VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-008-0088-x
Math. Slovaca **58** (2008), No. 4, 483–496

DEFORMATION PROPERTIES OF ONE REMARKABLE HYPERSURFACE BY H. TAKAGI IN $\mathbb{R}^4$

Z. Dušek* — O. Kowalski**

Dedicated to the memory of Professor H. Takagi

(*Communicated by Július Korbaš*)

ABSTRACT. We prove that the remarkable hypersurface found by H. Takagi in 1972 (as the first counter-example to the Nomizu conjecture on semi-symmetric spaces) is locally rigid.

1. Preface

A *semi-symmetric space* is a Riemannian manifold (M, g) satisfying the curvature condition $R(X, Y) \cdot R = 0$, where R denotes the curvature tensor and the dot denotes the derivation on the algebra of all tensor fields on M. It was already known to E. Cartan that any locally symmetric space and an arbitrary two-dimensional Riemannian manifold satisfy the identity above. In 1968 (see [9]), the Nomizu conjecture was formulated: *Every complete irreducible semi-symmetric Riemannian manifold of dimension $n \geq 3$ is locally symmetric.* The first counterexample to this conjecture was provided in 1972 by H. Takagi [14]. His example was a graph of a function of three variables defined on $\mathbb{R}^3$ and realized in $\mathbb{R}^4$. Almost the same time, other counter-examples were published

2000 Mathematics Subject Classification: Primary 53C20, 53C25, 53C40, 53C42.
Keywords: Riemannian manifold, hypersurface in Euclidean space, isometric deformation, semi-symmetric space.
The authors were supported by the grant GAČR 201/05/2707.
For the first author, this work is a part of the research project MSM 6198959214 financed by MŠMT.
For the second author, this work is a part of the research project MSM 0021620839 financed by MŠMT.

by K. Sekigawa. The systematic "rough" classification of all semi-symmetric spaces was provided by Z. Szabó in the eighties (see [11]–[13]). Finally, a summary of this theory, its further continuation and refinement, and more explicit classifications appeared in 1996 in the monograph [1].

But let us go back to the Takagi example. In 1995, the second author had the occasion to speak with Professor H. Takagi during the annual meeting of the Mathematical Society of Japan held at the Tôhoku University in Sendai. That time, H. Takagi called to his remembrance an episode saying that, after his counterexample was found, Professor K. Nomizu put the question about deformation properties of this three-dimensional hypersurface. Some attempts in this direction were done but without success. Now we want to fill in this gap.

Let us recall that the Takagi hypersurface is a hypersurface with *type number* 2. It means that the rank of the second fundamental form (i.e. that of the shape operator, or, still equivalently, that of the Gauss map) is equal to 2 everywhere. According to [6, Chap. VII, Sect. 6], this is equivalent with the property that the conullity of the curvature tensor of the induced metric is equal to 2. It is well-known that a hypersurface M ($\dim(M) \geq 3$) is locally rigid if the type number is ≥ 3, by the Beez-Killing Theorem, and locally flat and thus highly deformable if the type number is ≤ 1 everywhere (see [6]). The situation for the constant rank 2 is quite complex. Isometric deformations of such kind of hypersurface M (of dimension at least 3) were locally analyzed by V. Sbrana [10] and later by E. Cartan [2]. The main theorem says that we have just three main cases:

A) M is locally continuously deformable, and its isometric deformations depend either on arbitrary functions or arbitrary parameters.

B) M is locally rigid.

C) M is locally isometrically deformable in a unique way: to every sufficiently small domain $\Sigma \subset M$ there is (up to congruence) exactly one hypersurface $\bar{\Sigma} \subset \mathbb{R}^{n+1}$ which is isometric to Σ and which cannot be obtained from Σ by rigid motions or reflections.

E. Cartan used the extrinsic method in his work, namely the fact that every hypersurface $M \subset \mathbb{R}^{n+1}$ with type number 2 can be obtained as envelope of a two-parameter system of hyperplanes.

In recent times, much attention has been paid to the global study of hypersurfaces with type number two. For example, M. Dajczer and D. Gromoll [4] studied complete hypersurfaces of type number two and gave some conditions for continuous deformability or rigidity. H. Mori [8] also contributed to this topic. More recently, M. Dajczer, L. Florit and R. Tojeiro have given a revised version of the Cartan's paper (cf. [3]). The usual method used by all these authors is Cartan's method and the Gauss parametrization.

The authors of [1] offered a new approach to the study of local isometric deformations of hypersurfaces of type number two which can be called "intrinsic". This new method works as follows: take a Riemannian manifold of conullity two given in the explicit form in some local coordinates. Now, try to embed this manifold isometrically as a hypersurface in Euclidean space. This gives limiting conditions for the metric because a tensor field S of type $(1,1)$ ("a shape operator") must exist which is symmetric, of rank two, and satisfies the Gauss and Codazzi equations. If such S exists, then the corresponding hypersurface exists, its type number is two, and it is uniquely determined, at least locally, up to a congruence by this shape operator S (see [6, Chap. VII, Sect. 6]). If we start from an orientable hypersurface with type number two, we consider the induced Riemannian manifold (M, g), and the problem of isometric deformation is reduced to the question if there exist other potential shape operators S on (M, g) besides the two canonical ones, $\pm S_c$, belonging to the given hypersurface. If S is uniquely determined up to a sign, the hypersurface must be (locally) rigid.

