Mathematica Slovaca

DOI: 10.2478/s12175-008-0090-3
Math. Slovaca **58** (2008), No. 4, 509–520

SOME NOTES ON DENSELY CONTINUOUS FORMS

Peter Vadovič

(*Communicated by Ľubica Holá*)

ABSTRACT. We consider a special space of set-valued functions (multifunctions), the space of densely continuous forms $D(X,Y)$ between Hausdorff spaces X and Y, defined in [HAMMER, S. T.—McCOY, R. A.: *Spaces of densely continuous forms*, Set-Valued Anal. **5** (1997), 247–266] and investigated also in [HOLÁ, Ľ.: *Spaces of densely continuous forms, USCO and minimal USCO maps*, Set-Valued Anal. **11** (2003), 133–151]. We show some of its properties, completing the results from the papers [HOLÝ, D.—VADOVIČ, P.: *Densely continuous forms, pointwise topology and cardinal functions*, Czechoslovak Math. J. **58(133)** (2008), 79–92] and [HOLÝ, D.—VADOVIČ, P.: *Hausdorff graph topology, proximal graph topology and the uniform topology for densely continuous forms and minimal USCO maps*, Acta Math. Hungar. **116** (2007), 133–144], in particular concerning the structure of the space of real-valued locally bounded densely continuous forms $D_p^*(X)$ equipped with the topology of pointwise convergence in the product space of all nonempty-compact-valued multifunctions. The paper also contains a comparison of cardinal functions on $D_p^*(X)$ and on real-valued continuous functions $C_p(X)$ and a generalization of a sufficient condition for the countable cellularity of $D_p^*(X)$.

1. Introduction

For Hausdorff spaces X, Y, Z, we denote by $F(X,Y)$ the space of all functions from X to Y; by 2^Z, $\mathfrak{K}(Z)$ and $\mathfrak{F}(Z)$ we mean the family of all closed, nonempty compact and finite subsets of Z, respectively. Thus $F(X,2^Y)$ denotes the space of all closed-valued multifunctions from X to Y. Also, $\mathfrak{B}(x)$ denotes the local base of open sets at a point x; $\operatorname{int} A$ and $\overline{A}$ denote the interior and closure of A.

2000 Mathematics Subject Classification: Primary 54C60; Secondary 54A25, 54C35.
Keywords: locally bounded densely continuous form, pointwise convergence, Fréchet space, Volterra space.
This work was supported by Science and Technology Assistance Agency under the contract No. APVT-51-006904 and by the Eco-Net (EGIDE) programme of the Laboratoire de Mathématiques de l'Université de Saint-Etienne (LaMUSE), France.

Now we define the main object of our interest here, the space of densely continuous (or DC) forms $D(X,Y)$. First, a single-valued function f from X to Y is called *densely continuous* (or $f \in DC(X,Y)$), if the set $C(f)$ of points of continuity of f is dense in X. For any $f \in DC(X,Y)$ let ϕ_f denote the closure in $X \times Y$ of the graph of the restriction $f \upharpoonright C(f)$. The space $D(X,Y)$ of *densely continuous forms* then consists of the elements ϕ_f for all $f \in DC(X,Y)$. Of course, every densely continuous form has closed graph in $X \times Y$ and can be regarded as a *closed-valued multifunction*, if we put $\phi_f(x) = \{y \in Y : (x,y) \in \phi_f\}$ for each $x \in X$. Thus $D(X,Y) \subseteq F(X, 2^Y)$.

The subspace $D^*(X,Y)$ of *locally bounded densely continuous forms* is a selection of those forms $\phi_f \in D(X,Y)$ which satisfy: for each $x \in X$ there is an open neighborhood $O \in \mathfrak{B}(x)$ of x such that the closure of the set $\phi_f[O] = \bigcup \{\phi_f(x) : x \in O\}$ is compact in Y.

The spaces of densely continuous forms have probably emerged for the first time in the paper of Hammer and McCoy [HM]. Nowadays, densely continuous forms can be tracked down in various forms in different parts of mathematics, for example as the Argmin multifunction in variational calculus or others, see Holá [Ho1]. They also have an important relationship with minimal USCO maps (see Drewnowski and Labuda [DL]). For example, it has been shown in [HV2] (and earlier in [Ho1] for the real-valued case) that if X is Baire and Y is locally compact metric, then $D^*(X,Y)$ is the space of all minimal USCO maps from X to Y. It is well known that the minimal USCO maps are a very convenient tool in the theory of games (see Christensen [Ch] or Saint-Raymond [Sa]) or in functional analysis (see the paper of Borwein and Moors [BM] where a differentiability property of single-valued functions is characterized by their Clarke subdifferentials being convex minimal USCO maps).

