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OSCILLATION OF FOURTH ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS-II

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ABSTRACT. Oscillatory and asymptotic behaviour of solutions of a class of fourth order nonlinear neutral difference equations of the form

$$\Delta^2(r(n)\Delta^2(y(n)+p(n)y(n-m)))+q(n)G(y(n-k))=0$$

and

(E) $\Delta^2(r(n)\Delta^2(y(n)+p(n)y(n-m)))+q(n)G(y(n-k))=f(n)$

are studied under the assumption $\sum\limits_{n=0}^{\infty}\frac{n}{r(n)}<\infty$, for different ranges of $p(n)$. Sufficient conditions are obtained for the existence of positive bounded solutions of (E).

1. Introduction

In [2], Kusano and Naito have studied oscillatory behaviour of solutions of a class of fourth order nonlinear differential equations of the form

$$(r(t)y'')''+yF(y^2,t)=0,$$

where r and F are continuous and positive functions on $[0,\infty)$ and $[0,\infty)\times[0,\infty)$ respectively under the assumption that

$$(\mathrm{H}_0)\qquad \int\limits_0^{\infty}\frac{t}{r(t)}\,\mathrm{d}t<\infty.$$

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The object of this paper is to study the oscillatory and asymptotic properties of solutions of a class of fourth order nonlinear neutral difference equations of the form

$$\Delta^2(r(n)\Delta^2(y(n)+p(n)y(n-m)))+q(n)G(y(n-k))=0, \tag{1}$$

where Δ is the forward difference operator defined by $\Delta y(n)=y(n+1)-y(n)$, p, q are real valued functions defined on $N(n_0)=\{n_0,n_0+1,n_0+2,\dots\}$, $n_0\geq 0$ such that $q(n)\geq 0$, $G\in C(\mathbb{R},\mathbb{R})$ is non-decreasing and $uG(u)>0$ for $u\neq 0$ and $m>0$, $k\geq 0$ are integers, under the discrete analogue of the assumption (H_0) as

(A_0) $r(n)$ is a real valued function such that $r(n)>0$ and $\sum\limits_{n=0}^{\infty}\frac{n}{r(n)}<\infty$.

The associated forced equation

$$\Delta^2(r(n)\Delta^2(y(n)+p(n)y(n-m)))+q(n)G(y(n-k))=f(n), \tag{2}$$

where $f(n)$ is a real valued function, is also studied under the assumption (A_0). Different ranges of $p(n)$ and different types of forcing functions are considered. In [3], [4] and [8], Parhi and Tripathy have discussed oscillation and asymptotic behaviour of solutions of higher order difference equations of the form

$$\Delta^m(y(n)+p(n)y(n-s))+q(n)G(y(n-k))=0$$

and

$$\Delta^m(y(n)+p(n)y(n-s))+q(n)G(y(n-k))=f(n).$$

Equations (1) and (2) can not be termed as the particular case of the above equations in view of (A_0). Hence the study of (1) and (2) is very interesting. Necessary and sufficient conditions for oscillation of (1) and (2) are obtained in this paper.

By a solution of Eq. (1) on $N(n_0)$ we mean, a real valued function $y(n)$ defined on $N(-\rho)=\{-\rho,-\rho+1,\dots\}$, $\rho=\max\{m,k\}$, which satisfies (1) for sufficiently large n. If

$$y(n)=A_n, \qquad n=-\rho,-\rho+1,\dots,0,1,2,3\dots, \tag{3}$$

are given, then (1) admits a unique solution satisfying the initial conditions (3). A solution $y(n)$ of (1) is said to be oscillatory if, for every integer $N>0$, there exists an $n\geq N$ such that $y(n)y(n+1)\leq 0$; otherwise, it is called non oscillatory.

Equation (1) may be regarded as a discrete analogue of

$$\big(r(t)\,(y(t)+p(t)y(t-\tau))''\big)''+q(t)G(y(t-\sigma))=0, \qquad t\geq 0.$$

Oscillatory and asymptotic behaviour of solutions of this equation and the associated forced equation are studied in [6].

2. Some preparatory results

This section deals with the lemmas which play an important role in establishing the present work.

Remark. From (A_0) it follows that

$$\sum_{n=0}^{\infty} \frac{1}{r(n)} < \infty.$$

Lemma 2.1. *Let $u(n)$ be a real-valued function defined on $N(n_0)$ with $\Delta^2(r(n)\Delta^2 u(n)) \leq 0$ for large n. If $u(n) > 0$ eventually, then one of the following cases holds for all large n:*

(a) $\Delta u(n) > 0$, $\Delta^2 u(n) > 0$ *and* $\Delta(r(n)\Delta^2 u(n)) > 0$,

(b) $\Delta u(n) > 0$, $\Delta^2 u(n) < 0$ *and* $\Delta(r(n)\Delta^2 u(n)) > 0$,

(c) $\Delta u(n) > 0$, $\Delta^2 u(n) < 0$ *and* $\Delta(r(n)\Delta^2 u(n)) < 0$,

(d) $\Delta u(n) < 0$, $\Delta^2 u(n) > 0$ *and* $\Delta(r(n)\Delta^2 u(n)) > 0$,

If $u(n) < 0$ eventually, then either one of the following cases (b), (c), (d)

(e) $\Delta u(n) < 0$, $\Delta^2 u(n) < 0$ *and* $\Delta(r(n)\Delta^2 u(n)) > 0$,

(f) $\Delta u(n) < 0$, $\Delta^2 u(n) < 0$ *and* $\Delta(r(n)\Delta^2 u(n)) < 0$.

holds for all large n.

Proof. $\Delta^2(r(n)\Delta^2 u(n)) \leq 0$, for all large n implies that $u(n)$ is monotonic. Then $u(n) > 0$ or $u(n) < 0$. The rest of the proof is simple and hence is omitted. □

Lemma 2.2. *Let* (A_0) *hold. Assume that $u(n)$ is positive function defined on $N(n_0)$ such that $\Delta^2(r(n)\Delta^2 u(n)) \leq 0$ for all large n. Then:*

i) *Suppose that the case* (c) *of* Lemma 2.1 *holds. Then there is a constant $L \in (0,1)$ such that the following inequalities hold for all large n*
   (I_1) $\Delta u(n) \geq -\Delta(r(n)\Delta^2 u(n))R(n)$
   (I_2) $u(n) \geq Ln\Delta u(n)$
   (I_3) $u(n) \geq -L\Delta(r(n)\Delta^2 u(n))nR(n)$,
   where $R(n) = \sum_{s=n}^{\infty} \frac{s-n}{r(n)}$ *and*

ii) *$u(n) > r(n)\Delta^2 u(n)R(n)$ for all large n in case* (d).

P r o o f.

(i) For $s \geq n$, $\Delta(r(s)\Delta^2 u(s)) \leq \Delta(r(n)\Delta^2 u(n))$. Then

$$\begin{aligned}\sum_{i=n}^{s-1} \Delta\left(r(i)\Delta^2 u(i)\right) &\leq \sum_{i=n}^{s-1} \Delta\left(r(n)\Delta^2 u(n)\right)\\ &= (s-n)\Delta\left(r(n)\Delta^2 u(n)\right)\end{aligned}$$

that is,

$$r(s)\Delta^2 u(s) \leq r(n)\Delta^2 u(n) + (s-n)\Delta\left(r(n)\Delta^2 u(n)\right).$$

Consequently,

$$\sum_{i=n}^{s-1} \Delta^2 u(i) \leq r(n)\Delta^2 u(n) \sum_{i=n}^{s-1} \frac{1}{r(i)} + \Delta\left(r(n)\Delta^2 u(n)\right) \sum_{i=n}^{s-1} \frac{i-n}{r(i)},$$

that is,

$$0 < \Delta u(s) \leq \Delta u(n) + \Delta\left(r(n)\Delta^2 u(n)\right) \sum_{i=n}^{s-1} \frac{i-n}{r(i)}.$$