V. Hájková [5] (see also [1] and [7]) used this method systematically in order to present classes of examples of all kinds A), B) or C). The intrinsic geometry of all these examples was what we call "non-elliptic foliated spaces of conullity two". In the present paper we apply the intrinsic method for the first time to an example which belongs to the class of "elliptic foliated spaces of conullity two". (See [1] for the detailed explanation of these notions.)

The aim of this paper is to prove the following:

Main Theorem. *The Takagi hypersurface is locally rigid.*

Let us remark that Dajczer and Gromoll in [4] proved nice general rigidity theorems for *complete* hypersurfaces. Yet, the only result concerning the 3-dimensional case is [4, Theorem 3.11], which can be formulated (in this special dimension) as follows:

> *Let M be a complete and locally irreducible hypersurface in $\mathbb{R}^4$ (with type number 2) whose scalar curvature s satisfies either $s > 0$ or $s < \varepsilon < 0$. Then M is globally rigid.*

The Takagi example satisfies the first two assumptions but, as we shall see, not the last one because the scalar curvature is negative and *not* separated from zero. Thus we prove the global rigidity in the situation when the above theorem cannot be applied. Yet, in our paper we prove more — namely the local rigidity of this hypersurface, where the completeness is not assumed. In our computation we use the software Maple.

In the sequel we shall need the following lemma:

Lemma 1.1. *Let $M \subset \mathbb{R}^4$ be a hypersurface and $x \in M$. Suppose that the rank of the shape operator S_x at x is equal to two. Then the nullity space $T_{x,o} \subset T_xM$ of the curvature tensor R_x is of dimension one and, for every $U \in T_{x,o}$, we have $S_xU = 0$. Moreover, an isometric deformation of M in $\mathbb{R}^4$ does not change the rank of the shape operator at x.*

P r o o f. This follows immediately from [6, Chap. VII Sect. 6]. □

2. Proof of the Main Theorem

P r o o f. The Takagi example is a hypersurface M in $\mathbb{R}^4$ defined as the graph of the function

$$f(x,y,z) = \frac{(x^2-y^2)z - 2xy}{2(z^2+1)} \tag{1}$$

with the definition domain $\mathbb{R}^3[x,y,z]$. The type number of M is two, everywhere, and the induced Riemannian metric is complete, locally irreducible and such that the nullity space of the curvature tensor is one-dimensional. The components of the induced Riemannian metric (in the standard coordinates x, y, z, which are considered as global coordinates on M) are

$$\begin{aligned}
g_{1,1} &= 1 + \frac{(xz-y)^2}{(z^2+1)^2}\\
g_{1,2} &= -\frac{(xz-y)(yz+x)}{(z^2+1)^2}\\
g_{1,3} &= -\frac{(xz-y)\left((x^2-y^2)(z^2-1) - 4\,xyz\right)}{2\,(z^2+1)^3}\\
g_{2,2} &= 1 + \frac{(yz+x)^2}{(z^2+1)^2}\\
g_{2,3} &= \frac{(yz+x)\left((x^2-y^2)(z^2-1) - 4\,xyz\right)}{2\,(z^2+1)^3}\\
g_{3,3} &= 1 + \frac{\left((x^2-y^2)(z^2-1) - 4\,xyz\right)^2}{4\,(z^2+1)^4}.
\end{aligned} \tag{2}$$

The components of the Ricci operator are

$$\begin{bmatrix}
2\,\frac{(\alpha+\beta)(z^2+1)^3\delta}{(\alpha\beta)^2} & 2\,\frac{(\alpha-\beta)(z^2+1)^3\gamma}{(\alpha\beta)^2} & 8\,\frac{(-y+zx)(z^2+1)^4\gamma}{(\alpha\beta)^2}\\
-2\,\frac{(\alpha-\beta)(z^2+1)^3\delta}{(\alpha\beta)^2} & -2\,\frac{(\alpha+\beta)(z^2+1)^3\gamma}{(\alpha\beta)^2} & -8\,\frac{(zy+x)(z^2+1)^4\delta}{(\alpha\beta)^2}\\
-4\,\frac{(-y+zx)(z^2+1)^2\gamma\,\delta}{(\alpha\beta)^2} & -4\,\frac{(zy+x)(z^2+1)^2\gamma\,\delta}{(\alpha\beta)^2} & -8\,\frac{(z^2+1)^3\left(\alpha\,\beta-(z^2+1)^2(\alpha+\beta)\right)}{(\alpha\beta)^2}
\end{bmatrix}, \tag{3}$$

where

$$\begin{aligned}\alpha &= 2\,z^4 + 4\,z^2 + (x - y + z\,(x+y))^2 + 2,\\ \beta &= 2\,z^4 + 4\,z^2 + (-x - y + z\,(x-y))^2 + 2,\\ \gamma &= \frac{\alpha+\beta}{2} - 2\,(zy + x)^2\,,\\ \delta &= -\frac{\alpha+\beta}{2} + 2\,(-y + zx)^2\,.\end{aligned} \tag{4}$$