Other properties of DC forms might also be of interest, such as the cardinal functions of topologies on such spaces (see Holá and McCoy [HoM] or [HV1]) or the selection properties of DC forms (see Holá and Holý [HH]) or relations of various topologies on spaces of DC forms (see [HV2] or Holý [Ho]).

Moreover, there are other features of minimal USCO maps or densely continuous forms that are worth investigating for more general classes of maps. In particular, the concept of minimality of multifunctions has been successfully generalized and has found interesting applications (see for example the papers of Matejdes [Ma1] and [Ma2]). The concept of generalized (semi)continuity for multifunctions has also been intensively studied (see for example Holý and Matejíčka [HoMa]).

2. Properties of DC forms

To define the topology τ_p of pointwise convergence on $F(X, 2^Y)$ let (Y, d) be a metric space and consider the well-known Hausdorff (extended-valued) metric H_d induced by d on the hyperspace 2^Y of all closed subsets of Y. The topology τ_p then has a local base at any $\phi \in F(X, 2^Y)$ consisting of sets of the form

$$W(\phi, A, \varepsilon) = \big\{\psi \in F(X, 2^Y) : \ (\forall\, x \in A)\big(H_d(\phi(x), \psi(x)) < \varepsilon\big)\big\}$$

for all finite subsets $A \in \mathfrak{F}(X)$ and $\varepsilon > 0$. The subspace $D^*(X, Y)$ with the induced topology τ_p will be denoted by $D_p^*(X, Y)$. If $Y = \mathbb{R}$ is the space of reals with the usual topology, then the range space is omitted from the notation, i.e. we have $D^*(X)$ and $D_p^*(X)$, respectively.

Now, if $(\mathfrak{K}(\mathbb{R}), H)$ denotes the space of all nonempty compact subsets of the reals equipped with the Hausdorff metric, then $F_p\big(X, \mathfrak{K}(\mathbb{R})\big)$ can be seen as a subspace of $F_p\big(X, 2^{\mathbb{R}}\big)$. Moreover, we know that every form in $D^*(X)$ assumes only nonempty compact values (see [HV2, Lemma 2.2]), so $D^*(X) \subseteq F_p\big(X, \mathfrak{K}(\mathbb{R})\big)$. Then it is not very hard to deduce the following:

Theorem 2.1. *If the Hausdorff space X is without isolated points, then*

(a) $\operatorname{int} D(X) = \emptyset$ *in* $F_p\big(X, 2^{\mathbb{R}}\big)$.

(b) $\operatorname{int} D^*(X) = \emptyset$ *in* $F_p\big(X, \mathfrak{K}(\mathbb{R})\big)$.

Proof. It is sufficient to prove (a), since the proof of (b) uses the same basic idea. Suppose by the contrary that there is $\phi \in \operatorname{int} D(X)$ and $A \in \mathfrak{F}(X)$ and $\varepsilon > 0$ such that $W(\phi, A, \varepsilon) \subseteq D(X)$. Then $X \setminus A$ is nonempty open. By [Ho1, Proposition 2.2] of Holá, there is an open dense set $U \subseteq X$ such that ϕ is USCO at every point of U (i.e. ϕ is upper semicontinuous and has nonempty compact values at the points of U; see Beer [Be] or [HV1], [HV2] for definitions). Choose $x \in U \cap (X \setminus A)$. The set $\phi(x)$ is compact, so choose $y \notin \phi(x)$ and define

$$\psi(z) = \begin{cases} \phi(z), & z \neq x, \\ \phi(x) \cup \{y\}, & z = x. \end{cases}$$

Hence $\psi \in F_p\big(X, 2^{\mathbb{R}}\big)$. There are U, V disjoint open sets such that $\phi(x) \subseteq U$ and $y \in V$. Since ϕ is upper semicontinuous at x, there is an open neighborhood W of x such that $\phi\,[\,W\,] \subseteq U$. Since $y \notin U$, this contradicts the idea of ψ belonging to $D(X)$, i.e. of having a generating function, for such function would have to be similar to the generating function of ϕ. But because of $\psi\,[\,W \setminus \{x\}\,] = \phi\,[\,W \setminus \{x\}\,] \subseteq U$, no value outside the set U can be reached at x by such densely continuous form. Therefore $\psi \notin D(X)$ but $\psi \in W(\phi, A, \varepsilon)$, a contradiction. $\square$