Taking limit as $s \to \infty$, (I_1) is obtained. For $n > n_0 > 0$, we have

$$\begin{aligned}u(n) > u(n) - u(n_0) &= \sum_{s=n_0}^{n-1} \Delta u(s) > \Delta u(n) \sum_{s=n_0}^{n-1} 1\\ &= (n-n_0)\Delta u(n).\end{aligned}$$

Hence there exists a constant L, $0 < L < 1$, such that $u(n) > Ln\Delta u(n)$. (I_3) is the direct consequence of (I_1) and (I_2).

ii) For $s \geq t+1 > t > n$, $r(s)\Delta^2 u(s) > r(t)\Delta^2 u(t)$. Thus

$$\sum_{i=t}^{s-1} \Delta^2 u(i) > r(t)\Delta^2 u(t) \sum_{i=t}^{s-1} \frac{1}{r(i)}$$

that is,

$$-\Delta u(t) > \Delta u(s) - \Delta u(t) > r(t)\Delta^2 u(t) \sum_{i=t}^{s-1} \frac{1}{r(i)}.$$

Consequently, as $s \to \infty$,

$$-\Delta u(t) \geq r(t)\Delta^2 u(t) \sum_{i=t}^{\infty} \frac{1}{r(i)}.$$

Hence

$$-\sum_{t=n}^{s-1}\Delta u(t)\geq\sum_{t=n}^{s-1}r(t)\Delta^2u(t)\sum_{i=t}^{\infty}\frac{1}{r(i)}$$

implies that

$$-u(s)+u(n)\geq\left(r(n)\Delta^2u(n)\right)\sum_{t=n}^{s-1}\sum_{i=t}^{\infty}\frac{1}{r(i)},$$

that is,

$$u(n)>r(n)\Delta^2u(n)R(n)$$

which is the required inequality. This completes the proof of the lemma. □

Remark. The inequality (I_1) still holds for the case (c) if $u(n)$ is eventually negative.

Remark. Since $R(n)<\sum\limits_{s=n}^{\infty}\frac{s}{r(s)}$, then $R(n)\to 0$ as $n\to\infty$ in view of (A_0).

Lemma 2.3. *Let* (A_0) *hold. If the conditions of* Lemma 2.1 *hold, then there exist constants* $L_1>0$ *and* $L_2>0$ *such that* $L_1R(n)\leq u(n)\leq L_2n$ *for all large* n.

Proof. Suppose that the first four cases of Lemma 2.1 hold for $n\geq N>1$. Summing the inequality $\Delta^2\left(r(n)\Delta^2u(n)\right)\leq 0$ from N to $n-1$ four times we get,

$$u(n)\leq u(N)+(n-N)\Delta u(N)+r(N)\Delta^2u(N)\sum_{t=N}^{n-1}\sum_{i=N}^{t-1}\frac{1}{r(i)}$$
$$+\Delta\left(r(N)\Delta^2u(N)\right)\sum_{t=N}^{n-1}\sum_{i=N}^{t-1}\frac{i-n}{r(i)}.$$

If we denote $g(n)=\sum\limits_{t=N}^{n-1}\sum\limits_{i=N}^{t-1}\frac{1}{r(i)}$, then $\Delta g(n)=\sum\limits_{i=N}^{n-1}\frac{1}{r(i)}$ and $\Delta^2g(n)=1/r(n)$. Hence $g(n)$ is the increasing function and $\lim\limits_{n\to\infty}g(n)=\infty$. Similarly, if we denote $g_N(n)=\sum\limits_{t=N}^{n-1}\sum\limits_{i=N}^{t-1}\frac{i-1}{r(i)}$, then $\Delta g_N(n)=\sum\limits_{i=N}^{n-1}\frac{i-n}{r(i)}$ and $\Delta^2g_N(n)=(n-N)/r(n)>0$, that is, $g_N(n)$ is the increasing function and hence $\lim\limits_{n\to\infty}g_N(n)=\infty$. In cases

(a) and (d) of Lemma 2.1, the last inequality becomes

$$u(n) \leq n \left[\frac{u(N)}{n} + \frac{n-N}{n} \Delta u(N) + \frac{r(N)\Delta^2 u(N)}{n} \sum_{t=N}^{n-1} \sum_{i=N}^{t-1} \frac{1}{r(i)} + \frac{\Delta\left(r(N)\Delta^2 u(N)\right)}{n} \sum_{t=N}^{n-1} \sum_{i=N}^{t-1} \frac{i-n}{r(i)} \right].$$

Applying Stolz's theorem [1], it follows that

$$\lim_{n\to\infty} \frac{g(n)}{n} = \lim_{n\to\infty} \frac{\Delta g(n)}{\Delta(n)} = k_1$$

and

$$\lim_{n\to\infty} \frac{g_N(n)}{n} = \lim_{n\to\infty} \Delta g_N(n) = k_2.$$

Hence there exists a constant $L_2 > 0$ such that the last inequality reduces to $u(n) \leq nL_2$ for all large n. For cases (b) and (c), we have that there exists a constant $L_2 > 0$ such that

$$\begin{aligned} u(n) &\leq n \left[\frac{u(N)}{n} + \frac{n-N}{n} \Delta u(N) + \frac{\Delta\left(r(N)\Delta^2 u(N)\right)}{n} g_N(n) \right] \\ &\leq nL_2, \end{aligned}$$

for all large n. Further, $u(n) \geq L_1 R(n)$ for all large n in cases (a), (b) and (c) because, since $R(n) \to 0$ as $n \to \infty$ and $\Delta u(n) > 0$ for all large n, there exist $L_1 > 0$ and $n_2 > n_1 > 0$ such that $u(n) > u(n_1) > L_1 R(n)$ for all $n \geq n_2$. From Lemma 2.2, it follows that $u(n) \geq r(n)\Delta^2 u(n) R(n)$ in case (d). Hence for any $n \geq n_1, u(n) \geq r(n_1)\Delta^2 u(n_1)\Delta^2 u(n_1) R(n) > L_1 R(n)$. Thus the lemma is proved. □

Lemma 2.4. ([5]) *Let p, y, z be real valued functions such that $z(n) = y(n) + p(n)y(n-m)$, $n \geq m \geq 0$, $y(n) > 0$ for $n \geq n_1 > m$, $\liminf_{n\to\infty} y(n) = 0$ and $\lim_{n\to\infty} z(n) = L$ exist. Let $p(n)$ satisfy one of the following conditions*

(i) $0 \leq p(n) \leq p_1 < 1$,

(ii) $1 < p_2 \leq p(n) \leq p_3$,

(iii) $p_4 \leq p(n) \leq 0$,

where p_i, $1 \leq i \leq 4$, are constants. Then $L = 0$.

3. Oscillation results

Sufficient conditions are obtained for oscillation of solutions of Equations (1) and (2). We need the following conditions.

(A_1) For $u > 0$ and $\nu > 0$, there exists $\lambda > 0$ such that $G(u)+G(\nu) \geq \lambda G(u+\nu)$.

(A_2) $G(u\nu) = G(u)G(\nu)$ for u, $\nu \in \Re$.

(A_3) $Q(n) = \min\{q(n), q(n-m)\}$, $n \geq m$.

(A_4) For $u > 0$, $\nu > 0$, $G(u)G(\nu) \geq G(u\nu)$.

(A_5) $G(-u) = -G(u)$, $u \in \Re$.

(A_6) There exists a real valued function $F(n)$ such that $\Delta^2 \left(r(n)\Delta^2 F(n)\right) = f(n)$ and $F(n)$ changes sign.

(A_7) Suppose that F is same as in (A_6). In addition,

$$-\infty < \liminf_{n\to\infty} F(n) < 0 < \limsup_{n\to\infty} F(n) < \infty.$$

(A_8) There exists a real valued function $F(n)$ such that $\Delta^2 \left(r(n)\Delta^2 F(n)\right) = f(n)$ and $\lim\limits_{n\to\infty} F(n) = 0$.