It is clear that $\alpha > 0$ and $\beta > 0$. We choose an orthonormal moving frame $\{W_1, W_2, W_3\}$ on M as follows: for the components with respect to the coordinate basis $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$ we put

$$\begin{aligned}W_1 &= 2\,\left(z^2+1\right)(\alpha\beta)^{-\frac{1}{2}}\left[(xz-y), (yz+x), \left(z^2+1\right)\right],\\ W_2 &= (\alpha\beta)^{-\frac{1}{2}}\left[0, \left(z^2-1\right)\left(x^2-y^2\right) + 2\left(z^2+1\right)^2 - 4zxy - 2\left(z^2+1\right)(yz+x)\right],\\ W_3 &= (\alpha\beta)^{-\frac{1}{2}}\left[\left(z^2-1\right)\left(x^2-y^2\right) - 2\,\left(z^2+1\right)^2 - 4\,zxy, 0, 2\,\left(z^2+1\right)(xz-y)\right].\end{aligned}$$

It can be checked easily that the corresponding basis always consists of Ricci eigenvectors and the corresponding Ricci eigenvalues are

$$\lambda_1 = 0, \qquad \lambda_2 = \lambda_3 = -4\,\frac{\left(z^2+1\right)^3}{\alpha\beta}. \tag{5}$$

Moreover, we see easily that $\mathrm{span}(W_1)$ is the nullity space of the curvature tensor R at every point. For the covariant derivatives of the basic vector fields we obtain

$$\begin{aligned}&\nabla_{W_1} W_1 = \nabla_{W_1} W_2 = \nabla_{W_1} W_3 = 0\\ &\nabla_{W_2} W_1 = a \cdot W_2 + b \cdot W_3\\ &\nabla_{W_3} W_1 = -b \cdot W_2 + d \cdot W_3\\ &\nabla_{W_2} W_2 = -a \cdot W_1\\ &\nabla_{W_3} W_2 = b \cdot W_1\\ &\nabla_{W_2} W_3 = -b \cdot W_1\\ &\nabla_{W_3} W_3 = -d \cdot W_1\end{aligned} \tag{6}$$

where

$$\begin{aligned}a &= \frac{\left(z^2+1\right)\left(2\,z\,\alpha\,\beta + \alpha^2 - \beta^2\right)}{(\alpha\,\beta)^{3/2}},\\ d &= \frac{\left(z^2+1\right)\left(2\,z\,\alpha\,\beta - \alpha^2 + \beta^2\right)}{(\alpha\,\beta)^{3/2}},\\ b &= \frac{\left(z^2+1\right)\left(\alpha^2 + \beta^2\right)}{(\alpha\,\beta)^{3/2}}.\end{aligned} \tag{7}$$

Here we see that $b > 0$. Due to our Lemma 1.1 we see that, for every local isometric deformation of M in $\mathbb{R}^4$, the corresponding shape operators S must be (in the corresponding definition domain $U \subset M$) of the form

$$\begin{aligned} S(W_1) &= 0 \\ S(W_2) &= L \cdot W_2 + M \cdot W_3 \\ S(W_3) &= M \cdot W_2 + N \cdot W_3. \end{aligned} \tag{8}$$

For the canonical shape operator S_c of the Takagi hypersurface we obtain, using computer

$$L_c = \pm a, \qquad N_c = \mp d, \qquad M_c = \mp b. \tag{9}$$

Moreover, S must satisfy the Gauss equation and the Codazzi equation. The Codazzi equation has the form

$$\nabla_X(SY) - \nabla_Y(SX) - S(\nabla_X Y) + S(\nabla_Y X) = 0 \tag{10}$$

for all tangent vector fields X, Y. If we use W_i and W_j $(1 \leq i < j \leq 3)$ for X and Y in the equation (10) and use formulas (6) and (8), we get formally three vector equations. Every nontrivial coefficient of each of the vector fields W_i, $1 \leq i \leq 3$, in these vector equations must be equal to zero and we obtain the following differential equations:

$$W_1(L) + a \cdot L + b \cdot M = 0 \tag{11}$$

$$W_1(M) + a \cdot M + b \cdot N = 0 \tag{12}$$

$$W_1(M) - b \cdot L + d \cdot M = 0 \tag{13}$$

$$W_1(N) - b \cdot M + d \cdot N = 0 \tag{14}$$

$$M \cdot (d - a) - b \cdot (L + N) = 0 \tag{15}$$

$$W_2(M) - W_3(L) = 0 \tag{16}$$

$$W_2(N) - W_3(M) = 0. \tag{17}$$

Let us denote

$$m = \frac{d - a}{b} = -2 \, \frac{\left(\alpha^2 - \beta^2\right)}{\left(\alpha^2 + \beta^2\right)}. \tag{18}$$

From the equation (15) we obtain

$$N = m \cdot M - L \tag{19}$$

and by the differentiation in the directions W_2 and W_3 we get

$$W_3(N) = m \cdot W_3(M) + W_3(m) \cdot M - W_3(L), \tag{20}$$

$$W_2(N) = m \cdot W_2(M) + W_2(m) \cdot M - W_2(L). \tag{21}$$

It can be verified (using computer) that $W_1(m) = 0$. Then it is easy to see that using the equation (19) in the equations (12) and (14) we obtain linear combinations of the equations (11) and (13). Hence we can omit the equations (12) and (14).