Theorem 2.2. *If the Hausdorff space X is without isolated points, then*

(a) *$D(X)$ is not closed in $F_p\big(X, 2^{\mathbb{R}}\big)$.*

(b) *$D^*(X)$ is not closed in $F_p\big(X, \mathfrak{K}(\mathbb{R})\big)$.*

P r o o f. In this case, the proofs of (a) and (b) are actually the same. Choose a point $x \in X$ arbitrarily. Define a net of functions in the following way: for each basic open neighborhood $U \in \mathfrak{B}(x)$ let f_U be the characteristic function of U in X. For each $U \in \mathfrak{B}(x)$ the boundary of U is nowhere dense in X, so each f_U is a densely continuous function and its induced form ϕ_U belongs to $D^*(X)$. Consider now the (multi)function $\phi \in F_p\big(X, \mathfrak{K}(\mathbb{R})\big)$ defined by

$$\phi(y) = \begin{cases} \{0\}, & y \neq x, \\ \{1\}, & y = x. \end{cases}$$

Such ϕ clearly has no generating function, so $\phi \notin D(X)$. Now let $A \in \mathfrak{F}(X)$ and $\varepsilon > 0$ be arbitrary. Let $A^* = A \setminus \{x\}$. The set $X \setminus A^*$ is an open neighborhood of x, i.e. there is $U \in \mathfrak{B}(x)$ disjoint from A^*. Then for each $V \in \mathfrak{B}(x)$, $V \subseteq U$, we have $\phi_V \restriction A^* = \{0\} = \phi \restriction A^*$ and $\phi_V(x) = \{1\} = \phi(x)$, which implies $\phi_V \in W(\phi, A, \varepsilon)$. Therefore the net $\big\{\phi_U : U \in \mathfrak{B}(x)\big\}$, which is contained in $D^*(X)$, converges pointwise to $\phi \notin D(X)$ and both statements are proved. $\square$

Example 2.1. The essential difference between locally bounded and ordinary densely continuous forms is that in the general case we are not able to guarantee that all values of the generated densely continuous form are either nonempty (consider the value at 0 of the form generated by the function $1/x$ in the space $\mathbb{R}$) or are "nice" bounded or compact sets, as the following situation depicts: consider the generating function $f(x) = (1/x) \cdot \sin(1/x)$ for $x \neq 0$ which is a continuous single-valued function everywhere except for one point of the domain, and its generated form is

$$\phi(x) = \begin{cases} \big\{\big(\frac{1}{x}\big) \cdot \sin\big(\frac{1}{x}\big)\big\}, & x \neq 0, \\ \mathbb{R}, & x = 0. \end{cases}$$

This is the major obstacle (sometimes invincible) in finding analogs of the results of locally bounded densely continuous forms for ordinary densely continuous forms.

3. Cellularity of $D_p^*(X)$: Fréchet and Volterra spaces

In this section we investigate a property of first countable regular spaces which allows the construction of the desired densely continuous form in the proof of the density lemma ([HV1, Lemma 3.12]), saying that for a first countable regular space (without isolated points) $D_p^*(X)$ is dense in $F_p\big(X, \mathfrak{K}(\mathbb{R})\big)$. The corollary of this lemma ([HV1, 3.13]) then implies the countable cellularity of $D_p^*(X)$ for first countable regular spaces X.

Subsequently we shall discuss some applications of the investigated property for densely continuous forms on Fréchet and Volterra spaces as well as some improvements of the mentioned sufficient condition from [HV1] concerning countable cellularity of $D_p^*(X)$.

Definition 3.1. We say the topological space X has the property **P**, if:

$$\forall\, x \in X \ \ \forall\, O_x \in \mathfrak{B}(x) \ \ \forall\, l \in \mathbb{N} \ \ \exists\, \{U_i\}_{i=1}^{\infty} \text{ pairwise disjoint open sets}$$
$$\text{such that } U_i \subseteq O_x \text{ and } x \in \overline{U_i} \text{ for each } i = 1, \dots, l.$$

Recall that a topological space is a *Fréchet* space (see [En]) if for any point x and any subset with $x \in \overline{A}$, there is a sequence in A converging to x.

Lemma 3.1. *Let X be a Hausdorff and Fréchet space without isolated points. Then X has the property **P**.*

Proof. Let $l \in \mathbb{N}$, let $x \in X$ and, without the loss of generality, let $O_x \in \mathfrak{B}(x)$ be its non-trivial neighborhood. Let $\{x_n : \ n \in \mathbb{N}\}$ be a sequence of distinct points of O_x converging to x. Since the set $C_k = \{x\} \cup \{x_n : \ n > k\}$ is compact and closed for each $k \in \mathbb{N}$ we are able to define by induction the open disjoint sets O_k and V_k such that $C_k \subseteq V_k \subseteq V_{k-1}$ and $x_k \in O_k \subseteq V_{k-1}$ (of course put $V_0 = O_x$). Thus we have a sequence $\{O_n : \ n \in \mathbb{N}\}$ of pairwise disjoint open sets with $x_n \in O_n$. Now, for each $i = 1, \dots, l$ put

$$U_i = \bigcup \{O_n : \ n \in \mathbb{N}, \ n \bmod l = i - 1\}.$$

Then $x \in \overline{U_i}$, so $\{U_i\}_{i=1}^{l}$ is the desired pairwise disjoint system of open sets. $\square$