Remark. (A_2) implies (A_5). Indeed, $G(1)G(1) = G(1)$, so that $G(1) = 1$. Further, $G(-1)G(-1) = G(1) = 1$ gives $(G(-1))^2 = 1$. Because $G(-1) < 0$, then $G(-1) = -1$. Consequently, $G(-u) = G(-1)G(u) = -G(u)$. On the other hand $G(u\nu) = G(u)G(\nu)$ for $u > 0$, $\nu > 0$ and $G(-u) = -G(u)$ imply that $G(xy) = G(x)G(y)$ for every $x, y \in \Re$.

Remark. We may note that, if $y(n)$ is a solution of (1), then $x(n) = -y(n)$ is also a solution of (1), provided that G satisfies (A_2) or (A_5).

Remark. The prototype of G satisfying (A_1), (A_4), (A_5) is

$$G(u) = \left(a + b|u|^{\lambda}\right) |u|^{\mu} \operatorname{sgn} u,$$

where $a \geq 1$, $b \geq 1$, $\lambda \geq 0$ and $\mu \geq 0$. However, the prototype of G satisfying (A_1) and (A_2) is $G(u) = |u|^{\gamma} \operatorname{sgn} u$, where $\gamma > 0$. This G also satisfies the assumptions (A_1), (A_4) and (A_5).

Theorem 3.1. *Let $0 \leq p(n) \leq p < \infty$. Suppose that* ($A_0$)–($A_3$) *hold. If*

(A_9) $\sum\limits_{n=0}^{\infty} h(n)Q(n)G(R(n-k)) < \infty$,
where $h(n) = \min\{R^{\alpha}(n+1), R^{\alpha}(n-m+1)\}$ and $\alpha > 1$,

then all solutions of (1) *are oscillatory.*

P r o o f. Without any loss of generality we may suppose on the contrary that $y(n)$ is a non-oscillatory solution of (1) such that $y(n) > 0$ for $n \geq n_0$. Setting

$$z(n) = y(n) + p(n)y(n-m) \tag{4}$$

we obtain $z(n) < y(n) + py(n-m)$ and

$$\Delta^2 \left(r(n)\Delta^2 z(n)\right) = -q(n)G(y(n-m)) \leq 0, \tag{5}$$

but not identically zero for $n \geq n_0 + \rho$. Hence the four cases of Lemma 2.1 hold with $z(n)$. Suppose that one of the cases (a), (b), (d) of Lemma 2.1 holds. Then for $n \geq n_1 > n_0 + 2\rho$.

$$\begin{aligned}
0 &= \Delta^2 \left(r(n)\Delta^2 z(n)\right) + G(p)\Delta^2 \left(r(n-m)\Delta^2 z(n-m)\right) \\
&\quad + q(n)G(y(n-k)) + G(p)q(n-m)G(y(n-m-k)) \\
&\geq \Delta^2 \left(r(n)\Delta^2 z(n)\right) + G(p)\Delta^2 \left(r(n-m)\Delta^2 z(n-m)\right) + \lambda Q(n)G(z(n-k)) \\
&\geq \Delta^2 \left(r(n)\Delta^2 z(n)\right) + G(p)\Delta^2 \left(r(n-m)\Delta^2 z(n-m)\right) \\
&\quad + \lambda Q(n)G(L_1)G(R(n-m))
\end{aligned}$$

due to (A_1), (A_2), (A_3) and Lemma 2.3. Consequently,

$$\sum_{n=n_1}^{\infty} Q(n)G(R(n-m)) < \infty.$$

On the other hand, $h(n) \to 0$ as $n \to \infty$ and hence (A_9) yields the following contradiction

$$\sum_{n=n_1}^{\infty} Q(n)G(R(n-m)) = \infty.$$

Suppose that case (c) holds. From Lemma 2.2 and 2.3, it follows that there exist constants L, $L_2 > 0$ and $n \geq n_2 > n_1$ such that

$$L\Delta \left(-r(n)\Delta^2 z(n)\right) nR(n) \leq z(n) \leq nL_2, \qquad n \geq n_2. \tag{6}$$

Define $f \in C(\mathbb{R},\mathbb{R})$ such that $f(x) = x^{1-\alpha}$, $\alpha > 1$. Using the mean value theorem, we have that there exists $\beta \in \mathbb{R}$, such that $\Delta(-r(n)\Delta^2 z(n)) < \beta < \Delta(-r(n+1)\Delta^2 z(n+1))$ and

$$\begin{aligned}
&f\left(\Delta\left(-r(n+1)\Delta^2 z(n+1)\right)\right) - f\left(\Delta\left(-r(n)\Delta^2 z(n)\right)\right) \\
&= \left[\Delta\left(-r(n+1)\Delta^2 z(n+1)\right) - \Delta\left(-r(n)\Delta^2 z(n)\right)\right](1-\alpha)\beta^{-\alpha}.
\end{aligned}$$

Accordingly, we get

$$\begin{aligned}-\Delta\left[\Delta\left(-r(n)\Delta^2 z(n)\right)\right]^{1-\alpha} &= -(\alpha-1)\beta^{-\alpha}\left[\Delta^2\left(r(n)\Delta^2 z(n)\right)\right] \\ &> \frac{(\alpha-1)q(n)G(y(n-k))}{[-\Delta(r(n+1)\Delta^2 z(n+1)]^{\alpha}}\end{aligned}$$

and hence, using (6), we have

$$-\Delta\left[\Delta\left(-r(n)\Delta^2 z(n)\right)\right]^{1-\alpha} \geq (\alpha-1)R^{\alpha}(n+1)K^{\alpha}q(n)G(y(n-k)), \tag{7}$$

where $K = (L/L_2) > 0$. Thus

$$\begin{aligned}&-\Delta\left[\Delta\left(-r(n)\Delta^2 z(n)\right)\right]^{1-\alpha} - G(p)\Delta\left[\Delta\left(-r(n-m)\Delta^2 z(n-m)\right)\right]^{1-\alpha} \\ &> (\alpha-1)K^{\alpha}[R^{\alpha}(n+1)q(n)G(y(n-k)) \\ &\quad + G(p)R^{\alpha}(n-m+1)q(n-m)G(y(n-m-k))] \\ &> \lambda(\alpha-1)K^{\alpha}h(n)Q(n)G(z(n-k)) \\ &> \lambda(\alpha-1)K^{\alpha}G(L_1)h(n)Q(n)G(R(n-k))\end{aligned}$$

implies that

$$\sum_{n=n_2}^{\infty} h(n)Q(n)G(R(n-k)) < \infty,$$

which is a contradiction to (A_9). Hence the theorem is proved. □

Theorem 3.2. *Let $0 \leq p(n) \leq p < 1$. If (A_0) and (A_2) hold and if*

(A_{10}) $\sum\limits_{n=0}^{\infty} R^{\alpha}(n+1)G(R(n-k))q(n) = \infty$, $\alpha > 1$,

then every solution of (1) oscillates or tends to zero as $n \to \infty$.

