \wedge 1 = 1 : (a * 1) = 1 \odot (\tilde{a} \odot 1)^{-} = 1 \odot (\tilde{a})^{-} = (\tilde{a})^{-}$ (by equation (3) of Lemma 3.10) $= a$ (by equation (1) of Lemma 3.10). Hence $a \leq 1$ for all $a \in A$ so that the semi-ℓg-cone $(A; *, :, \oplus)$ is bounded. □

Lemma 3.12. *Let $(C; *, :, \oplus)$ be a* bounded *semi-ℓg-cone and let* 1 *denote the greatest element of C; and define $\bar{a} = 1 : a$ and $\tilde{a} = a * 1$. Then $(C; \oplus, ^{-}, ^{\sim}, 0, 1)$ is a pseudo MV-algebra.*

Proof. The *additional* binary operation of the Definition 3.9, (see (A0)) satisfies $b \odot a = (\bar{a} \oplus \bar{b})^{\sim} = (\bar{a} \oplus \bar{b}) * 1 = \bar{b} * (\bar{a} * 1) = \bar{b} * (\bar{a})^{\sim} = \bar{b} * ((1 : a) * 1) = \bar{b} * (a \wedge 1) = \bar{b} * a = (1 : b) * (1 : (a * 1)) = ((1 : b) * 1) : (a * 1) = (b \wedge 1) : \tilde{a} = b : \tilde{a}$. Hence $b \odot a = \bar{b} * a = b : \tilde{a}$. We now verify the axioms (A1) through (A8) of the Definition 3.9. Since $(C; \oplus, 0)$ is a monoid, we have (A1) and (A3); and since $1 \leq x \oplus 1$ and $1 \leq 1 \oplus x$ and 1 is the greatest element, we have (A3). (A4) holds because $\tilde{1} = 1 * 1 = 0 = 1 : 1 = \bar{1}$. Also, $(\tilde{x} \oplus \tilde{y})^{-} = 1 : (\tilde{x} \oplus \tilde{y}) = (1 : \tilde{y}) : \tilde{x} = y : \tilde{x} = (\bar{x} \oplus \bar{y})^{\sim} (= y \odot x)$ and hence (A5) holds.

Now $(\bar{x})^{\sim} = (1 : x) * 1 = x \wedge 1 = x$ and hence (A8); dually, $(\tilde{x})^{-} = x$. Now $\tilde{a} \odot b = (\bar{b} \oplus a)^{\sim} = (\bar{b} \oplus a) * 1 = a * (\bar{b} * 1) = a * (\bar{b})^{\sim} = a * b$ and similarly, $a \odot \bar{b} = a : b$. Now $x \oplus (\tilde{x} \odot y) = x \oplus (x * y) = x \vee y$ and similarly, each of the expressions in (A6) is $x \vee y$. Hence (A6) holds. Finally, $x \odot (\bar{x} \oplus y) = x \odot (\tilde{y} \odot x)^{-} = x : (y * x) = (y : x) * y = (y \odot \bar{x})^{\sim} \odot y = (x \oplus \tilde{y}) \odot y$. Hence (A7) is satisfied, and hence $(C; \oplus, ^{-}, ^{\sim}, 0, 1)$ is a pseudo MV-algebra. □

We now combine Lemmas 3.11 and 3.12 to obtain the following theorem.

Theorem 3.13.

(a) *Let $\mathcal{A} = (A; \oplus, ^-, ^\sim, 0, 1)$ be a pseudo MV-algebra and define $a * b = \tilde{a} \odot b$ and $a : b = a \odot \bar{b}$; then $(A; *, :, \oplus)$ is a* bounded *semi-ℓg-cone, which is denoted by $\mathfrak{C}(\mathcal{A})$.*

(b) *Let $\mathcal{C} = (C; *, :, \oplus)$ be a bounded semi-ℓg-cone and let 1 denote the greatest element of C; and define $\bar{a} = 1 : a$ and $\tilde{a} = a * 1$. Then $(C; \oplus, ^-, ^\sim, 0, 1)$ is a pseudo MV-algebra and we denote this by $\mathfrak{M}(\mathcal{C})$.*

(c) *With the above notation,*

$$\mathfrak{C}(\mathfrak{M}(\mathcal{C})) = \mathcal{C} \qquad \textit{and} \qquad \mathfrak{M}(\mathfrak{C}(\mathcal{A})) = \mathcal{A}.$$

P r o o f. We need only prove (c). Let $\mathcal{A} = (A; \oplus, ^-, ^\sim, 0, 1)$ be a pseudo MV-algebra; then $\mathfrak{C}(\mathcal{A})$ is a bounded semi-ℓg-cone with $a * b = \tilde{a} \odot b$ and $a : b = a \odot \bar{b}$. Hence $\mathfrak{M}(\mathfrak{C}(\mathcal{A}))$ is a pseudo MV-algebra $(A; \oplus, ^\smile, ^\wedge, 0, 1)$ where $\hat{a} = a * 1 = \tilde{a} \odot 1 = \tilde{a}$ and $\breve{a} = 1 : a = 1 \odot \bar{a} = \bar{a}$. Hence $\mathfrak{M}(\mathfrak{C}(\mathcal{A})) = \mathcal{A}$. Similarly, $\mathfrak{C}(\mathfrak{M}(\mathcal{C})) = \mathcal{C}$. $\square$

Summarizing the three special cases, we have proved that a semi-ℓg-cone is

(1) cancellative if and only if it is an ℓg-cone;

(2) Boolean cone if and only if every element is idempotent; and

(3) bounded if and only if it is term equivalent to a pseudo MV-algebra.

We now recall that a nonempty subset A of a pseudo MV-algebra $(C; \oplus, ^-, ^\sim 0, 1)$ is an *ideal* (by definition) if and only if

(i) $a, b \in A \implies a \oplus b \in A$, and

(ii) $b \in A$ and $a \le b \implies a \in A$.

Thus A is an ideal of a pseudo MV-algebra if and only if it is an ideal of the equivalent semi-ℓg-cone (see Definition 2.9). Hence by Theorem 2.13 we get:

Corollary 3.14. ([7, Lemma 3.2]) *Let $(C; \oplus, ^-, ^\sim 0, 1)$ be a pseudo MV-algebra and let A be an ideal; then A is a normal ideal if and only if*

$$c \oplus A = A \oplus c \qquad \textit{for all} \quad c \in C.$$

I am thankful to the referee for his/her helpful comments and to Prof. J. Kühr for a copy of his paper [9], which have led to a substantial improvement of the original version of Theorem 1.5.

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Received 19. 2. 2007
Revised 7. 12. 2007

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