Note 3.1. A topological space X is a *Volterra* space (see [GP]), if for each $f, g \in DC(X, \mathbb{R})$ the set $C(f) \cap C(g)$ is dense in X. It was shown in [GGP1] that X is Volterra if and only if for each pair $A, B \subseteq X$ of dense G_δ subsets the set $A \cap B$ is dense. Thus, every Baire space is Volterra, but there are Volterra spaces which are not of second category, i.e. not Baire spaces (see [GGP2]).

It can be shown (see Holá [Ho2] or McCoy [McC]) that if X is a Volterra (or Baire) space, then we can make the space $D(X)$ of densely continuous forms a vector space by defining: for densely continuous forms $\phi_f = \overline{f \restriction C\,(f)}$, $\phi_g = \overline{g \restriction C\,(g)}$ put $\phi_f + \phi_g := \overline{f + g \restriction C\,(f+g)} = \phi_{f+g}$ and for each $a \in \mathbb{R}$ put $a \cdot \phi_f := \overline{a \cdot f \restriction C\,(a \cdot f)} = \phi_{a \cdot f}$.

Theorem 3.1. *Let X be a Hausdorff, Fréchet and Volterra space with a non-isolated point. Then the addition is not continuous on $D_p^*(X)$, that is, $D_p^*(X)$ is not a linear topological space.*

Proof. Let $x \in X$ be the non-isolated point, let $O_x \in \mathfrak{B}(x)$ be its non-trivial open neighborhood, let $l = 2$. Then X has the property $\mathbf{P}$ at the point $x \in O_x$, i.e. using the procedure from Lemma 3.1 we get the disjoint open subsets $\{U_1, U_2\}$ in O_x such that $x \in \overline{U_1} \cap \overline{U_2}$. Define the functions $f, g : X \to \mathbb{R}$ by

$$f(y) = \begin{cases} 1, & y \in U_1, \\ -1, & y \notin U_1, \end{cases}$$

$$g(y) = \begin{cases} -1, & y \in U_1, \\ 1, & y \notin U_1. \end{cases}$$

Then $f, g \in DC(X)$ and we have $\phi_f = \overline{f \restriction C(f)}$, $\phi_g = \overline{g \restriction C(g)}$ in $D_p^*(X)$ such that $\phi_{f+g} \equiv 0 \equiv \phi_0$ is the zero (multi) function in the vector space with addition defined as in the remarks preceding the theorem.

Now if $A \in \mathfrak{F}(x)$ is any finite subset of X and $\varepsilon > 0$, we want to find an element $\phi_h \in D_p^*(X)$ such that $\phi_h \in W(\phi_f, A, \varepsilon)$ but $\phi_{h+g} \notin W(\phi_0, \{x\}, 1)$. So let $A' = A \setminus \{x\}$. Since A' is compact, there is an open neighborhood $U \in \mathfrak{B}(x)$ of x such that $\overline{U} \cap A' = \emptyset$.

Define the function $h \colon X \to \mathbb{R}$ by

$$h(y) = \begin{cases} g(y), & y \in U, \\ f(y), & y \notin U. \end{cases}$$

Then again $h \in DC(X)$ and we have $\phi_h \in D_p^*(X)$. If $y \in A'$, then $y \notin \overline{U}$, so $\phi_h(y) = \phi_f(y)$. Moreover, since $x \in U \in \mathfrak{B}(x)$, it is $\phi_h(x) = \phi_g(x) = \{-1, 1\} = \phi_f(x)$ and therefore $\phi_h \in W(\phi_f, A, \varepsilon)$.

On the other hand, $\phi_h + \phi_g = \phi_{h+g}$ and $(h+g)(y) = 2g(y)$ for $y \in U$ and therefore $(\phi_h + \phi_g)(x) = 2\phi_g(x) = \{-2, 2\}$, which implies $\phi_h + \phi_g \notin W(\phi_0, \{x\}, 1)$. This completes the proof of the theorem. □

On the other hand, one is able to see that the scalar multiplication is continuous on $D_p^*(X)$ even for non-Fréchet spaces. This is due to the fact that the finitely-determined basic sets $W(\phi_f, A, \varepsilon)$ in $D_p^*(X)$ are too "loose" in determining the properties of a contained form ϕ_g, but are still sufficient when the form is multiplied by a real scalar.