Proof. Since $R(n) \to 0$ as $n \to \infty$, then (A_{10}) implies that

$$\sum_{n=0}^{\infty} G(R(n-k))q(n) = \infty \tag{8}$$

and hence

$$\sum_{n=0}^{\infty} q(n) = \infty. \tag{9}$$

Without any loss of generality let us suppose that $y(n)$ is a nonoscillatory solution of (1) such that $y(n) > 0$ for $n \geq n_0 > 0$. Setting $z(n)$ as in (4) to obtain $z(n) > 0$ and (5) for $n \geq n_0 + \rho$. Consequently, the conclusion of Lemma 2.1

holds for $z(n)$. Consider the cases (a) and (b) of Lemma 2.1. In either of the cases, $z(n)$ is nondecreasing and hence

$$(1-p)z(n) < z(n)-p(n)z(n-m) = y(n)-p(n)p(n-m)y(n-2m) \leq y(n) \quad (10)$$

for $n \geq n_0 + 2\rho$. By Lemma 2.3, there exist $L_1 > 0$ and $n_1 > n_0 + 2\rho$ such that $z(n) > L_1 R(n)$, $n \geq n_1$ so that (10) yields $y(n) > (1-p)L_1 R(n)$, $n \geq n_1$. Consequently, from (5) we obtain

$$\sum_{n=n_2}^{\infty} q(n)G(R(n-k)) < \infty, \qquad n_2 > n_1 + k,$$

a contradiction to (8). For the case (c) of Lemma 2.1, we proceed as in the proof of Theorem 3.1 to obtain (7). Since $z(n)$ is nondecreasing and $y(n) > (1-p)L_1 R(n)$ for $n \geq n_1 > n_0 + 2\rho$, then

$$-\Delta \left[\Delta \left(-r(n)\Delta^2 z(n)\right)\right]^{1-\alpha} \geq (\alpha-1)R^{\alpha}(n+1)K^{\alpha}q(n)G((1-p)L_1)G(R(n-k))$$

for $n \geq n_2 > n_1 + \rho$. Consequently, summing of the above inequality we obtain

$$\sum_{n=n_2}^{\infty} R^{\alpha}(n+1)G(R(n-k))q(n) < \infty,$$

a contradiction to (A_{10}). In case (d) of Lemma 2.1, $\lim_{n\to\infty} z(n)$ exists. If $\liminf_{n\to\infty} y(n) > 0$, then from (5) it follows that

$$\sum_{n=0}^{\infty} q(n) < \infty,$$

which is a contradiction to (9). Hence $\liminf_{n\to\infty} y(n) = 0$. By Lemma 2.4, we conclude that $\lim_{n\to\infty} z(n) = 0$. Thus $y(n) \leq z(n)$ implies that $\lim_{n\to\infty} y(n) = 0$. This completes the proof of the theorem. □

Theorem 3.3. *Let $-1 < p \leq p(n) \leq 0$. If (A_0), (A_2) and (A_{10}) hold, then every solution of (1) oscillates or tends to zero as $n \to \infty$.*

Proof. Let $y(n)$ be a nonoscillatory solution of (1) such that $y(n) > 0$ for $n \geq n_0 > 0$. Setting $z(n)$ as in (4) we obtain (5) for $n \geq n_0 + \rho$. Consequently, the conclusion of Lemma 2.1 holds for $z(n)$. Hence $z(n) > 0$ or $z(n) < 0$ for $n \geq n_1 > n_0 + \rho$. Suppose the former holds for $n \geq n_1$. Assume that one of the cases(a), (b) and (d) of Lemma 2.1 holds. From Lemma 2.3 we have that there exist $L_1 > 0$ and $n_2 > n_1$ such that $y(n) \geq z(n) \geq L_1 R(n)$, $n \geq n_2$ and hence (5) yields

$$\sum_{n=n_3}^{\infty} G(R(n-k))q(n) < \infty, \qquad n_3 > n_2 + \rho,$$

a contradiction to (8). Suppose that the case (c) holds. Proceeding as in the proof of Theorem 3.1, we obtain (7). Consequently, for $n \geq n_3 > n_2 + \rho$.

$$-\Delta\left[\Delta\left(-r(n)\Delta^2 z(n)\right)\right]^{1-\alpha} \geq (\alpha-1)R^{\alpha}(n+1)K^{\alpha}q(n)G(L_1)G(R(n-k))$$

due to $y(n) \geq L_1 R(n)$. Hence the above inequality yields

$$\sum_{n=n_3}^{\infty} G(R(n-k))q(n)R^{\alpha}(n+1) < \infty,$$

a contradiction to (A_{10}).

Suppose the later holds for $n \geq n_1$. Then $y(n) < y(n-m)$, that is, $y(n)$ is bounded. Thus $z(n)$ is bounded. Indeed, in each case (e) and (f) of Lemma 2.1, $\lim_{n\to\infty} z(n) = -\infty$. Accordingly, none of the cases (e) and (f) holds. In the case (b) or (c), $-\infty < \lim_{n\to\infty} z(n) \leq 0$. Then

$$\begin{aligned} 0 \geq \lim_{n\to\infty} z(n) &= \limsup_{n\to\infty}[y(n) + p(n)y(n-m)] \\ &\geq \limsup_{n\to\infty}[y(n) + py(n-m)] \\ &\geq \limsup_{n\to\infty} y(n) + \liminf_{n\to\infty}(py(n-m)) \\ &= \limsup_{n\to\infty} y(n) + p\limsup_{n\to\infty} y(n-m) \\ &= (1+p)\limsup_{n\to\infty} y(n). \end{aligned}$$

Hence $\lim_{n\to\infty} y(n) = 0$. In the case (d), $z(n) < \mu < 0$ for $n \geq n_2 > n_1$. Then $z(n) > py(n-m)$ implies that $y(n-m) > (\mu/p)$ for $n \geq n_2$. Consequently, from (5) we obtain

$$G(\mu/p)\sum_{n=n_3}^{\infty} q(n) < \infty, \qquad n_3 > n_2 + k$$

a contradiction to (9). The case $y(n) < 0$ for $n \geq n_0$ may similarly be dealt with. Hence the proof of the theorem is complete. □

Theorem 3.4. *Suppose that $-\infty < p_1 \leq p(n) \leq p_2 < -1$. If (A_0), (A_2) and (A_{10}) hold, then every bounded solution of (1) oscillates or tends to zero as $n \to \infty$.*

P r o o f. Let $y(n)$ be a bounded nonoscillatory solution of (1) such that $y(n) > 0$ for $n \geq n_0 > 0$. Then from (5) it follows that $z(n) > 0$ or $z(n) < 0$ for $n \geq n_1 > n_0 + \rho$, where $z(n)$ is given by (4). If $z(n) > 0$ for $n \geq n_1$, then each of the cases (a), (b) of Lemma 2.1 holds for $z(n)$. Proceeding as in the proof of the Theorem 3.3, we arrive at a contradiction. Next, we suppose that $z(n) < 0$ for $n \geq n_1$. Since $R(n) \to 0$ as $n \to \infty$, then (A_{10}) implies (9) and

$$\sum_{n=0}^{\infty} R^{\alpha}(n+1)q(n) = \infty. \tag{11}$$

In the case (b) or (c) of Lemma 2.1, $-\infty < \lim\limits_{n\to\infty} z(n) \leq 0$. Let $-\infty < \lim\limits_{n\to\infty} z(n) < 0$. Then there exists $n_2 > n_1$ and $\beta < 0$ such that $p_1 y(n-m) < z(n) < \beta$, $n \geq n_2$ and hence in the case (b) of Lemma 2.1, it follows that

$$G(\beta/p_1)\sum_{n=n_3}^{\infty} q(n) < \infty,$$

a contradiction to (9). In the case (c), first inequality of Lemma 2.2(i) yields that

$$-\Delta\left(r(n)\Delta^2 z(n)\right) \leq \Delta z(n)/R(n) < -z(n)/R(n)$$

and hence

$$-\Delta\left[\Delta\left(-r(n)\Delta^2 z(n)\right)\right]^{1-\alpha} > \frac{(\alpha-1)q(n)G(y(n-k))}{\left[-\Delta\left(r(n+1)\Delta^2 z(n+1)\right)\right]^{\alpha}}$$

implies that

$$-\Delta\left[\Delta\left(-r(n)\Delta^2 z(n)\right)\right]^{1-\alpha} > \frac{(\alpha-1)q(n)G(\beta p_1^{-1})R^{\alpha}(n+1)}{[z(n+1)]^{\alpha}}$$

for $n \geq n_2 > n_1 + \rho$. Further, $\Delta z(n) > 0$ for $n \geq n_2$ implies that $0 > z(n) > z(n_2) = \gamma$. Consequently, the last inequality reduces to

$$-\Delta\left[\Delta\left(-r(n)\Delta^2 z(n)\right)\right]^{1-\alpha} > \frac{(\alpha-1)q(n)G(\beta p_1^{-1})R^{\alpha}(n+1)}{(-\gamma)^{\alpha}}$$