Now we shall use the Gauss equation in the form

$$g(R(X,Y)Z,U) = g(SY,Z)g(SX,U) - g(SX,Z)g(SY,U). \tag{22}$$

We denote $G = g(R(W_2, W_3)W_3, W_2)$. From (22) and (8) we obtain (for arbitrary admissible shape operator S)

$$G = LN - M^2. \tag{23}$$

If we take for S our canonical operators $\pm S_c$, we obtain especially, using (9) and (7),

$$G = -ad - b^2 = \frac{-4(z^2+1)^3}{\alpha\beta}. \tag{24}$$

Whence there follow, among others, the formulas (5) and the formula for the scalar curvature

$$s = -8\frac{(z^2+1)^3}{\alpha\beta} < 0.$$

Putting $z = y = 0$ and $x \to +\infty$ we get $s \to 0$. Thus, the assumption from [4, Theorem 3.11] is not satisfied.

Now, let us go back to the formula (23). By the differentiation in the directions W_2 and W_3 we have

$$W_2(G) = L \cdot W_2(N) + N \cdot W_2(L) - 2M \cdot W_2(M) \tag{25}$$

$$W_3(G) = L \cdot W_3(N) + N \cdot W_3(L) - 2M \cdot W_3(M) \tag{26}$$

and after using (17), (19) and (20), the last formulas take on the form

$$-2M \cdot W_2(M) + L \cdot W_3(M) + (mM - L) \cdot W_2(L) = W_2(G) \tag{27}$$

$$(mM - 2L) \cdot W_2(M) + (mL - 2M) \cdot W_3(M) = W_3(G) - LM \cdot W_3(m). \tag{28}$$

Using (17), we can rewrite (21) in the form

$$-m \cdot W_2(M) + W_3(M) + W_2(L) = M \cdot W_2(m). \tag{29}$$

Now we shall solve the system of equations (27), (28) and (29) by the Cramer's rule with respect to the derivatives $W_2(M)$, $W_3(M)$, $W_2(L)$. For the simplicity we write $m_2 = W_2(m)$, $m_3 = W_3(m)$, $G_2 = W_2(G)$, $G_3 = W_3(G)$. Using (18), (24), (23) we see that the determinant of this system is equal to

$$D = (m^2 - 4)G = 64\,\frac{\left(z^2+1\right)^3\alpha\,\beta}{\left(\alpha^2+\beta^2\right)^2} > 0. \tag{30}$$

For the solution of the system (27), (28), (29) we obtain

$$\begin{aligned} W_2(M) = \frac{1}{D}\Big(& (2\,m_3 + m_2\,m)\,ML^2 - \left(m_3\,m + m_2\,m^2 + 2\,m_2\right)M^2L \\ & + (mG_2 - 2\,G_3)\,L + 2\,mm_2M^3 + (mG_3 - 2\,G_2)\,M\Big) \end{aligned} \tag{31}$$

$$\begin{aligned} W_3(M) = \frac{1}{D}\Big(& (2\,m_2 + m_3\,m)\,ML^2 + \left(-3\,m_2\,m + 2\,m_3 - m_3\,m^2\right)M^2L \\ & + (2\,G_2 - mG_3)\,L + m^2m_2M^3 + \left(\left(m^2-2\right)G_3 - mG_2\right)M\Big) \end{aligned} \tag{32}$$

$$W_2(L) = \frac{1}{D}\Big((2\,m_2 + m_3\,m)\,ML^2 - (3\,m_2\,m + 2\,m_3)\,M^2L \\ + \big(\big(m^2 - 2\big)\,G_2 - mG_3\big)\,L + 4\,m_2M^3 + (2\,G_3 - mG_2)\,M\Big). \tag{33}$$

From the equation (16) we obtain

$$W_3(L) = \frac{1}{D}\Big((2\,m_3 + m_2\,m)\,ML^2 - \big(m_3\,m + m_2\,m^2 + 2\,m_2\big)\,M^2L \\ + (mG_2 - 2\,G_3)\,L + 2\,mm_2M^3 + (mG_3 - 2\,G_2)\,M\Big) \tag{34}$$

and we rewrite the equations (11) and (13) in the form

$$W_1(L) = -aL - bM \tag{35}$$

$$W_1(M) = bL - dM. \tag{36}$$

Here we obtained the system (31), ...,(36) of 6 PDE for 2 functions L and M. We are going to write down the integrability conditions of this system. We denote by $\tilde{m}_2, \tilde{m}_3, \tilde{l}_2, \tilde{l}_3, \tilde{l}_1, \tilde{m}_1$ the corresponding right-hand sides of these equations. For the Lie brackets of the vector fields W_1, W_2, W_3 we easily see from (6)

$$\begin{aligned} [W_1, W_2] &= -aW_2 - bW_3 \\ [W_1, W_3] &= bW_2 - dW_3 \\ [W_2, W_3] &= -2bW_1. \end{aligned} \tag{37}$$

Now, the integrability conditions of the system (31), ...,(36) are, due to (37)

$$\begin{aligned} W_1(\tilde{l}_2) - W_2(\tilde{l}_1) &= -a\tilde{l}_2 - b\tilde{l}_3 \\ W_1(\tilde{m}_2) - W_2(\tilde{m}_1) &= -a\tilde{m}_2 - b\tilde{m}_3 \\ W_1(\tilde{l}_3) - W_3(\tilde{l}_1) &= b\tilde{l}_2 - d\tilde{l}_3 \\ W_1(\tilde{m}_3) - W_3(\tilde{m}_1) &= b\tilde{m}_2 - d\tilde{m}_3 \\ W_2(\tilde{l}_3) - W_3(\tilde{l}_2) &= -2b\tilde{l}_1 \\ W_2(\tilde{m}_3) - W_3(\tilde{m}_2) &= -2b\tilde{m}_1. \end{aligned} \tag{38}$$