Theorem 3.2. *Let X be a Hausdorff and Volterra space. Then the scalar multiplication is continuous on $D_p^*(X)$.*

Proof. Let $\phi_f \in D_p^*(X)$, $A \in \mathfrak{F}(X)$ be finite and let $\varepsilon > 0$. Let $a \in \mathbb{R}$ be a scalar (without the loss of generality we may suppose that $a \neq 0$. It suffices to find $\delta > 0$ such that for each $b \in S_\delta(a) := (a-\delta, a+\delta)$ and each $\phi_g \in W(\phi_f, A, \delta)$ we have $\phi_{b\cdot g} \in W(\phi_{a\cdot f}, A, \varepsilon)$. Denote $S_\delta[A] := \bigcup\big\{S_\delta(a) :\ a \in A\big\}$.

Since ϕ_f assumes only nonempty compact values (see the remarks preceding Theorem 2.1), the set $\phi_f[A]$ is compact, so we can find

$$M = \max\Big\{\max\big\{|y| + \varepsilon :\ y \in \phi_f[A]\big\},\ |a| + 1\Big\}.$$

Moreover $M \geq \max\{1, \varepsilon\}$ and if we put $\delta = \frac{\varepsilon}{2M}$, then $\delta \leq \min\left\{\frac{\varepsilon}{2}, \frac{1}{2}\right\}$.

Now, let $\phi_g \in W(\phi_f, A, \delta)$, let $b \in S_\delta(a)$ and let $x \in A$. We want to show that $H(\phi_{b\cdot g}(x), \phi_{a\cdot f}(x)) < \varepsilon$ and for that it is sufficient to find for each $s \in \phi_{a\cdot f}(x)$ an element $t \in \phi_{b\cdot g}(x)$ such that $|s - t| < \varepsilon$ and vice versa (this argument will be symmetrical and will be dealt with later).

So if $s \in \phi_{a\cdot f}(x)$, then $(x, s) \in \phi_{a\cdot f} = \overline{a \cdot f \restriction C(f)}$ and for each $U \in \mathfrak{B}(x)$ and each $V \in \mathfrak{B}(s)$ we have $x_{(U,V)} \in U \cap C(f)$ such that $a \cdot f(x_{(U,V)}) \in V$. Thus we have a pair of double-indexed nets: $\{x_{(U,V)}\}$ converging to x and $\{a \cdot f(x_{(U,V)})\}$ converging to s. Put $s' = \frac{s}{a}$.

Since $Y = \mathbb{R}$ is a topological group, if $V \in \mathfrak{B}(s')$, then $a \cdot V \in \mathfrak{B}(s)$, i.e. the net $\{f(x_{(U,V)})\}$ converges to s' and therefore $(x, s') \in \overline{f \restriction C(f)} = \phi_f$ or equivalently $s' \in \phi_f(x)$. Recalling that $H(\phi_g(x), \phi_f(x)) < \delta$, we have

$$\phi_f(x) \subseteq S_\delta[\phi_g(x)] \qquad \text{and} \qquad \phi_g(x) \subseteq S_\delta[\phi_f(x)].$$

Thus there is $t' \in \phi_g(x)$ such that $|s' - t'| < \delta$. Since $(x, t') \in \phi_g = \overline{g \restriction C(g)}$, the same argument as before implies that $(x, b \cdot t') \in \phi_{b\cdot g} = \overline{b \cdot g \restriction C(g)}$, i.e $t := b \cdot t' \in \phi_{b\cdot g}(x)$.

Finally, since $t' \in S_\delta[\phi_f(x)]$, there is $y \in \phi_f(x)$ such that $|t'| \leq |y| + \delta < |y| + \varepsilon \leq M$. Now we have

$$\begin{aligned} |s - t| = |a \cdot s' - b \cdot t'| &\leq |a| \cdot |s' - t'| + |t'| \cdot |a - b| \\ &< M \cdot \delta + M \cdot \delta = 2M \cdot \frac{\varepsilon}{2M} = \varepsilon. \end{aligned}$$

Considering the reverse implication, we apply the same approach and for $t \in \phi_{b\cdot g}(x)$ we get $s \in \phi_{a\cdot f}(x)$ with

$$|t - s| = |b \cdot t' - a \cdot s'| \leq |b| \cdot |t' - s'| + |s'| \cdot |b - a| < \varepsilon,$$

because $|b| < |a| + \delta < |a| + 1 \leq M$. Therefore $H(\phi_{b\cdot g}(x), \phi_{a\cdot f}(x)) < \varepsilon$ and the theorem is proved. $\square$

The fact that a Fréchet space possesses the property **P** is also important for our next goal, the generalization of the countable cellularity of $D_p^*(X)$. First, realize the following theorem:

Theorem 3.3. *If X is a Hausdorff space with the property **P** and without isolated points, then $D_p^*(X)$ is dense in $F_p\big(X, \mathfrak{K}(\mathbb{R})\big)$.*

Proof. Let $W(\psi, A, \varepsilon)$ be an arbitrary nonempty open basic set in $F_p\big(X, \mathfrak{K}(\mathbb{R})\big)$ where $A = \{x_1, \dots, x_m\}$ is finite and $\varepsilon > 0$. Each $\psi(x_i)$ is compact, so for each $i = 1, \dots, m$ there is a finite subset $Y_i = \{y_1, \dots, y_{l_i}\}$ in $\psi(x_i)$ such that $\psi(x_i) \subseteq S_{\varepsilon/2}[Y_i]$.

Since A is finite, for each $i = 1, \dots, m$ there is a non-trivial open neighborhood $O(x_i)$ of x_i such that $O(x_i) \cap O(x_j) = \emptyset$, if $i \neq j$. Since X has the property **P**, $x_i \in O(x_i)$ and $l_i \in \mathbb{N}$, for each $i \in \{1, \dots, m\}$ we have the family $\{U_1, \dots, U_{l_i}\}$ of pairwise disjoint open sets in $O(x_i)$ such that $x_i \in \overline{U_i}$.

On each $O(x_i)$ define the function $f \colon X \to \mathbb{R}$ by

$$f(y) = \begin{cases} y_k, & y \in U_k, \ \ k = 1, \dots, l_i, \\ z, & y \notin \bigcup \{U_k : \ k = 1, \dots, l_i\}, \end{cases}$$

where $z \in Y_i = \{y_1, \dots, y_{l_i}\}$ is arbitrary fixed. Outside the set $\bigcup\limits_{i=1}^{m} O(x_i)$ the function f can be put arbitrarily constant.

Then $f \in DC(X)$ is densely continuous and let $\phi = \overline{f \restriction C\,(f)} \in D_p^*(X)$. The definition of f implies

$$\phi(x_i) = \{y_1, \dots, y_{l_i}\} = Y_i \subseteq \psi(x_i).$$

By our initial remarks: $\psi(x_i) \subseteq S_{\varepsilon/2}[\phi\,(x_i)]$ and therefore $H(\phi(x_i), \psi(x_i)) \leq \varepsilon/2 < \varepsilon$ for every $i = 1, \dots, m$, or equivalently, $\phi \in D_p^*(X) \cap W(\psi, A, \varepsilon)$ which proves the theorem. $\square$

Now we are ready to deliver the main result — a generalization of the sufficient condition for the countable cellularity of $D_p^*(X)$ (see [HV1, Theorem 3.13]). For the definition of cellularity see the next section of the paper.

Theorem 3.4. *Let X be a Hausdorff space which has the property **P**. Then $c\big(D_p^*(X)\big) = \aleph_0$.*

Proof. If X contains isolated points, then the construction described in the preceding theorem cannot be applied directly, because at an isolated point any densely continuous form is single-valued, i.e. no larger compact set can be generated as a value at such a point by a densely continuous form.

Thus consider the following: let $I(X)$ denote the set of all isolated points in X and denote by $Z = \prod_{x \in X} Y_x$ the product space where $Y_x = \mathbb{R}$ (with the usual topology) for $x \in I(X)$ and $Y_x = (\mathfrak{K}(\mathbb{R}), H)$ if $x \notin I(X)$.

We know by a result of Engelking (see [En, Corollary 2.3.18]) that the cellularity of Z is countable, and we know that cellularity is hereditary for dense subspaces. So let $W(\psi, A, \varepsilon)$ be an arbitrary nonempty open set in Z, i.e. $\psi \in Z$ is a form such that ψ is single-valued on $I(X)$ and compact-valued on $X \setminus I(X)$. Define the function $f\colon X \to \mathbb{R}$ by $f(x) = \psi(x)$ for $x \in I(X)$ and define f on $X \setminus I(X)$ as in the proof of the previous theorem. Then $\phi = \overline{f \restriction C(f)}$ is a locally bounded densely continuous form which is ε-close to ψ on A, i.e. $D_p^*(X)$ is dense in Z and hence $c\left(D_p^*(X)\right) = c(Z) = \aleph_0$. □

Corollary 3.1. *If X is a Hausdorff and Fréchet space, then $c\left(D_p^*(X)\right) = \aleph_0$.*

Note 3.2. It can be observed that if X is the product of any number of Hausdorff Fréchet spaces without isolated points, then X also has the property $\mathbf{P}$.