for $n \geq n_3 > n_2 + \rho$. Thus

$$\sum_{n=n_3}^{\infty} q(n)R^{\alpha}(n+1) < \infty,$$

a contradiction to (11). If $\lim_{n\to\infty} z(n) = 0$, then we obtain

$$\begin{aligned} 0 = \lim_{n\to\infty} z(n) &= \liminf_{n\to\infty}[y(n) + p(n)y(n-m)] \\ &\le \liminf_{n\to\infty}[y(n) + p_2 y(n-m)] \\ &\le \limsup_{n\to\infty} y(n) + \liminf_{n\to\infty}(p_2 y(n-m)) \\ &= \limsup_{n\to\infty} y(n) + p_2 \limsup_{n\to\infty} y(n-m) \\ &= (1+p_2)\limsup_{n\to\infty} y(n), \end{aligned}$$

where no sum is of the form $\infty-\infty$ due to bounded $y(n)$. Since $(1+p_2) < 0$, then $\lim_{n\to\infty} y(n) = 0$. In case (d), one may proceed as in the proof of the Theorem 3.3 to get a contradiction. However, such a contradiction can not be obtained either in the case (e) or in the case (f) due to $\lim_{n\to\infty} z(n) = -\infty$. In these two cases, since $z(n) > p_1 y(n-m)$, then $\lim_{n\to\infty} y(n) = \infty$, a contradiction to the boundedness of $y(n)$.

The case $y(n) < 0$ for $n \ge n_0$ is similar and hence is omitted. This completes the proof of the theorem. $\square$

Example. Consider

$$\Delta^2 \left[n e^n \Delta^2 (y(n) + p(n)y(n-1))\right] + q(n)y^3(n-2) = 0, \tag{12}$$

where $n \ge 1$, $p(n) = -(e^{-2} + e^{-n})$, $q(n) = (e^2+1)^2 e^{2(n-2)}[(e^2+e+2)n + 2e + 2]$. It is easy to see that (12) satisfies all the conditions of Theorem 3.3. Hence every solution of (12) oscillates or tends to zero as $n \to \infty$. In particular, $y(n) = (-1)^n e^{-n}$ is an oscillatory solution.

Theorem 3.5. *Let* $0 \le p(n) \le p < \infty$. *Suppose that* (A_0), (A_1), (A_3)–(A_6) *hold. If*

(A_{11}) $\sum_{n=k}^{\infty} h(n)Q(n)G(F^+(n-k)) = \infty = \sum_{n=k}^{\infty} h(n)Q(n)G(F^-(n-k))$, *where* $h(n) = \min\{R^\alpha(n+1), R^\alpha(n+1-m)\}$, $\alpha > 1$,

then all solutions of (2) *are oscillatory.*

P r o o f. (A_0) implies that $R(n) \to 0$ as $n \to \infty$. Then $h(n) \to 0$ as $n \to \infty$ and hence (A_{11}) implies that

$$\sum_{n=k}^{\infty} Q(n)G(F^+(n-k)) = \infty = \sum_{n=k}^{\infty} Q(n)G(F^-(n-k)). \tag{13}$$

Let $y(n)$ be a non oscillatory solution of (2) such that $y(n) > 0$ for $n \geq n_0 > 0$. Set $w(n) = z(n) - F(n)$ for $n \geq n_0 + \rho$, where $z(n)$ is given by (4). Hence $0 < z(n) \leq y(n) + py(n-m)$ for $n \geq n_0 + \rho$. Equation (2) may be written as

$$\Delta^2 \left[r(n)\Delta^2 w(n)\right] = -q(n)G(y(n-k)) \leq 0 \tag{14}$$

for $n \geq n_0 + \rho$. Hence $w(n)$ is monotonic. With $w(n)$ we have two cases, $w(n) > 0$ or $w(n) < 0$ for $n \geq n_1 > n_0 + \rho$. Ultimately, a contradiction is obtained to (A_6) if $w(n) < 0$, that is, $0 < z(n) < F(n)$. Further $w(n) > 0$ for $n \geq n_1$. Consequently, Lemma 2.1 holds for $w(n)$. Clearly, $w(n) > 0$ yields that $z(n) > F^+(n)$ for $n \geq n_1$. Using (A_1), (A_3) and (A_4) we get

$$\begin{aligned} 0 &\geq \Delta^2 \left(r(n)\Delta^2 w(n)\right) + G(p)\Delta^2 \left(r(n-m)\Delta^2 w(n-m)\right) + \lambda Q(n)G(z(n-k)) \\ &\geq \Delta^2 \left(r(n)\Delta^2 w(n)\right) + G(p)\Delta^2 \left(r(n-m)\Delta^2 w(n-m)\right) + \lambda Q(n)G(F^+(n-k)) \end{aligned}$$

for $n \geq n_2 > n_0 + 2\rho$. If one of the cases (a), (b), (d) of lemma 2.1 holds, then it follows from the above inequality that

$$\sum_{n=n_2+k}^{\infty} Q(n)G(F^+(n-k)) < \infty,$$

a contradiction to (13). Assume that the case (c) of Lemma 2.1 holds. Then the use of Lemma 2.2 and 2.3 yields that there exist positive constants L, L_2 and $n_3 > n_2$ such that

$$L\Delta\left(-r(n)\Delta^2 w(n)\right) nR(n) \leq w(n) \leq nL_2, n \geq n_3$$

and hence

$$-\Delta\left[\Delta\left(-r(n)\Delta^2 z(n)\right)\right]^{1-\alpha} \geq (\alpha-1)R^{\alpha}(n+1)K^{\alpha}q(n)G(y(n-k)), \tag{15}$$

where $K = (L/L_2) > 0$. Thus

$$\begin{aligned} -\Delta\left[\Delta\left(-r(n)\Delta^2 z(n)\right)\right]^{1-\alpha} &- G(p)\Delta\left[\Delta\left(-r(n-m)\Delta^2 z(n-m)\right)\right]^{1-\alpha} \\ &> \lambda(\alpha-1)K^{\alpha}R^{\alpha}(n+1)Q(n)G(z(n-k)) \\ &> \lambda(\alpha-1)K^{\alpha}R^{\alpha}(n+1)Q(n)G(F^+(n-k)). \end{aligned}$$

Summing the above inequality we obtain

$$\sum_{n=n_3+k}^{\infty} h(n)Q(n)G(F^+(n-k)) < \infty,$$

a contradiction to (A_{11}). If $y(n) < 0$ for $n \geq n_0$, then we set $x(n) = -y(n)$ to obtain $x(n) > 0$ for $n \geq n_0$ and

$$\Delta^2(r(n)\Delta^2(x(n) + p(n)x(n-m))) + q(n)G(x(n-k)) = \tilde{f}(n),$$

where $\tilde{f}(n) = -f(n)$. If $\tilde{F}(n) = -F(n)$, then $\Delta^2(r(n)\Delta^2\tilde{F}(n)) = -f(n) = \tilde{f}(n)$ and $\tilde{F}(n)$ changes sign. Further, $\tilde{F}^+(n) = F^-(n)$ and $\tilde{F}^-(n)) = F^+(n)$. Proceeding as above we obtain a contradiction. Hence the theorem is proved. □

Theorem 3.6. *Let* $0 \le p(n) \le p < \infty$. *Suppose that* ($A_0$), ($A_1$), ($A_3$)–($A_5$), ($A_8$) *and* ($A_{11}$) *hold. Then every solution of* (2) *oscillates or tends to zero as* $n \to \infty$.