Writing these conditions explicitly (using computer) we obtain the following six *algebraic* equations for L and M with coefficients which are known functions of x, y, z. (For the simplicity, we write $m_{23} = W_3(m_2)$, $G_{23} = W_3(G_2)$ and so on.)

$$\begin{aligned} & b\,(2\,m_2 + mm_3)\,L^3 - b\,\big(3\,mm_2 + \big(m^2 + 2\big)\,m_3\big)\,L^2M \\ & + b\,\big(\big(m^2 + 2\big)\,m_2 + 3\,mm_3\big)\,LM^2 - b\,(mm_2 + 2\,m_3)\,M^3 \\ & + \big(\big(-a\,\big(m^2 - 2\big) - bm\,\big(m^2 - 3\big)\big)\,G_2 + \big(am + b\,\big(m^2 - 2\big)\big)\,G_3 + D\,a_2\big)\,L \\ & + \big(\big(am + b\,\big(m^2 - 2\big)\big)\,G_2 - (bm + 2\,a)\,G_3 + D\,b_2\big)\,M = 0, \end{aligned} \tag{39}$$

$$\begin{aligned}&b\,(mm_2+2\,m_3)\,L^3-b\left(\left(m^2+2\right)m_2+3\,mm_3\right)L^2M\\&+b\left(3\,mm_2+\left(m^2+2\right)m_3\right)LM^2-b\,(2\,m_2+mm_3)\,M^3\\&+\left(-\left(am+b\left(m^2-2\right)\right)G_2+(bm+2\,a)\,G_3-D\,b_2\right)L\\&+((bm+2\,a)\,G_2-(2\,b+am)\,G_3+D\,d_2)\,M=0,\end{aligned}\tag{40}$$

$$\begin{aligned}&b\,(mm_2+2\,m_3)\,L^3-b\left(\left(m^2+2\right)m_2+3\,mm_3\right)L^2M\\&+b\left(3\,mm_2+\left(m^2+2\right)m_3\right)LM^2-b\,(2\,m_2+mm_3)\,M^3\\&+\left(-\left(am+b\left(m^2-2\right)\right)G_2+(bm+2\,a)\,G_3+D\,a_3\right)L\\&+((bm+2\,a)\,G_2-(2\,b+am)\,G_3+D\,b_3)\,M=0,\end{aligned}\tag{41}$$

$$\begin{aligned}&b\,(2\,m_2+mm_3)\,L^3-b\left(3\,mm_2+\left(m^2+2\right)m_3\right)L^2M\\&+b\left(\left(m^2+2\right)m_2+3\,mm_3\right)LM^2-b\,(mm_2+2\,m_3)\,M^3\\&+(-(bm+2\,a)\,G_2+(2\,b+am)\,G_3-D\,b_3)\,L\\&+\left((2\,b+am)\,G_2-\left(a\left(m^2-2\right)+bm\right)G_3+D\,d_3\right)M=0,\end{aligned}\tag{42}$$

$$\begin{aligned}&\left(m^2-4\right)\Big[\left({m_2}^2-{m_3}^2\right)L^4M+\left(-3\,m{m_2}^2+2\,m_2\,m_3+m{m_3}^2\right)L^3M^2\\&\quad+(m_2\,G_2+m_3\,G_3)\,L^3+\left(\left(2\,m^2+3\right){m_2}^2-2\,mm_2\,m_3-{m_3}^2\right)L^2M^3\\&\quad+\left(-4\,m{m_2}^2+2\,m_2\,m_3\right)LM^4+2\,{m_2}^2M^5\Big]\\&+\Big(\left(-\left(m^2-4\right)(mG_2+3\,G_3)-D_2\,m+2\,D_3\right)m_2\\&\qquad+\left(\left(m^2-4\right)(G_2-mG_3)-2\,D_2+D_3\,m\right)m_3\\&\qquad+\left({m_2}^2-{m_3}^2+m\,(m_{22}-m_{33})-2\,m_{23}+2\,m_{32}\right)D\Big)L^2M\\&+(((m^2-4)(G_2+3\,mG_3)+(m^2+2)D_2-3\,D_3\,m)m_2\\&\qquad+((m^2-4)(-mG_2+G_3)+D_2\,m-2\,D_3)m_3\\&\qquad+(-2\,m{m_2}^2+2\,m_2\,m_3-(m^2+2)m_{22}+2\,m_{33}\\&\qquad+m(3\,m_{23}-m_{32}))D)LM^2\\&+\Big((-mG_2+2\,G_3)\,D_2+\left(\left(m^2-2\right)G_2-mG_3\right)D_3\\&\qquad+((m_2-2\,mm_3)G_2+m_3\,G_3+m(G_{22}+G_{33})\\&\qquad-(m^2-2)G_{23}-2\,G_{32})D-2\,bD^2a)L\\&+((2\,G_2-mG_3)\,D_2+(-mG_2+2\,G_3)\,D_3\\&\qquad+(m_3\,G_2+m_2\,G_3-2\,G_{22}-2\,G_{33}+m\,(G_{23}+G_{32}))\,D-2\,b^2D^2\Big)M\\&+\Big(\left(-3\,\left(m^2-4\right)G_3+4\,D_3-2\,D_2\,m\right)m_2+\left(m^2-4\right)G_2\,m_3\\&\qquad+2\,\left({m_2}^2+mm_{22}-2\,m_{23}\right)D\Big)M^3=0,\end{aligned}\tag{43}$$