Corollary 3.2. *If X is the product of any number of Hausdorff Fréchet spaces, then $c\left(D_p^*(X)\right) = \aleph_0$.*

4. Cardinal functions: $D_p^*(X)$ vs. $C_p(X)$

To close this paper of remarks we intend to perform a short comparison of cardinal functions on two spaces equipped with the topology of pointwise convergence: the space $D_p^*(X)$ of real-valued locally bounded densely continuous forms on X and the space $C_p(X)$ of continuous real-valued functions on X.

For the first part, the most common cardinal functions for $D_p^*(X)$ were determined (or have had boundaries found) in the paper [HV1]. Concerning $C_p(X)$, we will use the standard arguments and the knowledge of C_p-theory (see for example the books of Arhangelskii [Ar] or McCoy and Ntantu [MN]). Moreover, all cardinal functions are supposed to assume only infinite values (as in Engelking [En]) and to conform with the assumptions of C_p-theory, the space X is throughout the section supposed to be Tychonoff (completely regular T_1).

We shall consider classic cardinal functions like the *weight* $w(Z)$, the *network weight* $nw(Z)$ or the *density* $d(Z)$ of a space Z which are defined as the minimal cardinalities of all bases, networks or dense subsets, respectively. The *cellularity* $c(Z)$, on the other hand, is defined as the supremum of the cardinalities of all systems of pairwise disjoint open sets. Firstly, it follows from [HV1, Theorem 3.8] and from [MN, Theorem 4.5.2] of McCoy and Ntantu that the weight satisfies:

Corollary 4.1. *For every Tychonoff space X, $w\left(D_p^*(X)\right) = |X| = w\left(C_p(X)\right)$.*

Next, if we consider the *pseudocharacter* of the space Z defined by $\Psi(Z) = \sup\bigl\{\Psi(Z,z) : z \in Z\bigr\}$, where $\Psi(Z,z)$ is the minimum of the cardinalities of all families $\mathfrak{G}$ of open sets in Z such that $\bigcap \mathfrak{G} = \{z\}$, and if we consider the *diagonal degree* $\Delta(Z)$ of Z defined as the minimum of the cardinalities of all families $\mathfrak{G}$ of open sets in $Z \times Z$ such that $\bigcap \mathfrak{G} = \bigl\{(z,z) : z \in Z\bigr\}$, then it follows from [HV1, Theorem 3.2] and from [MN, Theorem 4.3.1] that:

Corollary 4.2. *For every Tychonoff space X,*

$$\Psi\left(D_p^*(X)\right) = \Psi\left(C_p(X)\right) = \Delta\left(D_p^*(X)\right) = \Delta\left(C_p(X)\right) = d\left(X\right).$$

Continuing along this line, recall that a family $\mathfrak{P}(z)$ of nonempty open sets is called a *local π-base* (local pseudo-base) at $z \in Z$, if for each $U \in \mathfrak{B}(z)$ there is $P \in \mathfrak{P}(z)$ with $P \subseteq U$. Then the *π-character* of Z is defined by $\pi_\chi(Z) = \sup\bigl\{\pi_\chi(Z,z) : z \in Z\bigr\}$, where $\pi_\chi(Z,z)$ is the minimum of the cardinalities of all local π-bases at $z \in Z$. The *character* of Z is defined by $\chi(Z) = \sup\bigl\{\chi(Z,z) : z \in Z\bigr\}$, where $\chi(Z,z)$ is the minimum of the cardinalities of all local bases at $z \in Z$. It then follows from [HV1, Theorem 3.5] and from [MN, Theorem 4.4.1] that:

Corollary 4.3. *For every Tychonoff space X,*

$$\pi_\chi\left(D_p^*(X)\right) = \pi_\chi\left(C_p(X)\right) = \chi\left(D_p^*(X)\right) = \chi\left(C_p(X)\right) = |X|.$$

Concerning the cellularity, from the results of the previous section and from the results of C_p-theory we have the following corollary.

Corollary 4.4. *If X is the product of Fréchet Tychonoff spaces, then we have $c\left(D_p^*(X)\right) = \aleph_0 = c\left(C_p(X)\right)$.*

These results might lead to the conclusion that all the cardinals on $D_p^*(X)$ and $C_p(X)$ are in fact the same. Such conclusion, however, is quickly proved wrong by the virtue of [HV1, Example 4.1], which states that if $X = \mathbb{R}$ is the space of all reals with the usual topology, then $d\left(D_p^*(\mathbb{R})\right) > \aleph_0$. It follows that:

Corollary 4.5. *Whenever I is a compact interval of the reals, then the density satisfies:*

$$d\left(D_p^*(I)\right) > \aleph_0 = d\left(C_p(I)\right).$$

And directly from [HV1, Example 4.1] and [MN, Theorem 4.1.2] it also follows that:

Corollary 4.6. *The network weight satisfies*

$$nw\left(D_p^*(\mathbb{R})\right) > \aleph_0 = nw\left(C_p(\mathbb{R})\right).$$

Note 4.1. The collection of all results of [HV1] together with [HV2] and some previous results concerning function-space topologies can be found in [Va].