Proof. As in the proof of the Theorem 3.5, we obtain $w(n) > 0$ or $w(n) < 0$ for $n \ge n_1 > n_0 + \rho$. If $w(n) > 0$ for $n \ge n_1$, then by the similar steps of the Theorem 3.5 we have a contradiction. Let $w(n) < 0$ for $n \ge n_1$. Then $y(n) \le z(n) < F(n)$ and hence $\limsup_{n\to\infty} y(n) \le 0$ by (A_8). Consequently, $\lim_{n\to\infty} y(n) = 0$. The proof of the theorem is therefore completed. □

Theorem 3.7. *Let* $0 \le p(n) \le p < \infty$. *Let* ($A_0$), ($A_1$), ($A_3$)–($A_5$) *and* ($A_8$) *hold. If*

(A_{12}) $\sum\limits_{n=k}^{\infty} h(n)Q(n)G(|F(n-k)|) = \infty,$

then every bounded solution of (2) *oscillates or tends to zero as* $n \to \infty$.

Proof. As in the proof of the Theorem 3.5, we obtain (14). Hence $w(n) > 0$ or $w(n) < 0$ for $n \ge n_1 > n_0 + \rho$. Let $F(n) \ge 0$. If $w(n) > 0$ for $n \ge n_1$, there exists $n_2 > n_1$ such that $z(n) > F(n)$, $n \ge n_2$. Using (A_1), (A_3) and (A_4) we get

$$0 \ge \Delta^2(r(n)\Delta^2 w(n)) + G(p)\Delta^2(r(n-m)\Delta^2 w(n-m)) + \lambda Q(n)G(F(n-k))$$

for $n \ge n_3 > n_2 + \rho$. If one of the cases (a), (b), (d) of Lemma 2.1 holds, then

$$\sum_{n=n_3+k}^{\infty} Q(n)G(F(n-k)) < \infty,$$

which is a contradiction to (A_{12}), because (A_{12}) implies that

$$\sum_{n=k}^{\infty} Q(n)G(F(n-k)) = \infty.$$

In case (c) of Lemma 2.1, we may proceed as in the proof of Theorem 3.5 to obtain

$$\sum_{n=n_3+k}^{\infty} h(n)Q(n)G(F(n-k)) < \infty,$$

a contradiction to (A_{12}). Hence $w(n) < 0$ for $n \geq n_1$, that is, $y(n) < F(n)$. Consequently, $\liminf_{n\to\infty} y(n) = 0$. Further, in each of the cases (b) and (c) of Lemma 2.1, $\lim_{n\to\infty} w(n)$ exists and hence $\lim_{n\to\infty} z(n)$ exists. Since $y(n)$ is bounded, then $w(n)$ is bounded. In the case (d) of Lemma 2.1, $\lim_{n\to\infty} w(n)$ exists and hence $\lim_{n\to\infty} z(n)$ exists. On the other hand, the cases (e) and (f) of Lemma 2.1 do not hold here due to bounded $w(n)$. From Lemma 2.4, it follows that $\lim_{n\to\infty} z(n) = 0$. As $z(n) > y(n)$, then $\lim_{n\to\infty} y(n) = 0$.

Next, we suppose that $F(n) < 0$ for $n \geq n_2$. In this case $w(n) < 0$ implies that $0 < z(n) < F(n)$, a contradiction. Hence $w(n) > 0$ for $n \geq n_1$. Since $w(n)$ is bounded, the case (a) of Lemma 2.1 does not hold. Further, in each of the cases (b), (c) and (d) $\lim_{n\to\infty} w(n)$ exists. From (14) it follows that

$$\sum_{n=n_2}^{\infty} q(n)G(y(n-k)) < \infty$$

in each of the cases (b) and (d). We claim that $\liminf_{n\to\infty} y(n) = 0$. If not, we can find $\gamma > 0$ and $n^* > 0$ such that $y(n) > \gamma$ for $n > n^*$. Let $n_3 > \max\{n_2 + k, n^*\}$. Accordingly, the last inequality gives

$$G(\gamma) \sum_{n=n_3}^{\infty} q(n) < \infty,$$

which contradicts the assumption (A_{12}) due to $Q(n) \leq q(n)$. So our claim holds. In case (c) of Lemma 2.1, we obtain (15) which yields

$$\sum_{n=n_2}^{\infty} h(n)q(n)G(y(n-k)) < \infty.$$

Hence $\lim_{n\to\infty} y(n) = 0$; otherwise $\sum_{n=n_2}^{\infty} h(n)q(n) < \infty$, which contradicts the assumption (A_{12}). From Lemma 2.4, it follows that $\lim_{n\to\infty} z(n) = 0$ and hence $\lim_{n\to\infty} y(n) = 0$. The case $y(n) < 0$ for $n \geq n_0$ is similar. Thus the theorem is proved. $\square$

Remark. Equation (2) does not admit a non oscillatory solution due to Theorem 3.5, where $F(n)$ changes sign only. However, when the assumption (A_8) hold, Theorem 3.6 implies that only some oscillatory solutions of (2) could tend to zero as $n \to \infty$. Without the assumption (A_{11}), Theorem 3.7 predicts differently to that of the Theorems 3.5 and 3.6. Hence it seems that the nature of the forcing term influence the behaviour of the solutions of (2).

Theorem 3.8. *Let $-1 < p \le p(n) \le 0$. Suppose that* (A_0), (A_7) *and* (10) *hold. If*

$$(\mathrm{A}_{13}) \quad \sum_{n=k}^{\infty} R^{\alpha}(n+1)q(n)G(F^{+}(n-k)) = \infty = \sum_{n=k}^{\infty} q(n)G(F^{-}(n+m-k)),$$

and

$$(\mathrm{A}_{14}) \quad \sum_{n=k}^{\infty} R^{\alpha}(n+1)q(n)G(-F^{-}(n-k)) = -\infty = \sum_{n=k}^{\infty} q(n)G(-F^{+}(n+m-k)),$$

then a solution of (2) *oscillates.*

Proof. As in the proof of the Theorem 3.5, we obtain $w(n) > 0$ or $w(n) < 0$ for $n \ge n_1 > n_0 + \rho$. If $w(n) > 0$, then $y(n) > F(n)$ and hence $y(n) > F^{+}(n)$, $n \ge n_1$. Consequently, in each of the cases (a), (b) and (d) of Lemma 2.1, we obtain from (14) that

$$\sum_{n=n_1+k}^{\infty} q(n)G(F^{+}(n-k)) < \infty,$$

which contradicts the assumption (A_{13}). Using (15) in the case (c) of Lemma 2.1, we get

$$\sum_{n=n_2+k}^{\infty} R^{\alpha}(n+1)q(n)G(F^{+}(n-k)) < \infty,$$

which contradicts the assumption (A_{13}). Hence $w(n) < 0$ for $n \ge n_1$. We claim that $y(n)$ is bounded. If not, then there exists a sub sequence $\{n'_j\}$ of {n} such that $n'_j \to \infty$ and $y(n'_j) \to \infty$ as $j \to \infty$ and $y(n'_j) = \max\{y(n):\ n_1 \le n \le n'_j\}$. Hence

$$\begin{aligned} w(n'_j) &\ge y(n'_j) + py(n'_j - m) - F(n'_j) \\ &\ge (1+p)y(n'_j) - F(n'_j) \end{aligned}$$

and using (A_7), we come to the following contradiction that $w(n'_j) > 0$ for all large j. So our claim holds and $w(n)$ is bounded. Hence none of the cases (e) and (f) of Lemma 2.1 holds. Since $w(n) < 0$, then $y(n) > F^{-}(n+m)$. Thus in each of the cases (b) and (d) of Lemma 2.1, we obtain from (14) that

$$\sum_{n=n_1+k}^{\infty} q(n)G(F^{-}(n+m-k)) < \infty,$$

which contradicts the assumption (A_{13}). Let the case (c) of Lemma 2.1 hold. Then proceeding as in the proof of the Theorem 3.4 for the case (c) when $z(n) < 0$, replacing $z(n)$ by $w(n)$ and using (A_7), we get a contradiction to (11).