$$
\begin{aligned}
&\left(m^2-4\right)\Big[(m_3\,G_2+m_2\,G_3)\,L^3+\left(-3\,{m_2}^2+{m_3}^2\right)L^3M^2\\
&\quad+\left(5\,m{m_2}^2-2\,m_2\,m_3-m{m_3}^2\right)L^2M^3\\
&\quad+\left(-\left(2\,m^2+3\right){m_2}^2+2\,mm_2\,m_3+{m_3}^2\right)LM^4\\
&\quad+\left(2\,m{m_2}^2-2\,m_2\,m_3\right)M^5\Big]\\
&+\Big(\big(-\left(m^2-4\right)(G_2+2\,mG_3)-2\,D_2+D_3\,m\big)\,m_2\\
&\qquad+\left(-\left(m^2-4\right)G_3-D_2\,m+2\,D_3\right)m_3\\
&\qquad+(2\,m_{22}-2\,m_{33}+m\,(-m_{23}+m_{32}))\,D\Big)\,L^2M\\
&+\Big(\Big(\left(m^2-4\right)\left(mG_2+\left(m^2+1\right)G_3\right)+3\,D_2\,m-\left(m^2+2\right)D_3\Big)\,m_2\\
&\qquad+\Big(\left(m^2-4\right)\left(-\left(m^2-1\right)G_2+mG_3\right)+\left(m^2-2\right)D_2-D_3\,m\Big)\,m_3\\
&\qquad+(-3\,{m_2}^2+{m_3}^2-3\,mm_{22}+mm_{33}+(m^2+2)m_{23}\\
&\qquad+(-m^2+2)m_{32})D\Big)LM^2\\
&+\Big((-2\,G_2+mG_3)\,D_2+(mG_2-2\,G_3)\,D_3\\
&\qquad+(-m_3\,G_2-m_2\,G_3+2\,G_{22}+2\,G_{33}-m\,(G_{23}+G_{32}))\,D+2\,b^2D^2\Big)\,L\\
&+\Big(\left(-\left(m^2-4\right)(G_2+mG_3)-D_2\,m^2+2\,D_3\,m\right)m_2\\
&\qquad+\left(m^2-4\right)(mG_2-G_3)\,m_3\\
&\qquad+\left(2\,m{m_2}^2-2\,m_2\,m_3+m_{22}\,m^2-2\,mm_{23}\right)D\Big)\,M^3\\
&+\Big(\left(mG_2-\left(m^2-2\right)G_3\right)D_2+(-2\,G_2+mG_3)\,D_3\\
&\qquad+(-m_2\,G_2+(2\,mm_2-m_3)G_3-m(G_{22}+G_{33})+2\,G_{23}\\
&\qquad+(m^2-2)G_{32})D-2\,b(bm+a)D^2\Big)M=0. \qquad (44)
\end{aligned}
$$

In what follows we shall use the following procedure: into each term of the form L^iM^j in the equations (43) and (44), where $j>1$, we substitute for M^2 the expression

$$M^2=m\,L\,M-L^2-G \qquad (45)$$

which follows from (19) and (23). (We will not use the equations (39),…,(42), because they are consequences of the Gauss equation. We show this at the end of this section.) We iterate this operation until the symbol M occurs everywhere at most in the first power. After three iterations, the degree of the original equations will be reduced to three and we obtain the equations

$$
\begin{aligned}
P_1\,L^3+P_2\,L^2\,M+P_3\,L+P_4\,M&=0,\\
Q_1\,L^3+Q_2\,L^2\,M+Q_3\,L+Q_4\,M&=0.
\end{aligned} \qquad (46)
$$

Here $P_1, \dots, P_4$ and $Q_1, \dots, Q_4$ are again the functions in x, y, z. We can express them using the functions α, β, z as follows:

$$\begin{aligned}
P_1 &= \Big[2^{13}\,(z\,(\alpha+\beta)+(\alpha-\beta))\,(z\,(-\alpha+\beta)+(\alpha+\beta))\,\cdot\\
&\quad\cdot(\alpha+\beta)\,(\alpha\,\beta)^3\,(z^2+1)^3\Big]/(\alpha^2+\beta^2)^6,\\
P_2 &= \Big[-2^{13}\,(z\,(-\alpha^2+\beta^2+2\,\alpha\,\beta)+(\alpha^2-\beta^2+2\,\alpha\,\beta))\,\cdot\\
&\quad\cdot(z\,(-\alpha^2+\beta^2-2\,\alpha\,\beta)+(-\alpha^2+\beta^2+2\,\alpha\,\beta))\,\cdot\\
&\quad\cdot(\alpha\,\beta)^3\,(\alpha+\beta)\,(z^2+1)^3\Big]/(\alpha^2+\beta^2)^7,\\
P_3 &= \Big[2^{13}\,(2\,z\,(\alpha\,\beta)\,(-\alpha^2+\beta^2)+(\alpha^4+\beta^4+6\,(\alpha\,\beta)^2))\,\cdot\\
&\quad\cdot(-2\,z\,\alpha\,\beta-\alpha^2+\beta^2)\,(\alpha+\beta)\,(z^2+1)^6\Big]/(\alpha^2+\beta^2)^6,\\
P_4 &= \Big[-2^{13}\,(-2\,z\,\alpha\,\beta-\alpha^2+\beta^2)^2\,(\alpha+\beta)\,(z^2+1)^6\Big]/(\alpha^2+\beta^2)^5,\\
Q_1 &= \Big[2^{13}\,(z\,(-\alpha^2+\beta^2+2\,\alpha\,\beta)+(\alpha^2-\beta^2+2\,\alpha\,\beta))\,\cdot\\
&\quad\cdot(z\,(-\alpha^2+\beta^2-2\,\alpha\,\beta)+(-\alpha^2+\beta^2+2\,\alpha\,\beta))\,\cdot\\
&\quad\cdot(\alpha\,\beta)^3\,(\alpha+\beta)\,(z^2+1)^3\Big]/(\alpha^2+\beta^2)^7,\\
Q_2 &= \Big[-2^{13}\,(z\,(-\alpha^3+\beta^3+3\,\alpha\,\beta\,(-\alpha+\beta))+(-\alpha^3-\beta^3+3\,\alpha\,\beta\,(\alpha+\beta)))\,\cdot\\
&\quad\cdot(z\,(+\alpha^3+\beta^3-3\,\alpha\,\beta\,(\alpha+\beta))+(-\alpha^3+\beta^3+3\,\alpha\,\beta\,(-\alpha+\beta)))\,\cdot\\
&\quad\cdot(\alpha\,\beta)^3\,(\alpha+\beta)\,(z^2+1)^3\Big]/(\alpha^2+\beta^2)^8,\\
Q_3 &= \Big[2^{13}\,(2\,z\,\alpha\,\beta\,(\alpha^4+\beta^4-6\,(\alpha\,\beta)^2)+(-\alpha^6+\beta^6+9\,(\alpha\,\beta)^2\,(-\alpha^2+\beta^2)))\,\cdot\\
&\quad\cdot(-2\,z\,\alpha\,\beta-\alpha^2+\beta^2)\,(\alpha+\beta)\,(z^2+1)^6\Big]/(\alpha^2+\beta^2)^7,\\
Q_4 &= \Big[-2^{13}\,(-2\,z\,\alpha\,\beta-\alpha^2+\beta^2)^2\,(-\alpha+\beta)\,(\alpha+\beta)^2\,(z^2+1)^6\Big]/(\alpha^2+\beta^2)^6.
\end{aligned}\tag{47}$$

Let us denote by $f_1(L, M)$ and $f_2(L, M)$ the polynomials on the left-hand sides of the equations (46). We denote by $R_1(f_1(L), f_2(L))$ the resultant of f_1 and f_2 considered as polynomials in variable L and we denote by $R_2(f_1(M), f_2(M))$ the resultant of f_1 and f_2 considered as polynomials in variable M. Here R_1 is the polynomial of degree 5 in variable M and R_2 is the polynomial of degree 5 in variable L. We do not write explicitly these resultants, because first of them is too long and their computation is straightforward.

Now, necessary conditions for the functions L and M to be a solution of the equations (46) are

$$R_1(M) = R_2(L) = 0. \tag{48}$$

Using the expressions (47) for $P_1, \dots, P_4$, $Q_1, \dots, Q_4$ in the resultants R_1, R_2, the computer check easily verifies the equalities

$$\begin{aligned} R_1(M) &= h_1 M (M^2 - M_c^2)^2, \\ R_2(L) &= h_2 L (L^2 - L_c^2)^2. \end{aligned} \tag{49}$$

Here h_1, h_2 are the functions

$$\begin{aligned} h_1 &= 2^{84}\, \frac{\alpha^{16}\beta^{16}\,(\alpha+\beta)^6\,\left(z^2+1\right)^{26}\,\left(2\,z\alpha\,\beta+\alpha^2-\beta^2\right)^6\,\left(z\alpha^2-z\beta^2-2\,\alpha\,\beta\right)^2}{\left(\alpha^2+\beta^2\right)^{38}}, \\ h_2 &= -2^{28}\, \frac{\alpha^8\beta^8\,(\alpha+\beta)^2\,\left(z^2+1\right)^8}{\left(\alpha^2+\beta^2\right)^{12}} \end{aligned} \tag{50}$$

and L_c, M_c are the components of the canonical shape operator, given by the formulas (9) and (7). Now, the solution $L = 0$ implies (using the equations (46)) $M = 0$ and vice versa. But $L = M = 0$ is a contradiction with the Gauss equation

$$-L^2 - M^2 + m\,L\,M = G. \tag{51}$$

The remaining four possible solutions of the equations (48) are given by

$$L^2 = L_c^2, \qquad M^2 = M_c^2. \tag{52}$$

It is easy to show that only the two solutions

$$L = L_c, \quad M = M_c \qquad \text{and} \qquad L = -L_c, \quad M = -M_c \tag{53}$$

satisfy the Gauss equation. We conclude that the canonical shape operators $\pm S_c$ are the only admissible shape operators at every point, and the given hypersurface is locally rigid. □

Remark 1. Now we show that the equations (39),...,(42) are consequences of the Gauss equation. First, we compare the equations (40) and (41). We easily verify (using computer to the expressions in x, y, z of the functions a_3, d_2, b_2, b_3) that

$$a_3 + b_2 = 0, \qquad d_2 - b_3 = 0 \tag{54}$$

and hence the equations (40) and (41) are the same. To compare the equations (39) and (42) we use the following identities, which can be checked easily using computer:

$$\begin{aligned} \left(m^2-4\right)\left((a+b\,m)\,G_2 - b\,G_3\right) - (a_2+b_3)\,D = 0, \\ \left(m^2-4\right)\left(b\,G_2 + a\,G_3\right) + (b_2-d_3)\,D = 0. \end{aligned} \tag{55}$$

From here we see that the equations (39) and (42) are the same.