Acknowledgement. The author would like to thank Ľubica Holá for suggestions and comments and also Georges Grekos of LaMUSE, France.

REFERENCES

[Ar] ARHANGELSKII, A. V.: *Topological Function Spaces*, Kluwer Academic Publishers, Dordrecht, 1992.

[Be] BEER, G.: *Topologies on Closed and Closed Convex Sets*, Kluwer Academic Publishers, Dordrecht, 1993.

[BM] BORWEIN, J. M.—MOORS, W. B.: *Essentially smooth Lipschitz functions*, J. Funct. Anal. **149** (1997), 305–351.

[Ch] CHRISTENSEN, J. P. R.: *Theorems of Namioka and R. E. Johnson type for upper semicontinuous and compact valued set-valued mappings*, Proc. Amer. Math. Soc. **86** (1982), 649–655.

[DL] DREWNOWSKI, L.—LABUDA, I.: *On minimal upper semicontinuous compact valued maps*, Rocky Mountain J. Math. **20** (1990), 737–752.

[En] ENGELKING, R.: *General Topology*, Polish Scientific Publishers PWN, Warsaw, 1977.

[GP] GAULD, D. B.—PIOTROWSKI, Z.: *On Volterra spaces*, Far East J. Math. Sci. (FJMS) **1** (1993), 209–214.

[GGP1] GAULD, D. B.—GREENWOOD, S.—PIOTROWSKI, Z.: *On Volterra spaces II*, Ann. New York Acad. Sci. **806** (1996), 169–173.

[GGP2] GAULD, D. B.—GREENWOOD, S.—PIOTROWSKI, Z.: *On Volterra spaces III*, Topology Proc. **23** (1998), 167–182.

[HM] HAMMER, S. T.—McCOY, R. A.: *Spaces of densely continuous forms*, Set-Valued Anal. **5** (1997), 247–266.

[Ho1] HOLÁ, Ľ.: *Spaces of densely continuous forms, USCO and minimal USCO maps*, Set-Valued Anal. **11** (2003), 133–151.

[Ho2] HOLÁ, Ľ.: *Spaces of densely continuous forms*, Topology Proc. **31** (2007), 145–150.

[HH] HOLÁ, Ľ.—HOLÝ, D.: *Minimal USCO maps, densly continuous forms and upper semicontinuous functions*, Rocky Mountain J. Math. (To appear).

[HoM] HOLÁ, Ľ.—McCOY, R. A.: *Cardinal invariants of the topology of uniform convergence on compact sets on the space of minimal USCO maps*, Rocky Mountain J. Math. **37** (2007), 229–246.

[Ho] HOLÝ, D.: *Uniform convergence on spaces of multifunctions*, Math. Slovaca **57** (2007), 561–570.

[HoMa] HOLÝ, D.—MATEJÍČKA, L.: *C-upper semicontinuous and C^*-upper semicontinuous multifunctions*, Tatra Mt. Math. Publ. **34** (2006), 159–165.

[HV1] HOLÝ, D.—VADOVIČ, P.: *Densely continuous forms, pointwise topology and cardinal functions*, Czechoslovak Math. J. **58(133)** (2008), 79–92.

[HV2] HOLÝ, D.—VADOVIČ, P.: *Hausdorff graph topology, proximal graph topology and the uniform topology for densely continuous forms and minimal USCO maps*, Acta Math. Hungar. **116** (2007), 133–144.

[Ma1] MATEJDES, M.: *Minimal multifunctions and the cluster sets*, Tatra Mt. Math. Publ. **34** (2001), 71–76.

[Ma2] MATEJDES, M.: *Minimality of multifunctions*, Real Anal. Exchange (To appear).

[McC] McCOY, R. A.: *Spaces of semicontinuous forms*, Topology Proc. **23** (1998), 249–275.

[MN] McCOY, R. A.—NTANTU, I.: *Topological Properties of Spaces of Continuous Functions.* Lecture Notes in Math. 1315, Springer-Verlag, Berlin, 1988.

[Sa] SAINT-RAYMOND, J.: *Jeux topologiques et espaces de Namioka*, Proc. Amer. Math. Soc. **87** (1983), 499–504.

[Va] VADOVIČ, P.: *Topologies on Function Spaces and Hyperspace Topologies.* PhD Thesis, Mathematical Institute, Slovak Academy of Sciences, Bratislava, 2006.

Received 29. 12. 2006
Revised 6. 9. 2007

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK–81473 Bratislava
SLOVAKIA