If $y(n) < 0$ for $n \geq n_0$, then one may proceed as above. This completes the proof of the theorem. $\square$

Theorem 3.9. *Suppose that all the conditions of* Theorem 3.8 *are satisfied except* (A_7)*, which is replaced by* (A_8)*. Then every solution of* (2) *oscillates or tends to zero as* $n \to \infty$*.*

Proof. If $w(n) > 0$, then a contradiction is obtained in each of the cases (a)–(d) of Lemma 2.1. Hence $w(n) < 0$ for $n \geq n_1 > n_0 + \rho$, that is, $z(n) < F(n)$. Since $z(n) \geq y(n) + py(n-m)$, $(1+p) > 0$ and $\limsup_{n\to\infty} z(n) \leq 0$, then $\lim_{n\to\infty} y(n) = 0$. Hence the proof is complete. $\square$

Theorem 3.10. *Let* $-\infty < p \leq p(n) \leq 0$*. If* (A_0), (A_2), (A_7), (A_{13}) *and* (A_{14}) *hold, then a solution* $y(n)$ *of* (2) *oscillates or* $|y(n)| \to \infty$ *as* $n \to \infty$*.*

The proof is similar to that of Theorem 3.8 and hence is omitted.

Theorem 3.11. *Let* $-1 < p \leq p(n) \leq 0$*. Suppose that* (A_0), (A_2) *and* (A_8) *hold. If*

(A_{15}) $\sum_{n=k}^{\infty} q(n)R^{\alpha}(n+1)G\left(|F(n-k)|\right) = \infty$, $\alpha > 1$,

then every solution of (2) *oscillates or tends to zero or tends to* $\pm\infty$ *as* $n \to \infty$*.*

Proof. As in the proof of the Theorem 3.5, we get $w(n) > 0$ or $w(n) < 0$ for $n \geq n_1 > n_0 + \rho$. Assume that $w(n) > 0$ for $n \geq n_1$. Then $y(n) \geq F(n)$. From (A_{15}) it follows that $\sum_{n=k}^{\infty} q(n)G\left(|F(n-k)|\right) = \infty$, $\sum_{n=k}^{\infty} q(n)R^{\alpha}(n+1) = \infty$ and $\sum_{n=k}^{\infty} q(n) = \infty$, due to $F(n) \to 0$ and $R^{\alpha}(n+1) \to 0$ as $n \to \infty$. If we suppose that $F(n) \geq 0$, for $n \geq n_2 > n_1$, then in each of the cases (a), (b) and (d) of Lemma 2.1, we have from (14) that

$$\sum_{n=n_2+k}^{\infty} q(n)G(F(n-k)) < \infty,$$

a contradiction. In the case (c) of Lemma 2.1, we obtain from (15) that

$$\sum_{n=n_2+k}^{\infty} q(n)R^{\alpha}(n+1)G(F(n-k)) < \infty,$$

which contradicts the assumption (A_{15}). Accordingly, $F(n) < 0$ for $n \geq n_2 > n_1$. In the case (a) for $w(n)$, $\lim_{n\to\infty} w(n) = \infty$ and accordingly, using (A_8), we get $\lim_{n\to\infty} z(n) = \infty$. Hence, $y(n) > z(n)$ yields $\lim_{n\to\infty} y(n) = \infty$. In each of the cases

(b) and (c) of Lemma 2.1, $\lim_{n\to\infty} w(n) = \beta$, $0 < \beta \le \infty$. If $\beta = \infty$, then as in the previous case we get that $\lim_{n\to\infty} y(n) = \infty$. Otherwise, if $\beta \in (0, \infty)$, using (A_8), we have that $\lim_{n\to\infty} z(n) = \beta$. From (14) we get

$$\sum_{n=n_2+k}^{\infty} q(n)G(y(n-k)) < \infty, \tag{16}$$

in the case (b). In the case (c), (14) yields

$$\sum_{n=n_2+k}^{\infty} q(n)R^{\alpha}(n+1)G(y(n-k)) < \infty.$$

Hence $\liminf_{n\to\infty} y(n) = 0$. From Lemma 2.4 it follows that $\beta = 0$, a contradiction. In the case (d) of Lemma 2.1, (16) holds and $\lim_{n\to\infty} w(n) = \beta \in [0, \infty)$ implying that $\lim_{n\to\infty} z(n) = \beta \in [0, \infty)$. If $\beta \in (0, \infty)$, as in the case (b) we come to the contradiction. If $\beta = 0$, since $z(n) \ge y(n) + py(n-m)$ and $(1 + p) > 0$, we conclude that $y(n)$ is bounded. Accordingly, $\limsup_{n\to\infty} y(n) = 0$, which implies that $\lim_{n\to\infty} y(n) = 0$.

Let $w(n) < 0$ for $n \ge n_1$. Then the following analysis holds for $F(n) \ge 0$ or $F(n) \le 0$. As in the proof of Theorem 3.8, we may show that $y(n)$ is bounded and accordingly $w(n)$ is bounded. Consequently, the cases (e) and (f) of Lemma 2.1 do not hold. Since $z(n) < F(n)$, $n \ge n_1$, we have that $\limsup_{n\to\infty} z(n) \le 0$. Thus in the cases (b), (c) and (d), we conclude from

$$\begin{aligned} 0 \ge \limsup_{n\to\infty}[y(n) + py(n-m)] &\ge \limsup_{n\to\infty} y(n) + \liminf_{n\to\infty}(py(n-m)) \\ &= (1+p)\limsup_{n\to\infty} y(n). \end{aligned}$$

that $\lim_{n\to\infty} y(n) = 0$. Similar conclusions can be obtained for the case $y(n) < 0$, $n \ge n_0$. Thus the theorem is proved. □

Corollary 3.12. *Suppose that the conditions of* Theorem 3.10 *hold. Then every bounded solution of* (2) *oscillates or tends to zero as* $n \to \infty$.

Remark. Theorems 3.8, 3.11 and Corollary 3.12 do not hold for homogeneous equation (1).

Example. Consider

$$\begin{aligned} \Delta^2\left[e^n\Delta^2(y(n) + (1+e^{-n})y(n-1))\right] + (ae^{-n} + be^{-2n} + e^3)y^3(n-1) \\ = (-1)^{n+1}e^{3n}, \qquad n \ge 0, \end{aligned} \tag{17}$$

where $a = \mathrm{e}^2(\mathrm{e}-1)(\mathrm{e}+1)^2(\mathrm{e}^2+1)^2$, $b = -4\mathrm{e}^2(\mathrm{e}+1)^2$, $R(n) = \mathrm{e}(\mathrm{e}-1)^{-2}\mathrm{e}^{-n}$ and $Q(n) = (a\mathrm{e}^{-n} + b\mathrm{e}^{-2n} + \mathrm{e}^3)$. Taking $\alpha = 2$, we get $h(n) = (\mathrm{e}-1)^{-2}\mathrm{e}^{-(2n+1)}$. Setting $F(n) = (-1)^{n+1}\mathrm{e}^{2n}/[(\mathrm{e}^3+1)(\mathrm{e}^2+1]^4$, we obtain $\Delta^2[\mathrm{e}^n\Delta^2 F(n)] = f(n) = (-1)^{n+1}\mathrm{e}^{3n}$. Since

$$\begin{aligned} F^+(n) &= \mathrm{e}^{2n}/[(\mathrm{e}^3+1)(\mathrm{e}^2+1)]^4, \qquad \text{if } n \text{ is odd} \\ &= 0, \qquad \text{if } n \text{ is even} \end{aligned}$$

and

$$\begin{aligned} F^-(n) &= \mathrm{e}^{2n}/[(\mathrm{e}^3+1)(\mathrm{e}^2+1)]^4, \qquad \text{if } n \text{ is even} \\ &= 0, \qquad \text{if } n \text{ is odd} \end{aligned}$$

then

$$\begin{aligned} &\sum_{n=1}^{\infty}(\mathrm{e}-1)^{-2}\mathrm{e}^{-(2n+1)}(a\mathrm{e}^{-n} + b\mathrm{e}^{-2n} + \mathrm{e}^3)G(F^+(n-1)) \\ =& \sum_{n=1}^{\infty}\mathrm{e}^{-1}(\mathrm{e}-1)^{-2}(a\mathrm{e}^{-3n} + b\mathrm{e}^{-4n} + \mathrm{e}^3\mathrm{e}^{-2n})\mathrm{e}^{6n}/\left[(\mathrm{e}^3+1)(\mathrm{e}^2+1)\right]^{12} \\ =& \sum_{n=1}^{\infty}\mathrm{e}^{-1}(\mathrm{e}-1)^{-2}\left[(\mathrm{e}^3+1)(\mathrm{e}^2+1)\right]^{-12}(a\mathrm{e}^{3n} + b\mathrm{e}^{2n} + \mathrm{e}^3\mathrm{e}^{4n}) \\ =& \ \infty. \end{aligned}$$

Hence every solution of (17) oscillates by Theorem 3.5. In particular, $y(n) = (-1)^n\mathrm{e}^n$ is such a solution of (17).