Now we make the substitution (45) into the equations (39) and (40). After two iterations we are left with two linear equations with respect to L and M, namely

$$\begin{aligned}
&\Big[\big(-2\,b^3 m_2 + b^2\,(a-d)\,m_3\big)\,G \\
&\qquad + \big((-a+3\,d)\,b^2 - d\,(a-d)^2\big) G_2 - b\,\big(d\,(a-d) + 2\,b^2\big)\,G_3 + b^2\,a_2\,D\Big] L \\
&+ \Big[b^2\,((d-a)\,m_2 + 2\,b m_3)\,G \\
&\qquad - b\,\big(d\,(a-d) + 2\,b^2\big)\,G_2 - b^2\,(a+d)\,G_3 + b^2\,b_2\,D\Big] M = 0, \\
&- \Big[b^2\,((d-a)\,m_2 + 2\,b m_3)\,G \\
&\qquad - b\,\big(d\,(a-d) + 2\,b^2\big)\,G_2 - b^2\,(a+d)\,G_3 + b^2\,b_2\,D\Big] L \\
&+ \Big[b^2\,(2\,b\,m_2 + b\,(d-a)\,m_3)\,G \\
&\qquad + b^2\,(a+d)\,G_2 + b\,\big(a\,(a-d) - 2\,b^2\big)\,G_3 + b^2\,d_2\,D\Big] M = 0.
\end{aligned} \tag{56}$$

Another computer check shows that all coefficients of these linear equations also vanish identically. Hence we conclude that the equations (39) and (40) are the algebraic consequences of the Gauss equation and they do not give new information.

Acknowledgement. The authors wish to thank to Professor Idzhad Sabitov for the idea which simplifies the final part of the proof.

REFERENCES

[1] BOECKX, E.—KOWALSKI, O.—VANHECKE, L.: *Riemannian Manifolds of Conullity Two*, World Scientific, Singapore, 1996.

[2] CARTAN, E.: *La déformation des hypersurfaces dans l'espace euclidien réel à n dimensions*, Bull. Soc. Math. France **44** (1916), 65–99.

[3] DAJCZER, M.—FLORIT, L.—TOJEIRO, R.: *On deformable hypersurfaces in space forms*, Ann. Mat. Pura Appl. (4) **174** (1998), 361–390.

[4] DAJCZER, M.—GROMOLL, D.: *Rigidity of complete Euclidean hypersurfaces*, J. Differential Geom. **31** (1990), 401–416.

[5] HÁJKOVÁ, V.: *Foliated Semi-Symmetric Spaces in Dimension Three*. Doctoral Dissertation, Charles University, Prague, 1994 (Czech).

[6] KOBAYASHI, S.—NOMIZU, K.: *Foundations of Differential Geometry II*, Interscience Publishers, New York, 1969.

[7] KOWALSKI, O.: *A new approach to the theory of hypersurfaces whose second fundamental form has rank two*. Abstract. Annual Meeting of the Math. Soc. of Japan, Sendai, September 1995.

[8] MORI, H.: *Remarks on complete deformable hpersurfaces in* $\mathbb{R}^4$, J. Differential Geom. **40** (1994), 1–6.

[9] NOMIZU, K.: *On hypersurfaces satisfying a certain condition on the curvature tensor*, Tohoku Math. J. **20** (1968), 46–59.

[10] SBRANA, V.: *Sulle varietà ad* $n-1$ *dimensioni defomabili nello spazio euclideo ad* n *dimensioni*, Rend. Circ. Mat. Palermo **27** (1909), 1–45.

[11] SZABÓ, Z. I.: *Structure theorems on Riemannian manifolds satisfying* $R(X,Y)\cdot R=0$*, I, Local version*, J. Differential Geom. **17** (1982), 531–582.
[12] SZABÓ, Z. I.: *Structure theorems on Riemannian manifolds satisfying* $R(X,Y)\cdot R=0$*, II, Global version*, Geom. Dedicata **19** (1985), 65–108.
[13] SZABÓ, Z. I.: *Classification and construction of complete hypersurfaces satisfying* $R(X,Y)\cdot R=0$, Acta Sci. Math. (Szeged) **47** (1984), 321–348.
[14] TAKAGI, H.: *An example of Riemannian manifold satisfying* $R(X,Y)\cdot R=0$ *but not* $\nabla R=0$, Tohoku Mah. J. **24** (1972), 105–108.

Received 19. 10. 2006
Revised 5. 2. 2007

* *Department of Algebra and Geometry*
Palacky University
Tomkova 40
CZ–779 00 Olomouc
CZECH REPUBLIC
E-mail: dusek@prfnw.upol.cz

** *Faculty of Mathematics and Physics*
Charles University
Sokolovská 83
CZ–186 75 Prague
CZECH REPUBLIC
Fax: +420 222 323 394
E-mail: kowalski@karlin.mff.cuni.cz