4. Existence of positive solutions

In this section some conditions are obtained for the existence of bounded positive solutions of (2).

Theorem 4.1. *Let $0 \le p(n) \le p < 1$. Assume that G is Lipschitzian on the intervals of the form $[a, b]$, $0 < a < b < \infty$ and $F(n)$ changes sign such that $-(1-p)/8 \le F(n) \le (1-p)/2$, where F is same as in* (A_6). *If*

(A_0') $\sum\limits_{n=0}^{\infty} \frac{n+1}{r(n)} < \infty$ *and*

(A_{16}) $\sum\limits_{n=0}^{\infty} (n+1)q(n) < \infty$,

then (2) *admits a positive bounded solution.*

P r o o f. It is possible to choose a positive integer N_1 such that

$$L\sum_{n=N_1}^{\infty}(n+1)q(n)<(1-p/2),\qquad \sum_{n=N_1}^{\infty}\frac{n+1}{r(n)}<\frac{1}{2},$$

where $L=\max\{L_1,G(1)\}$ and L_1 is the Lipschitz constant of G on $[(1-p)/8,1]$. Let $X=\ell_{\infty}^{N_1}$ be the Banach space of all real valued functions $x(n)$, $n\geq N_1$ with supremum norm

$$\|x\|=\sup\bigl\{|x(n)|:\ n\geq N_1\bigr\}.$$

We define a subset S of X as follows:

$$S=\bigl\{x\in X:\ (1-p)/8\leq x(n)\leq 1,\ \ n\geq N_1\bigr\}.$$

Hence S is a complete metric space with the metric induced by the norm on X. Let us define the mapping $T\colon S\to X$ as follows:

$$(Ty)(n)=\begin{cases}Ty(N_1+\rho), & N_1\leq n<N_1+\rho\\ -p(n)y(n-m)+\frac{1+p}{2}+F(n) & \\ -\sum\limits_{i=n}^{\infty}\frac{i-n+1}{r(i)}\sum\limits_{s=i}^{\infty}(s-i+1)q(s)G(y(s-k)), & n\geq N_1+\rho.\end{cases}$$

Hence for $n\geq N_1$, $Ty(n)\leq(1+p)/2+(1-p)/2=1$ and

$$Ty(n)\geq -p+(1+p)/2-(1-p)/8-\frac{(1-p)}{4}=(1-p)/8$$

because for $n\geq N_1$,

$$\begin{aligned}&\sum_{i=n}^{\infty}\frac{i-n+1}{r(i)}\sum_{s=i}^{\infty}(s-i+1)q(s)G(y(s-k))\\ &\leq G(1)\sum_{i=n}^{\infty}\frac{i-n+1}{r(i)}\sum_{s=i}^{\infty}(s-i+1)q(s)\\ &\leq G(1)\sum_{i=n}^{\infty}\frac{i+1}{r(i)}\sum_{s=i}^{\infty}(s+1)q(s)\\ &\leq G(1)\sum_{i=N_1}^{\infty}\frac{i+1}{r(i)}\sum_{s=N_1}^{\infty}(s+1)q(s)\\ &\leq(1-p)/4.\end{aligned}$$

Thus $Ty\in S$, that is, $T\colon S\to S$. Further, for $x,y\in S$,

$$|Ty(n)-Tx(n)|\leq p\|x-y\|+\frac{1}{4}(1-p)\|x-y\|=\frac{1}{4}(3p+1)\|x-y\|$$

Hence $\|Ty-Tx\| \le \frac{3p+1}{4}\|x-y\|$, for every $x, y \in S$ implies that T is a contraction. Consequently, T has a unique fixed point y in S which is the required solution of (2) such that $(1-p)/8 \le y(n) \le 1$, $n \ge N_1$. The proof of the theorem is complete. □

Theorem 4.2. *Let $-1 < p \le p(n) \le 0$. If (A_0') and (A_{16}) hold, G is Lipschitzian on the intervals of the form $[a,b]$, $0 < a < b < \infty$ and $F(n)$ changes sign such that $-(1+p)/8 \le F(n) \le (1+p)/2$, then* (2) *admits a positive bounded solution.*

P r o o f. It is possible to choose N_1, sufficiently large such that

$$L\sum_{n=N_1}^{\infty}(n+1)q(n) < \frac{(1+p)}{2}, \qquad \sum_{n=N_1}^{\infty}\frac{n+1}{r(n)} < \frac{1}{2},$$

where $L = \max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $[(1+p)/8, 1]$. In this case we define the subset S and the mapping T as follows:

$$S = \big\{x \in X:\ (1+p)/8 \le x(n) \le 1,\ \ n \ge N_1\big\}.$$

$$(Ty)(n) = \begin{cases} Ty(N_1+\rho), & N_1 \le n < N_1+\rho \\ -p(n)y(n-m) + \frac{1+p}{2} + F(n) & \\ -\sum\limits_{i=n}^{\infty}\frac{i-n+1}{r(i)}\sum\limits_{s=i}^{\infty}(s-i+1)q(s)G(y(s-k)), & n \ge N_1+\rho. \end{cases}$$

Rest of the analysis is similar to that of Theorem 4.1. □

Theorem 4.3. *Let $0 \le p(n) \le p < 1$. Assume that G is Lipschitzian on the intervals of the form $[a,b]$, $0 < a < b < \infty$. If (A_0'), (A_8) and (A_{16}) hold, then* (2) *admits a positive bounded solution.*

P r o o f. We choose N_1 sufficiently large so that

$$|F(n)| < (1-p)/10, \qquad n \ge N_1;$$

$$L\sum_{n=N_1}^{\infty}(n+1)q(n) < (1-p)/10, \qquad \sum_{n=N_1}^{\infty}\frac{n+1}{r(n)} < \frac{1}{2},$$

where $L = \max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $[(1-p)/20, 1]$. For this case we define the subset S and the mapping T as follows:

$$S = \big\{x \in X:\ (1-p)/20 \le x(n) \le 1,\ \ n \ge N_1\big\}.$$

$$(Ty)(n) = \begin{cases} Ty(N_1+\rho), & N_1 \le n < N_1+\rho \\ -p(n)y(n-m) + \frac{1+4p}{5} + F(n) & \\ -\sum\limits_{i=n}^{\infty} \frac{i-n+1}{r(i)} \sum\limits_{s=i}^{\infty} (s-i+1)q(s)G(y(s-k)), & n \ge N_1+\rho. \end{cases}$$

Rest of the analysis can be followed from the Theorem 4.1. This completes the proof of the theorem. □

Similar theorems may be obtained in other ranges of $p(n)$.

5. Summary

In this work, no super linearity or sub-linearity conditions are imposed on G. It is interesting to observe that the nature of the function $r(n)$ influences the behaviour of solutions of (1) or (2). This influence is more explicit in case of unforced equation (1). However, if $p(n) \le 0$, the results are not satisfactory. It seems that some extra conditions could help in this case. Equations (1) and (2) are studied under the assumption $\sum\limits_{n=0}^{\infty} \frac{n}{r(n)} = \infty$ in [7].

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